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A remark on sensitivity in linear programming and Gale-Samuelson nonsubstitution theorem*

by

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Abstract: The main purpose of this paper is to show that David Gale's result (1960, Lemma 9.3 on sensitivity in linear programming) is not generally valid. In this lemma, additional assumptions, that are instead required, are not made. We give some correct versions of the above mentioned lemma, and with these an elementary proof of the Gale-Samuelson nonsubstitution theorem.

Keywords: linear programming, sensitivity analysis, nonsubstitution theorem.

1. Introduction

David Gale (1960) proved a version of the Samuelson nonsubstitution theorem by means of a result (Lemma 9.3 in Gale, 1960) on sensitivity in linear programming problems. This result, also considered in a similar way by Hadley (1962), is not generally valid, as its conclusion requires some additional assumptions to those made by Gale. Due to the autonomous interest of Gale's sensitivity result, we first point out a mistake in Gale's proof and present a counterexample. This mistake was also noted in an unpublished paper by Maiti (1971). Finally, we shall give an elementary proof of Gale's version of Samuelson nonsubstitution theorem.

We now introduce some notations and definitions.

Let A be a real matrix of order (m, n), with rows A_i , $i = 1, \dots, m$ and columns A^j, $j = 1, \dots, n$; b is a column-vector of R^m; c is a row-vector of R^n, $x \geqq [0]$ is a non-negative column-vector of R^n.

The standard linear programming problem considered by Gale is the following:

$$P(A, b, c): \quad \begin{cases} \min\limits_x cx \\ Ax = b \\ x \geqq [0]. \end{cases}$$

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A vector $\overline{x}$ is said to be a *feasible solution* for $P(A, b, c)$ if $A\overline{x} = b$, $\bar{x} \geqq [0]$. Let $\mathcal{B}$ be a subset of the set of columns A^j of A, B the matrix formed with $A^j \in \mathcal{B}$, and $c_{\mathcal{B}}$ the vector whose components are c_j with $A^j \in \mathcal{B}$. A feasible solution x of $P(A, b, c)$ *depends* on the set $\mathcal{B}$ if $x_j = 0$ for $A^j \notin \mathcal{B}$.

The basis $\mathcal{B}$ is said to be a *feasible basis* for $P(A, b, c)$ if:

(i) There is a feasible solution x of $P(A, b, c)$ depending on $\mathcal{B}$;

(ii) The columns $A^j \in \mathcal{B}$ are linearly independent.

The same terminology applies to matrix B.

We note that it is always possible to assume that $rank(A) = m$ and that the matrix B (of $rank\ m$) is formed by the first m columns of A. The same permutations must be performed both on columns of A and on the vector x.

A feasible basis $\mathcal{B}$ for $P(A, b, c)$ is *degenerate* if for the feasible solution x of $P(A, b, c)$ depending on $\mathcal{B}$ we have $x_j = 0$ for some $A^j \in \mathcal{B}$. Otherwise, $\mathcal{B}$ is a *nondegenerate* basis.

A feasible basis $\mathcal{B}$ is an *optimal basis* for $P(A, b, c)$ if the feasible solution x of $P(A, b, c)$ depending on $\mathcal{B}$ is optimal for $P(A, b, c)$, i.e. minimizes cx subject to $Ax = b$, $x \geqq [0]$. So, an optimal solution x depending on an optimal basis $\mathcal{B}$ is called a *basic solution*, and if $\mathcal{B}$ is degenerate, x is a *degenerate solution*.

We now describe Gale's result on sensitivity for $P(A, b, c)$.

Let $\mathcal{B}$ be an optimal basis for $P(A, b, c)$. Consider a new problem $P'(A, b', c)$, obtained by replacing b in $P(A, b, c)$ by another vector $b' \neq b$. Gale's lemma asserts that if $\mathcal{B}$ remains a feasible basis for $P'(A, b', c)$, it is also optimal for it.

For the proof of this assertion, Gale considers a vector $\overline{x}$ as an optimal solution of $P(A, b, c)$ depending on $\mathcal{B}$, a vector $\overline{y}$ as an optimal solution of $D(A, b, c)$, the dual of $P(A, b, c)$, and x', as a feasible solution of $P'(A, b', c)$, depending on $\mathcal{B}$.

