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On stability of some lexicographic integer optimization problem*

by

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Abstract: A lexicographic integer optimization problem with criteria represented by absolute values of linear functions is considered. Five types of stability for the set of lexicographic optima under small changes of the parameters of the vector criterion are investigated.

Keywords: multicriteria optimization, lexicographic integer optimization problem, absolute values of linear functions, lexicographic optima, stability, strong stability, quasi-stability, strong quasi-stability, unalterability.

1. Introduction

Many problems of design, planning and management in technical and organizational systems have a pronounced multicriteria character. Multiobjective models appearing in these cases are reduced to the choice of "best" (in a certain sense) values of variable parameters from some discrete aggregate of the given quantities. Therefore, recent interest of mathematicians to multicriteria discrete optimization problems has been very high, as confirmed by the intensive publishing activity (see, e.g., the bibliography by Ehrgott and Gandibleux, 2000, which contains 234 references).

While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty such as lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors etc. Therefore, widespread use of discrete optimization models in the last decades stimulated many experts to investigate various aspects of problems under uncertainties and, in particular, the questions of stability.

Stability of a multicriteria optimization problem in a classical sense is usually understood as a property of continuity or semicontinuity of a multi-valued

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mapping, which determines the choice function. Numerous studies are devoted to the analysis of conditions, under which a problem possesses one or another property of stability (see, e.g., Sawaragi et al., 1985; Tanino, 1988; Fiacco, 1998; Tanino and Sawaragi, 1980). Such good properties as continuity, convexity, connectivity, which make solving many problems easier, are usually not applicable to discrete optimization problems. However, in the study of stability of discrete problems such concepts find their application, since the set of initial data of a problem can be endowed with a nontrivial, non-discrete topology.

Many publications are devoted to the investigation of stability of scalar or multicriteria discrete optimization problems (see, e.g., Sotskov et al., 1995; Chakravarti and Wagelmans, 1998; Greenberg, 1998; Libura et al., 1998, 2004, 2007; van Hoesel and Wagelmans, 1999; Sergienko and Shilo, 2003; Kozeratska et al., 2004). This paper continues our study of various types of stability for the problems with different kinds of partial criteria and optimality principles (see, e.g., Emelichev et al., 2002, 2004, 2005, 2006, 2007, 2007b; Bukhtoyarov and Emelichev, 2006). Stability of the lexicographic set for a multicriteria integer problem of minimizing absolute values of linear functions is investigated in this work. Necessary and sufficient conditions for five best known (see, e.g., Emelichev et al., 2002; Sergienko and Shilo, 2003) types of stability are obtained. In this paper we show that the structure of the lexicographic set and the image of this set in the criterion space are closely connected with the solvability of some corresponding system of integer linear equations (SILE). In other words, each element of such lexicographic set can be considered as an approximation in the case of unsolvability of such SILE or a solution otherwise. Our research is related to the behavior analysis of this kind of approximations (solutions) under small changes of SILE parameters. The estimations of stability radius for an analogous Boolean problem with lexicographic and Pareto principles of optimality were obtained earlier in Emelichev and Gurevsky (2007a), Gurevskii and Emelichev (2006), respectively. The formula of the stability radius for a fixed Pareto-optimal Boolean solution was given in Gurevskii and Emelichev (2007).

2. Basic definitions and notations

Let us consider a multicriteria integer programming problem in the following formulation.

Let m be the number of criteria, n be the number of variables, C_i denote the i-th row of matrix $C = [c_{ij}]_{m\times n} \in \mathbf{R}^{mn}$, $m \geq 1$, $n \geq 2$, $i \in N_m = \{1, 2, \ldots, m\}$. Let $X \subset \mathbf{Z}^n$ be the set of (feasible) solutions, $1 < |X| < \infty$. We define a vector objective function $f(x, C) = (f_1(x, C_1), f_2(x, C_2), \ldots, f_m(x, C_m))^T$ on X with criteria

$$f_i(x, C_i) = |C_i x| \to \min_{x\in X},$$

where $x = (x_1, x_2, \ldots, x_n)^T$.

In the criterion space $\mathbf{R}^m$ we define the binary relation of lexicographic order $\prec$ between any pair of vectors $y = (y_1, y_2, \ldots, y_m)^T$ and $y' = (y'_1, y'_2, \ldots, y'_m)^T$, as follows

$$y \prec y' \iff \exists k \in N_m \quad (y_k < y'_k \ \& \ k = \min\{i \in N_m : \ y_i \neq y'_i\}).$$

Under the lexicographic integer optimization problem

$$Z^m(C): \ \operatorname{lex\,min}\{f(x, C): \ x \in X\}$$

we understand the problem of finding the set of lexicographic optima, which is defined by the following formula:

$$L^m(C) = \{x \in X : \ \forall x' \in X \quad (f(x', C) \ \overline{\prec} \ f(x, C))\},$$

where $\overline{\prec}$ is the negation of $\prec$. It is easy to see that this set is a subset of the Pareto set and is nonempty for any $C \in \mathbf{R}^{mn}$.

It is obvious that $f(x, C)$ can serve as a measure of inconsistency of the following homogenous system of linear equations

$$Cx = \mathbf{0}_{(m)}, \quad x \in X, \tag{1}$$

where $\mathbf{0}_{(m)} = (0, 0, \ldots, 0)^T \in \mathbf{R}^m$. Since $f(x, C)$ is a discrepancy function of system (1), then this system is consistent if and only if the set of vector estimations of $L^m(C)$ defined by

$$F(L^m(C)) = \{y \in \mathbf{R}^m : \ y = f(x, C), \ x \in L^m(C)\}$$

contains $\mathbf{0}_{(m)}$ only.

It is easy to see that a partial case of homogenous system (1) is the heterogeneous system

$$Ax = b, \ x \in X,$$

where $X \subset \mathbf{Z}^{n-1}, \ A \in \mathbf{R}^{m(n-1)}, \ b \in \mathbf{R}^m$. Therefore all the results obtained in this paper also hold for the problem with vector function

$$f(x, A, b) = \big(|A_1 x + b_1|, |A_2 x + b_2|, \ldots, |A_m x + b_m|\big)^T.$$

It is known (see, e.g., Ehrgott, 2005) that $L^m(C)$ can be defined as the result of solving the sequence of m scalar problems:

$$L^m_i(C) = \operatorname{Arg\,min}\{|C_i x| : \ x \in L^m_{i-1}(C)\}, \ i \in N_m, \tag{2}$$

where $L^m_0(C) = X$, $\operatorname{Arg\,min}\{\cdot\}$ is as usual the set of all optimal solutions of the respective minimization problem. Thus, we have a sequence of sets

$$X \supseteq L^m_1(C) \supseteq L^m_2(C) \supseteq \ldots \supseteq L^m_m(C) = L^m(C). \tag{3}$$

Therefore, the problem of finding $L^m(C)$ can be considered as a problem of sequential optimization.

We investigate five types of stability of the problem $Z^m(C)$ under independent perturbations of the parameters of vector function $f(x, C)$, i.e. the elements of C. To do that we define the norm l_∞ in $\mathbf{R}^k$ for an arbitrary dimension $k \in \mathbf{N}$:

$$||z|| = \max_{j \in N_k} |z_j|, \quad z = (z_1, z_2, \dots, z_k) \in \mathbf{R}^k.$$

Under the norm of a matrix we will understand the norm l_∞ of the vector composed of all elements of a matrix.

3. Stability and strong stability

The most frequently used definition of stability of discrete problems (see, e.g., Sotskov et al., 1995; Emelichev et al., 2002, 2007a, 2007b; Sergienko and Shilo, 2003; Libura and Nikulin, 2004; Bukhtoyarov and Emelichev, 2006; Gurevskii and Emelichev, 2006) is the following one.

Problem $Z^m(C)$, $m \geq 1$, is called stable (under perturbations of the elements of C) if the set

$$\{\varepsilon > 0 : \forall C' \in \Omega(\varepsilon) \quad (L^m(C + C') \subseteq L^m(C))\}$$

is nonempty, where $\Omega(\varepsilon) = \{C' \in \mathbf{R}^{mn} : \ ||C'|| < \varepsilon\}$.

In other words, stability of $Z^m(C)$ is the property of nonappearance of new lexicographic optima under any small perturbations of the parameters of the problem. Therefore stability of $Z^m(C)$ is a discrete analogue of the Hausdorff upper semicontinuity (Tanino, 1988) at point C of the multi-valued optimal mapping

$$L^m : \ \mathbf{R}^{mn} \to 2^X, \tag{4}$$

which assigns the set of lexicographic optima to each matrix of $\mathbf{R}^{mn}$. Here and hereafter a problem $Z^m(C + C')$ will be called a perturbed problem, and a matrix $C' \in \Omega(\varepsilon)$ a perturbing matrix.