Therefore, since $\overline{x}$ and $\overline{y}$ are optimal for $P(A, b, c)$ and $D(A, b, c)$ respectively, by the canonical equilibrium theorem (Theorem 3.2 in Gale 1960, p. 82), we have

(i) $\overline{x}_j = 0$ whenever $\overline{y}A^j < c_j$ and, "since by hypothesis $x'_j = 0$ whenever $\bar{x}_j = 0$", one has also

(ii) $x'_j = 0$ whenever $\overline{y}A^j < c_j$,

so that by the same theorem, x' and $\bar{y}$ are optimal for $P'(A, b', c)$ and its dual $D'(A, b', c)$ respectively.

Now, the assertion quoted in *(i)* is false. Indeed, the hypothesis that $\mathcal{B}$ is an optimal basis for $P(A, b, c)$ and a feasible basis for $P'(A, b', c)$ only implies that, if $A^j \notin \mathcal{B}$, then $\overline{x}_j = x'_j = 0$, and not that $x'_j = 0$ whenever $\overline{x}_j = 0$. Clearly, if $\mathcal{B}$ is degenerate for $P(A, b, c)$, then $x_j = 0$, for some $A^j \in \mathcal{B}$, so we cannot assert that for this A^j, $x'_j = 0$. Hence, there is no reason why *(ii)* should be satisfied even when *(i)* is satisfied. On the other hand, if $\mathcal{B}$ is a nondegenerate optimal basis for $P(A, b, c)$, then $x_j = 0$ is equivalent to the statement that $A^j \notin \mathcal{B}$. Thus, Gale's conclusion is valid under some assumptions, i.e. that $\mathcal{B}$ is a nondegenerate optimal basis for $P(A, b, c)$ or some other (weaker) assumptions.

We remark that the incorrect version of Gale's lemma is quoted also in Hadley (1962), Heal et al. (1974), Lancaster (1968), Murata (1977) and Nicola (2000).

2. Counterexample to Gale's result and new versions

The following example (see De Giuli, 1995) shows that the conclusion of Gale's lemma may fail. We note that Gale's lemma has a general validity and is not therefore restricted to nonnegative matrices. The same holds for vector b and c.

Let

$$A = \begin{bmatrix} 10 & 5 & 0 & 0 & -120 \\ 110 & 111 & 118 & 0 & 0 \\ 0 & 1 & 0 & 147 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 10 & 110 & 0 \end{bmatrix}^T, \quad b' = \begin{bmatrix} 10 & 220 & 1 \end{bmatrix}^T$$

$$c = \begin{bmatrix} -90 & -90 & -90 & -90 & 100 \end{bmatrix}.$$

Then, both $P(A, b, c)$ and $P'(A, b', c)$ admit one optimal solution, namely

$$\overline{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad \text{for } P(A, b, c)$$
$$\overline{x}' = \begin{bmatrix} \frac{1}{2} & 1 & \frac{27}{59} & 0 & 0 \end{bmatrix}^T \quad \text{for } P'(A, b', c)$$

$\overline{x}$ is degenerate and may be associated with the following five bases:

$$B^{(1)} = \begin{bmatrix} A^1 & A^2 & A^3 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} A^1 & A^2 & A^4 \end{bmatrix},$$
$$B^{(3)} = \begin{bmatrix} A^1 & A^2 & A^5 \end{bmatrix}, \quad B^{(4)} = \begin{bmatrix} A^1 & A^3 & A^4 \end{bmatrix},$$
$$B^{(5)} = \begin{bmatrix} A^1 & A^4 & A^5 \end{bmatrix},$$

whereas $\overline{x}'$ is nondegenerate and may be associated with $B^{(1)}$ only. All five bases above, except for $B^{(2)}$, are feasible. Therefore, Gale's lemma leads to stating that each vector depending on a feasible basis, namely

$$x^1 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{27}{59} & 0 & 0 \end{bmatrix}^T \quad x^3 = \begin{bmatrix} \frac{100}{110} & 1 & 0 & 0 & \frac{9}{220} \end{bmatrix}^T$$
$$x^4 = \begin{bmatrix} 1 & 0 & \frac{55}{59} & \frac{1}{147} & 0 \end{bmatrix}^T \quad x^5 = \begin{bmatrix} 2 & 0 & 0 & \frac{1}{147} & \frac{1}{12} \end{bmatrix}^T,$$

is optimal for $P'(A, b', c)$, whereas only the first one is optimal.