Relaxing the requirement of nonappearance of new lexicographic optima, we logically come to the concept of strong stability of the problem. According to Emelichev et al. (2002), $Z^m(C)$ is called strongly stable if the set

$$\{\varepsilon > 0 : \forall C' \in \Omega(\varepsilon) \quad (L^m(C + C') \cap L^m(C) \neq \emptyset)\}$$

is nonempty. Thus, $Z^m(C)$ is strongly stable only in the case where for any small perturbations of parameters at least one solution of $L^m(C)$ preserves its lexicographic optimality (not necessarily the same solution for different perturbations). It is obvious that $Z^m(C)$ is strongly stable if it is stable. Below we will show (see Theorem 1) that the inverse statement also holds.

Let us introduce the set

$$V^m(C) = \{x \in L_1^m(C) : \ \exists x' \in L^m(C) \ \exists p \in \mathbf{R} \quad (x' = px)\}.$$

Thus, $V^m(C)$ is the set of solutions of $L_1^m(C)$ each of which has at least one collinear solution in $L^m(C)$. It is easy to see that any $x \in L^m(C) \subseteq L_1^m(C)$ belongs to $V^m(C)$. So, for any $C \in \mathbf{R}^{mn}$ we have

$$L^m(C) \subseteq V^m(C) \subseteq L_1^m(C). \tag{5}$$

Let us use the following notation

$$\overline{L^m}(C) = X \backslash L^m(C).$$

THEOREM 1 *For lexicographic problem $Z^m(C)$, $m \geq 1$, the following statements are equivalent:*

(i) $Z^m(C)$ *is stable,*

(ii) $Z^m(C)$ *is strongly stable,*

(iii) $V^m(C) = L_1^m(C)$.

Proof. (i)$\Rightarrow$(ii). This implication is evident.

(ii)$\Rightarrow$(iii). Assume the contrary. Let $Z^m(C)$ be strongly stable, but $V^m(C) \neq L_1^m(C)$. Then $\mathbf{0}_{(n)} \notin X$. Let $x^0 \in L_1^m(C) \backslash V^m(C)$. Then, in view of (5) we have $x^0 \in \overline{L^m}(C)$, and therefore for any $x \in L^m(C)$ we obtain $f(x, C) \prec f(x^0, C)$. Taking into account $x^0 \in L_1^m(C)$ we derive

$$\forall x \in L^m(C) \quad \exists k = k(x) \in N_m \backslash \{1\} \quad \forall i \in N_{k-1}$$
$$\big(|C_i x| = |C_i x^0| \ \& \ |C_k x| < |C_k x^0|\big).$$

Using the fact that $f(x, C) = f(x', C)$ for any x, $x' \in L^m(C)$ we can write this formula in the form

$$\exists k \in N_m \backslash \{1\} \ \ \forall i \in N_{k-1} \ \ \forall x \in L^m(C) \ \ \big(|C_i x| = |C_i x^0| \ \& \ |C_k x| < |C_k x^0|\big). \tag{6}$$

It follows that $C_k \neq \mathbf{0}_{(n)}^T$, i.e.

$$||C_k|| > 0. \tag{7}$$

We use inclusion $x^0 \in L_1^m(C) \setminus V^m(C)$ to construct a perturbing matrix C^* to show that $Z^m(C)$ is not strongly stable.

Let $\varepsilon > 0$. In view of (6) two cases are possible.

Case 1: for any $x \in L^m(C)$ the following equalities hold

$$C_1 x^0 = C_1 x = 0. \tag{8}$$

Since $x^0 \notin V^m(C)$, x^0 does not have a collinear vector in $L^m(C)$. Therefore, there is a hyperplane

$$H = \{x \in \mathbf{R}^n : \ ax = 0\},$$

such that (in view of $\mathbf{0}_{(n)} \notin X$)

$$\forall x \in L^m(C) \ \ (0 = ax^0 \neq ax). \tag{9}$$

Next, we assign $||a|| = 1$, and define the rows of $C^* \in \mathbf{R}^{mn}$ by

$$C_i^* = \begin{cases} \delta a, & \text{if } i = 1, \\ \mathbf{0}_{(n)}^T, & \text{if } i \neq 1, \end{cases}$$

where $0 < \delta < \varepsilon$. Then, $C^* \in \Omega(\varepsilon)$. Moreover, (8) and (9) imply

$$|(C_1 + C_1^*)x^0| = |\delta a x^0| = 0 < |\delta a x| = |(C_1 + C_1^*)x|, \tag{10}$$

which is true for any $x \in L^m(C)$.

Case 2: for any $x \in L^m(C)$ we have

$$|C_1 x^0| = |C_1 x| > 0. \tag{11}$$

Then define the rows C_i^*, $i \in N_m$, by

$$C_i^* = \begin{cases} \delta C_k, & \text{if } i = 1, \\ \mathbf{0}_{(n)}^T, & \text{if } i \neq 1, \end{cases}$$

(here and hereafter, k is the same as in (6)), and set δ such that

$$||\delta C_k|| < \varepsilon,$$

$$\text{sign } \delta = \begin{cases} 1, & \text{if } \ \text{sign } C_1 x^0 \neq \text{sign } C_k x^0, \\ -1, & \text{if } \ \text{sign } C_1 x^0 = \text{sign } C_k x^0, \end{cases} \tag{12}$$

$$0 < |\delta| < \frac{|C_1 x^0|}{|C_k x^0|}. \tag{13}$$

From this, using (7) and $||\delta C_k|| > 0$, we obtain $C^* \in \Omega(\varepsilon)$. Moreover, taking into account (by virtue of (6) and (11)) that $C_1 x^0$ and $C_k x^0$ are nonzero, by (12) and (13) we derive

$$|C_1 x^0 + \delta C_k x^0| = |C_1 x^0| - |\delta| \cdot |C_k x^0|.$$

Taking into account the construction of matrix C^*, we conclude that

$$|(C_1 + C_1^*)x^0| - |(C_1 + C_1^*)x| = |C_1 x^0 + \delta C_k x^0| - |(C_1 + C_1^*)x| \leq$$
$$\leq |C_1 x^0| - |\delta| \cdot |C_k x^0| - |C_1 x| + |C_1^* x| = -|\delta| \cdot |C_k x^0| + |\delta| \cdot |C_k x| < 0,$$

for any $x \in L^m(C)$, which, together with (10), implies

$$\forall \varepsilon > 0 \;\; \exists C^* \in \Omega(\varepsilon) \;\; \forall x \in L^m(C) \;\; (|(C_1 + C_1^*)x^0| < |(C_1 + C_1^*)x|).$$

Therefore, we have

$$L^m(C) \cap L_1^m(C + C^*) = \emptyset,$$

and in view of

$$L^m(C + C^*) \subseteq L_1^m(C + C^*),$$

which holds due to (3), we conclude that $L^m(C) \cap L^m(C + C^*) = \emptyset$. Hence, $Z^m(C)$ is not strongly stable. The obtained contradiction proves that implication (ii)$\Rightarrow$(iii) is true.

(iii)$\Rightarrow$(i). If $\overline{L^m}(C) = \emptyset$, then it is obvious that $Z^m(C)$ is stable. Let $x \in \overline{L^m}(C)$. Then there are two possible cases: $x \in V^m(C)$ and $x \in \overline{L^m}(C) \setminus V^m(C)$.

Case 1: $x \in V^m(C)$. According to the definition of $V^m(C)$, we have

$$\exists x^0 \in L^m(C) \;\; \exists p \in \mathbf{R} \qquad (x^0 = px). \tag{14}$$

In view of $x \in \overline{L^m}(C)$ there exists $k \in N_m \setminus \{1\}$ such that

$$|C_k x| > |C_k x^0| = |C_k p x|. \tag{15}$$

Consequently, we obtain

$$|p| < 1.$$

Using (14), for any $C' \in \mathbf{R}^{mn}$ we derive

$$|(C + C')x^0| - |(C + C')x| = (|p| - 1)|(C + C')x| \leq 0. \tag{16}$$

By virtue of the continuity of function $|C_k x|$ over $\mathbf{R}^n$ and taking into account (15), we conclude that there exists $\varepsilon_1(x) > 0$, such that

$$|(C_k + C'_k)x^0| < |(C_k + C'_k)x|$$

for any $C' \in \Omega(\varepsilon_1(x))$.