Here we put forward some theorems which overcome these drawbacks.

Theorem 1 *Let x^* and y^* be optimal vectors, respectively, for the canonical problem $P(A, b, c)$ and for its dual. We denote by F the matrix made of all columns A^j of A associated with the constraints of the dual which y^* makes active:*

$$A^j \in F \iff y^* A^j = c_j. \tag{1}$$

Choose a vector $b' \neq b$ and assume that the system

$$\begin{cases} Fq = b' \\ q \geqq [0] \end{cases} \tag{2}$$

admits a solution q. Then the following propositions hold:
i) *Each vector $x' \in R^n$ obtained by filling with zeroes a solution q of system (2) is optimal for $P'(A, b', c)$;*
ii) *y^* is optimal for the dual of $P'(A, b', c)$ also;*
iii) *when b turns into b', the optimal value of the objective function exhibits the variation*

$$(cx' - cx^*) = y^*(b' - b).$$

Proof. After a suitable permutation of its columns, A displays first the columns of F, followed by those of the matrix G, which collects together the other columns of A :

$$A = \begin{bmatrix} F & G \end{bmatrix}.$$

Apply the same perturbation to the entries of x and consider the vector

$$x^* = \begin{bmatrix} q \\ [0] \end{bmatrix}, \tag{3}$$

which is feasible for $P'(A, b', c)$, as the relations

$$x' \geqq [0], \quad Ax' = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} q \\ [0] \end{bmatrix} = Fq = b' \tag{4}$$

obviously hold.
The dual problems of $P(A, b, c)$ and of $P'(A, b', c)$ have the same feasible set, hence y^* is feasible for both. Relation (1) states that $(y^* A^j - c_j)$ vanishes as soon as $A^j \in F$, whereas relation (3) shows that $A^j \notin F$ implies $x'_j = 0$. Therefore

$$(y^* A - c)x' = \sum_{A^j \in F} (y^* A^j - c_j)x'_j + \sum_{A^j \notin F} (y^* A^j - c_j)x'_j = 0. \tag{5}$$

As (4) gives $(Ax' - b') = [0]$, this yields

$$y^*(Ax' - b') = (y^* A - c)x' = [0]. \tag{6}$$

As x' and y^* are feasible for $P'(A, b', c)$ and its dual, respectively, the complementarity theorem shows their optimality for these problems. Item *(iii)* directly follows from the duality theorem. ∎

Theorem 1 holds for a maximization problem and only provides a sufficient condition for the validity of its thesis.

Theorem 2 reported below is taken from an unpublished report by Maiti (1971). Maiti also pointed out the flaw in Gale's proof of Lemma 9.3.

Let $c_{\mathcal{B}}$ be the vector, whose components are c_j, for those j such that $A^j \in \mathcal{B}$, therefore the next theorem holds.

THEOREM 2 *Assume that $\mathcal{B}$ is a feasible basis for both problems $P(A, b, c)$ and $P'(A, b', c)$. Let*

$$c_{\mathcal{B}} B^{-1} A \leqq c. \tag{7}$$

Then, $\mathcal{B}$ is an optimal basis for both problems.

Proof. Let x^* and x' be the solutions of $P(A, b, c)$, and $P'(A, b', c)$, respectively, depending on $\mathcal{B}$. Define $y^* = c_{\mathcal{B}} B^{-1}$. Due to (7), y^* is a feasible solution of the dual problem of both $P(A, b, c)$ and $P'(A, b', c)$. Further, if $A^j \in \mathcal{B}$, we have $y^* A^j = c_{\mathcal{B}} B^{-1} A^j = c_j$. Therefore $y^* A^j < c_j$ implies that $A^j \notin \mathcal{B}$, which in turn, implies $x^*_j = x'_j = 0$. By the canonical equilibrium theorem, (x^*, y^*) and (x', y^*) are optimal for $P(A, b, c)$ and its dual and for $P'(A, b', c)$ and its dual. ■

We observe that if $\mathcal{B}$ is a nondegenerate optimal basis for $P(A, b, c)$, then the optimal solution of the dual problem is unique and is equal to $c_{\mathcal{B}} B^{-1}$. Hence relation (7) holds in this case, as it is implied by the assumption of a nondegenerate optimal basis. The converse need not be true, thus Theorem 2 is a weaker version of Gale's lemma.

We now consider the following example suggested by one of the two referees in order to show that Theorem 1 does not work, whereas Theorem 2 works, or vice-versa.

Suppose $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ is a column vector representing levels of operation of the three activities

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Upon solving the primal problem $P(A, b, c)$ we have an $x \geqq [0]$ such that $x_1 + x_2 + x_3$ is a minimum subject to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3 = b,$$

where $b = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

Upon solving the corresponding dual we have a row vector $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$ such that $y_1 + y_2$ is a maximum subject to

$$y_1 \leq 1, \quad y_1 + y_2 \leq 1, \text{ and } y_2 \leq 1.$$

Problem P has a unique optimal solution, $x_2^* = 1, x_1^* = x_3^* = 0$, while its dual has an infinite number of optimal solutions: any point (within the first quadrant) on the line $y_1 + y_2 = 1$, including the two extreme points, $y_{e_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $y_{e_2} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is an optimal solution. Let us now take a non-basic optimal solution $y^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, so $y_1^* a_1 < 1$, $y^* a_3 < 1$ while $y^* a_2 = 1$. Following Theorem 1 the subsystem

$$S: \qquad a_2 x_2 = b$$

has a unique solution $x_2^* = 1$. If, instead, we consider another primal problem P' in which b is replaced by $b' = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, since the subsystem S does not admit any solution for b', Theorem 1 cannot be applied here. If P had a basic solution, $x_2^* = 1$, $x_3^* = 0$, depending on the basis $[a_2, a_3]$, then this basis would be feasible and optimal for P', yielding a unique optimal solution $x_2' = 1$, $x_3' = 1$. According to Theorem 2, referring to this basis, we note that the dual vector is given by $c_{\mathcal{B}} B^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and that $\begin{bmatrix} 0 & 1 \end{bmatrix} a_1 \leq 1$, $\begin{bmatrix} 0 & 1 \end{bmatrix} a_2 \leq 1$, $\begin{bmatrix} 0 & 1 \end{bmatrix} a_3 \leq 1$.

Besides, if non-degeneracy is assumed, our Theorem 1 reduces into Gale's lemma.

In order to provide a thorough analysis, we now state a similar sensitivity result for a linear programming problem in a general canonical form (*i.e.* with inequality constraints).

Consider the problem

$$P_1(A,b,c): \quad \begin{cases} \min\limits_x cx \\ Ax \geqq b \\ x \geqq [0] \end{cases}$$

and let E be the index set of the constraints which makes a feasible vector x^* active (or effective) for $P_1(A,b,c)$:

$$E = \{i \mid A_i x^* = b_i\}\,. \tag{8}$$

Theorem 3 *Let x^* and y^* be optimal vectors, respectively, for the general problem $P_1(A,b,c)$ and for its dual. Define the matrix F and the set E of active constraints at x^* as in (2) and (8):*

$$A^j \in F \Longleftrightarrow y^* A^j = c_j; \quad E = \{i \mid A_i x^* = b_i\}\,.$$

Choose a vector $b' \neq b$ and assume that the system

$$\begin{cases} Fq \geqq b' \\ F_i q = b_i' \quad \forall i \in E \\ q \geqq [0] \end{cases} \tag{9}$$

admits a solution q. Then all propositions of Theorem 1 hold.