Then, (16) implies

$$f(x^0, C + C') \prec f(x, C + C').$$

As a result we obtain

$$\forall x \in V^m(C) \;\; \exists \varepsilon_1(x) > 0 \;\; \forall C' \in \Omega(\varepsilon_1(x)) \qquad (x \in \overline{L^m}(C + C')). \tag{17}$$

Case 2: $x \in \overline{L^m}(C) \setminus V^m(C)$. In view of (iii) we have

$$|C_1 x^0| < |C_1 x|,$$

where $x^0 \in L^m(C)$. Therefore, there exists $\varepsilon_2(x) > 0$ such that for any $C' \in \Omega(\varepsilon_2(x))$ the following inequality holds:

$$|(C_1 + C_1')x^0| < |(C_1 + C_1')x|,$$

it follows that $f(x^0, C + C') \prec f(x, C + C')$. Thus, we have

$$\forall x \in \overline{L^m}(C) \setminus V^m(C) \;\; \exists \varepsilon_2(x) > 0 \;\; \forall C' \in \Omega(\varepsilon_2(x)) \qquad (x \in \overline{L^m}(C + C')). \tag{18}$$

Let

$$\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\},$$

where

$$\varepsilon_1 = \min\{\varepsilon_1(x) : \; x \in V^m(C)\},$$
$$\varepsilon_2 = \min\{\varepsilon_2(x) : \; x \in \overline{L^m}(C) \setminus V^m(C)\}.$$

Then, (17) and (18) imply

$$\forall C' \in \Omega(\varepsilon^*) \qquad (\overline{L^m}(C) \subseteq \overline{L^m}(C + C')),$$

which implies that $Z^m(C)$ is stable.

Theorem 1 is proved. ■

Theorem 1 implies several evident corollaries.

Corollary 1 *Problem $Z^m(C)$ is stable (strongly stable) if at least one of the following conditions holds:*
(i) $\mathbf{0}_{(n)} \in X$,
(ii) $L^m(C) = L_1^m(C)$.

Corollary 2 *Scalar problem $Z^1(C)$ is stable (strongly stable) for any row $C \in \mathbf{R}^n$.*

We call problem $Z^m(C)$ Boolean and denote by $Z_B^m(C)$ if $X \subseteq \{0,1\}^n$.

Corollary 3 *(Emelichev and Gurevsky, 2007b) Let $\mathbf{0}_{(n)} \notin X$, $m \geq 1$. Then the following statements are equivalent:*
(i) *$Z_B^m(C)$ is stable,*
(ii) *$Z_B^m(C)$ is strongly stable,*
(iii) $L^m(C) = L_1^m(C)$.

Corollary 4 *(Emelichev and Gurevsky, 2007b) If $\mathbf{0}_{(n)} \in X$, then $Z_B^m(C)$, $m \geq 1$, is stable (strongly stable) for any $C \in \mathbf{R}^{mn}$.*

4. Quasi-stability and unalterability

By analogy with Emelichev et al. (2002, 2005), we call the problem $Z^m(C)$ quasi-stable if the set

$$\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \quad (L^m(C) \subseteq L^m(C + C'))\}$$

is nonempty, where $\Omega(\varepsilon) = \{C' \in \mathbf{R}^{mn} : \ ||C'|| < \varepsilon\}$, as before.

In other words, the problem is quasi-stable if the lexicographic optima of the initial problem do not disappear for any small perturbations of the initial data, but new optima can appear. Thus, quasi-stability of the problem $Z^m(C)$ is a discrete analogue of the Hausdorff lower semicontinuity (Tanino, 1988) at C of the multi-valued optimal mapping (4).

Let us also introduce the concept of unalterability of the problem. Problem $Z^m(C)$ is called unalterable if the set

$$\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \quad (L^m(C) = L^m(C + C'))\}$$

is nonempty. Thus, unalterability of $Z^m(C)$ is a discrete analogue of the Hausdorff continuity at C of the multi-valued optimal mapping (4). It is evident that $Z^m(C)$ is unalterable if and only if it is stable and quasi-stable at once. Below (see Theorem 2) we will prove that unalterability is equivalent to quasi-stability for the problem $Z^m(C)$.

Suppose

$$W^m(C) = \{x \in L^m(C) : \ \forall x' \in L_1^m(C) \ \exists p \in [-1, 1] \quad (x = px')\}.$$

Thus, $W^m(C)$ is the set of solutions of $L^m(C)$, each of which is collinear to any solution of $L_1^m(C)$. It is evident that $W^m(C) \subseteq L^m(C)$ for any $C \in \mathbf{R}^{mn}$. Note that $W^m(C)$ can be empty, but $L^m(C) \neq \emptyset$ for any $C \in \mathbf{R}^{mn}$. It is easy to see that $W^m(C)$ is nonempty if and only if the elements of set $L_1^m(C)$ are pairwise collinear or $\mathbf{0}_{(n)} \in X$.