Proof. The proof follows the same steps as that of Theorem 1. We consider the matrix $A = [F; G]$ and observe that the vector

$$x' = \begin{bmatrix} q \\ [0] \end{bmatrix}$$

is feasible for $P_1'(A, b', c)$, as

$$x' \geqq [0], \quad Ax' = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} q \\ [0] \end{bmatrix} = Fq \geqq b', \tag{10}$$

and that the dual of $P_1(A, b, c)$ and $P_1'(A, b', c)$ both have the same feasible set. Hence, y^* is feasible for both. The vanishing of $(y^*A - c)x'$ may be proved as it was for relation (5). Moreover, thanks to (10) the following relation holds:

$$y^*(Ax' - b') = y^*(Fq - b') = \sum_{i \in E} y_i^*(F_i q - b_i') + \sum_{i \notin E} y_i^*(F_i q - b_i') = 0,$$

as (9) gives $(F_i q - b_i') = 0$, $\forall i \in E$, whereas the complementarity theorem gives $y_i^* = 0$ if $i \notin E$. Hence, (6) holds and the proof can be completed in the same way as the one of Theorem 1. ■

For other considerations on Theorems 1 and 3 the reader is referred to De Giuli (1995) and De Giuli and Magnani (1996). Also with reference to problem $P_1(A, b, c,)$, when non-degeneracy is assumed, our Theorem 3 reduces to the result of K. Lancaster (1968), called *basis theorem*, and also reported by Heal et al. (1974).

We note that Theorems 1 and 3 do not assume that x^* is basic, nor that A has a full row-rank (and therefore that $m \leqq n$). Moreover, they afford a constructive way to get an optimum x' for the new problems $P'(A, b', c,)$ and $P_1'(A, b', c,)$ starting from a solution q of system (2) or (9). They also ensure the stability of the optimum y^* for the dual problem. Finally, they qualify y^* as the marginal contribution of an additional unit in b_i to the optimal level cx^* of the objective function; this meaning is often summarized by the relation

$$y_i^* = \frac{\partial(cx^*)}{\partial b_i}$$

and is complementary to the classical one, deduced from the duality theorem $(cx^* = y^*b)$ which identifies y_i^* as the average contribution of each unit of b_i to the same value cx^*.

3. The Gale-Samuelson nonsubstitution theorem

There are many nonsubstitution (or also substitution) theorems in economic analysis. The first statement of these theorems was proposed by Samuelson

(1951). Moreover, in the same collection there were both an algebraic proof by Koopmans (1951) for the case of three industries and an algebraic proof by Arrow (1951) for the general case. For a detailed analysis of several versions of the nonsubstitution theorems, see Pasinetti (1977).

Here we briefly sketch the Gale's version (1960). The classical Leontief model is generalized by considering the possibility that more than one industry can produce the same good, i.e. there is a certain number of alternative ways of producing a given good. If the total number of goods produced by the system is n, let us suppose that m $(m > n)$ linear activities exist, each of them producing a single good. More precisely, the set $M = \{1, 2, \ldots, m\}$ is partitioned into n subsets M_k, $k = 1, 2, \ldots, n$, such that $M_k \cap M_{k'} = \emptyset$ for $k \neq k'$, $M = \cup_{k=1}^{n} M_k$. Let us suppose that the activity $j \in M_k$ produces the good k and consider the (n, m) technological matrix $\widehat{A}$ whose column $\widehat{A}_{i_k}^{j}$ are the inputs for the good i, $i = 1, 2, \ldots, n$, used in the activity j, to produce one unit of good k, being $j \in M_k$.

Instead of the usual output matrix I, here the output matrix is not square, it is denoted by $\widehat{I}$, of dimension (n, m), and is formed by ones and zeroes:

$$\widehat{I} = \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Let $x = [x_1, x_2, \cdots, x_m]^T$ be the vector of activity levels of the generalized Leontief system, then the inputs of the production system are $\widehat{A}x$ and the total production is the vector $\widehat{I}x$. The fundamental equilibrium equation of the model is given by

$$\widehat{I}x - \widehat{A}x = c, \tag{11}$$

where c is the consumption vector. We suppose that there is one primary factor (labour) which is used in every activity, i.e. there is a (row) vector $l = [l_1, l_2, \cdots, l_m]$, $l > [0]$, whose elements are the direct labour requirements per unit intensity of each activity, and that the available amount of labour is fixed at $\bar{l}$. The output space Y is defined as the set of all nonnegative c satisfying (11) and using no more labour than $\bar{l}$, i.e.

$$Y = \left\{ c : c = \left(\widehat{I} - \widehat{A} \right) x, \quad x \geqq [0], \quad lx \leqq \bar{l} \right\}.$$

We remark that the Leontief model can produce any c in the non-negative orthant if no constraint on the amount of labour is imposed.