THEOREM 2 *For lexicographic problem $Z^m(C)$, $m \geq 1$, the following statements are equivalent:*

(i) *$Z^m(C)$ is unalterable,*

(ii) *$Z^m(C)$ is quasi-stable,*

(iii) *$W^m(C) = L^m(C)$.*

Proof. (i)⇒(ii). This implication is evident.

(ii)⇒(iii). Let $Z^m(C)$ be quasi-stable. Then, for any $x \in L^m(C)$ we have

$$\exists \varepsilon > 0 \ \ \forall C' \in \Omega(\varepsilon) \ \ (x \in L^m(C + C')). \tag{19}$$

Assume the contrary, i.e. that (iii) does not hold. Then there exists a solution $x^0 \in L^m(C) \setminus W^m(C)$, which means that

$$\exists x^0 \in L^m(C) \ \ \exists x^* \in L_1^m(C) \ \ \forall p \in [-1, 1] \quad (x^0 \neq px^*). \tag{20}$$

Besides that, from (2) and $i = 1$ we have

$$|C_1 x^0| = |C_1 x^*|. \tag{21}$$

We will construct a matrix C^*, which contradicts the formula (19). Let $\varepsilon > 0$. Taking into account (20) there are two possible cases.

Case 1: $x^0 \neq px^*$ for any $p \in \mathbf{R}$. Then there exists a vector-row $a \in \mathbf{R}^n$ such that

$$ax^0 \neq ax^* = 0, \quad ||a|| = 1. \tag{22}$$

Due to (21) two subcases are possible.

Subcase 1.1: $C_1 x^0 = C_1 x^* = 0$. Define the rows C_i^*, $i \in N_m$, of $C^* \in \mathbf{R}^{mn}$ by

$$C_i^* = \begin{cases} \delta a, & \text{if } i = 1, \\ \mathbf{0}_{(n)}^T, & \text{if } i \neq 1, \end{cases} \tag{23}$$

where $0 < \delta < \varepsilon$. Taking into account (22), we obtain

$$|(C_1 + C_1^*)x^*| = |\delta a x^*| = 0 < |\delta a x^0| = |(C_1 + C_1^*)x^0|, \quad C^* \in \Omega(\varepsilon).$$

Subcase 1.2: $|C_1 x^0| = |C_1 x^*| > 0$. Then, setting the rows of $C^* \in \mathbf{R}^{mn}$ by (23) and taking δ such that

$$0 < |\delta| < \varepsilon,$$
$$\operatorname{sign} \delta a x^0 = \operatorname{sign} C_1 x^0,$$

we have

$$|(C_1 + C_1^*)x^*| - |(C_1 + C_1^*)x^0| = |C_1 x^*| - |C_1 x^0| - |\delta a x^0| = -|\delta a x^0| < 0.$$

Resuming the two subcases, we conclude that in Case 1 the following formula is true:

$$\forall \varepsilon > 0 \;\; \exists C^* \in \Omega(\varepsilon) \;\; (x^0 \in \overline{L^m}(C + C^*)), \tag{24}$$

which contradicts (19).

Case 2: there exists p, $|p| > 1$, such that $x^0 = px^*$. In view of (21) we obtain

$$C_1 x^0 = C_1 x^* = 0, \tag{25}$$

and from (20) we have

$$x^0 \neq \mathbf{0}_{(n)}. \tag{26}$$

Define the rows C_i^*, $i \in N_m$, of $C^* \in \mathbf{R}^{mn}$ by

$$C_i^* = \begin{cases} \delta y, & \text{if } i = 1, \\ \mathbf{0}_{(n)}^T, & \text{if } i \neq 1, \end{cases}$$

where $0 < \delta ||x^0|| < \varepsilon$, $y = {x^0}^T$. Then, using (25) and (26), we have

$$|(C_1 + C_1^*)x^*| = |\delta y x^*| = \frac{\delta}{|p|}|yx^0| < \delta|yx^0| = |(C_1 + C_1^*)x^0|, \quad C^* \in \Omega(\varepsilon).$$

It follows that in Case 2 formula (24) is true, which contradicts (19).