Let $\sigma = [j_1, j_2, \cdots, j_n]$, where $j_k \in M_k$, $k = 1, 2, \ldots, n$, be a subset of the set of indices M, formed by choosing one activity in each M_k. A_σ denotes the matrix formed by the columns of $\widehat{A}$ indexed by the elements of σ and by the same operation we obtain $I_\sigma = I$. The matrices A_σ and $I_\sigma = I$ are square matrices of order n, A_σ is a usual Leontief sub-model of the generalized Leontief model considered.

Let us assume that the model $(\widehat{A}, \widehat{I})$ is *productive*, i.e. there exists $x > [0]$ such that

$$\left(\widehat{I} - \widehat{A}\right) x > [0].$$

The Gale-Samuelson nonsubstitution theorem can be stated as follows.

THEOREM 4 *(Gale-Samuelson nonsubstitution theorem). Let all the previous assumptions hold. If in system (11), $x \geqq [0]$ produces a strictly positive consumption vector, a productive technique σ exists such that the simple Leontief model (A_σ, I) has the same output space Y as that defined by (11).*

The proof given by Gale of the above theorem relies on his (incorrect) sensitivity Lemma 9.3; *however, the proof works*, due to the productivity assumption made on $\widehat{A}$, the fact that the production model is simple (each activity produces one output) and the fact that the final demand vector is assumed nonnegative (and nonzero). These assumptions are sufficient to make both problems $P(A, b, c,)$ and $P_1(A, b, c,)$ solvable for each b. This means that they admit basic non-degenerate optima. Moreover, these assumptions also ensure that the matrix F is square and has no positive off-diagonal elements. This is enough to get a nonnegative (nonzero) inverse F^{-1}, which makes $q = F^{-1}b'$ feasible, no matter how b' is chosen.

The same proof of the Gale-Samuelson nonsubstitution theorem can be found in Heal et al. (1974), Lancaster (1968), Murata (1977) and Nicola (2000).

There are also other proofs which do not appeal to Gale's lemma on sensitivity, see e.g., Achamanov (1984), Bapat and Raghavan (1997), Bose (1972) and Manara and Nicola (1967). However, the proof of the last two authors is performed under the restrictive assumption of indecomposability of the matrix A_σ. Among all these proofs, the one by Gale is the most neat and concise.

We now give a more complete version of Theorem 4, with an elementary proof.

We first consider the linear programming problem

$$(P): \quad \begin{cases} \min\limits_{x} lx \\ \left(\widehat{I} - \widehat{A}\right) x \geqq c \\ x \geqq [0] \end{cases}$$

whose straightforward economic interpretation may be to find a minimum quantity of total labour requirements allowing to obtain a final demand $c > [0]$.

THEOREM 5 *Let be given the generalized Leontief model $(\widehat{A}, \widehat{I})$, $l > [0]$, Y nonempty containing a strictly positive c and let $(\widehat{A}, \widehat{I})$ be productive. Then for problem (P) the following properties hold:*

i) (P) *admits solution for any* $c > [0]$*;*

ii) *Every optimal vector* x^* *is such that over-productions are excluded:*

$$x^* \text{ optimal for } (P) \Longrightarrow \left(\widehat{I} - \widehat{A}\right) x^* = c;$$

iii) *The optimal vectors* x^* *vary with the choice of* c*. However the* n *techniques which are optimal at a given vector* c *remain the same and they are optimal also at every other vector of final demands. Hence, when final demands change, there is no need to substitute any production process with a different technique.*

Proof. We recall that the productivity assumption of $(\widehat{A}, \widehat{I})$ means that there exists $x^0 > [0]$ such that

$$\begin{cases} \left(\widehat{I} - \widehat{A}\right) x^0 = y^0 > [0] \\ x^0 > [0]. \end{cases}$$

If $y^0 \geqq c$, (P) is feasible; otherwise we consider αx^0, $\alpha > 0$ and sufficiently large. The dual of (P) is

$$(D): \quad \begin{cases} \max \pi c \\ \pi \left(\widehat{I} - \widehat{A}\right) \leqq l \\ \pi \geqq [0] \end{cases}$$

and it is always feasible, as, e.g., $\pi = [0]$ satisfies all the constraints and the sign conditions. Thanks to the existence and duality theorems, (P) and (D) both admit solutions.

Let r be the rank of $(\widehat{I} - \widehat{A})$. Then, (P) admits a basic optimal vector x^*, which therefore contains at least $(m - r)$ zero components. We denote the optimal basis by $(I^+ - A^+)$.

We now consider an optimal vector x^* associated to a vector $\overline{c} > [0]$. In this case at least n components x_j^* of x^* have to be positive. These n components will form the vector $x^+ > [0]$. The optimal basis $(I^+ - A^+)$ associated to x^* contains therefore at least n columns. Since $m > n$, we can get only $r = n$, so $(I^+ - A^+)$ contains n columns, i.e. every good is produced by a single activity and x^+ contains n positive components, so that $(I^+ - A^+)$ is a nondegenerate optimal basis. As every column of I^+ contains one positive element, $(I^+ - A^+)$ is a Z-matrix and a K-matrix (see De Giuli et al., 2008, and Fiedler and Ptàk, 1962), so we have

$$\begin{cases} (\widehat{I} - \widehat{A})x^* = (I^+ - A^+)x^+ \geqq c > [0] \\ x^+ \geqq [0]. \end{cases}$$

Therefore, we have $(I^+ - A^+)^{-1} \geqq [0]$ and so, for any $c > [0]$, the system

$$\begin{cases} (\widehat{I} - \widehat{A})x \geqq c \\ x \geqq [0] \end{cases}$$

always has the solution $x = \overline{x}$ formed by setting $(m-n)$ components equal to zero (the ones not associated to the basis $(I^+ - A^+)$) and n nonnegative components equal to the vector

$$x^+ = (I^+ - A^+)^{-1}c \geqq [0]. \tag{12}$$

Therefore, the matrix $(\widehat{I} - \widehat{A})$ contains a basis $(I^+ - A^+)$ which is optimal for the chosen vector $\overline{c} > [0]$ and remains feasible for any choice of $c > [0]$. Thanks to Theorem 3, problem (P) remains solvable when $c > [0]$ varies and among its solutions there is always the one associated with the same basis $(I^+ - A^+)$. So, *i)* and *iii)* are proved.

Let us now prove *ii)*. We define the vector of net productions generated by the optimal solution $\overline{x}$:

$$\overline{y} = (\widehat{I} - \widehat{A})\overline{x} = (I^+ - A^+)x^+$$

and the vector of over-productions, which can be only nonnegative quantities:

$$z = \overline{y} - c = (I^+ - A^+)\overline{x} - c \geqq [0].$$

We define the vector l^+, whose components are associated to the basis $(I^+ - A^+)$. The total labour requirement $l\overline{x}$ is given by

$$\begin{aligned} l\overline{x} &= l^+x^+ = l^+(I^+ - A^+)^{-1}\overline{y} \\ &= l^+(I^+ - A^+)^{-1}(c+z), \end{aligned}$$

i.e., thanks to (12)

$$\begin{aligned} l\overline{x} &= l^+(I^+ - A^+)^{-1}c + l^+(I^+ - A^+)^{-1}z \\ &= l^+x^+ + l^+(I^+ - A^+)^{-1}z. \end{aligned}$$

The total labour requirement $l\overline{x}$ is the sum of l^+x^+, the amount of labour requirements strictly necessary to meet the demand vector c, with $l^+(I^+ - A^+)^{-1}z$, the requirements of labour which are not necessary, as reserved to over-productions. This second term will be zero if the over productions z is the zero vector. But, as problem (P) aims only at minimizing total labour requirements, it is obvious that in each optimal solution of (P) there are no over-productions. ∎

For other considerations on the Gale-Samuelson nonsubstitution theorem, also with regard to financial models, the reader is referred to De Giuli et al. (2008) and De Giuli and Magnani (1996).

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