(iii)$\Rightarrow$(i). Let $W^m(C) = L^m(C)$. Then any two solutions $x \in L^m(C)$ and $x' \in L_1^m(C)$ are collinear to each other and according to the definition of $V^m(C)$ we have $V^m(C) = L_1^m(C)$. In view of Theorem 1, problem $Z^m(C)$ is stable.

Moreover, $W^m(C) = L^m(C)$ implies that either $|L^m(C)| = 1$ or $L^m(C)$ contains only two solutions, which differ from each other only by a sign. Therefore, $|F(L^m(C))| = 1$ and hence problem $Z^m(C)$ is quasi-stable.

Resuming the said above, we conclude that $Z^m(C)$ is unalterable.

Theorem 2 is proved. ∎

Theorem 2 implies a few following evident corollaries.

Corollary 5 *If $|L_1^m(C)| = 1$, then $Z^m(C)$ is unalterable.*

Corollary 6 *If $Z^m(C)$ is quasi-stable, then it is stable (strongly stable).*

Corollary 7 *If $Z^m(C)$ has more than two lexicographic optima, then it is not quasi-stable.*

Corollary 8 *Scalar problem $Z^1(C)$ is quasi-stable (unalterable) if and only if at least one of the following conditions holds:*

(i) *$Z^1(C)$ has a unique optimal solution,*

(ii) *$Z^1(C)$ has two optimal solutions that differ from each other only by a sign.*

Corollary 9 *(Emelichev and Gurevsky, 2007b) Let $\mathbf{0}_{(n)} \in X$. Then, the Boolean problem $Z_B^m(C)$, $m \geq 1$, is quasi-stable (unalterable) if and only if $L^m(C) = \{\mathbf{0}_{(n)}\}$.*

Corollary 10 *(Emelichev and Gurevsky, 2007b) Let $\mathbf{0}_{(n)} \notin X$. Then, for the Boolean problem $Z_B^m(C)$, $m \geq 1$, the following statements are equivalent:*

(i) *$Z_B^m(C)$ is unalterable,*

(ii) *$Z_B^m(C)$ is quasi-stable,*

(iii) $\big|\operatorname{Arg\,min}\{|C_1 x| \ : \ x \in X\}\big| = 1$.

5. Strong quasi-stability

The quasi-stability of the problem $Z^m(C)$ introduced in Section 4 requires preserving the whole set of lexicographic optima under small perturbations of the initial data. By relaxing this demand we obtain the concept of strong quasi-stability. This type of stability means preserving the lexicographic optimality for at least one of the solutions for any small perturbations of the parameters. Thus, the problem $Z^m(C)$ is called strongly quasi-stable (Emelichev et al., 2002) if the set

$$\{\varepsilon > 0 : \ \exists x^0 \in L^m(C) \ \ \forall C' \in \Omega(\varepsilon) \quad (x^0 \in L^m(C + C'))\}$$

is nonempty. It is evident that any quasi-stable problem is strongly quasi-stable.

THEOREM 3 *Lexicographic problem $Z^m(C)$, $m \geq 1$, is strongly quasi-stable if and only if $W^m(C) \neq \emptyset$.*

Proof. Necessity. Assume the contrary, i.e. that $Z^m(C)$ is strongly quasi-stable, but $W^m(C) = \emptyset$. Then for any $x \in L^m(C)$ we have

$$\exists x^* \in L_1^m(C) \ \ \forall p \in [-1, 1] \quad (x \neq px^*).$$

Using the same reasoning as when proving the implication (ii)⇒(iii) in Theorem 2, we conclude

$$\forall x \in L^m(C) \ \ \forall \varepsilon > 0 \ \ \exists C^* \in \Omega(\varepsilon) \quad (x \in \overline{L^m}(C + C^*)),$$

which contradicts the strong quasi-stability of $Z^m(C)$.

Sufficiency. Let $x^0 \in W^m(C)$. We will show that

$$\exists \varepsilon^0 > 0 \ \ \forall C' \in \Omega(\varepsilon^0) \quad (x^0 \in L^m(C + C')), \tag{27}$$

which implies the strong quasi-stability of $Z^m(C)$.

Let $x \in X$. Consider two possible cases.

Case 1: $x \in L_1^m(C)$. Then, in view of $x^0 \in W^m(C)$, there exists $p \in [-1, 1]$ such that $x^0 = px$. Taking into account $C' \in \mathbf{R}^{mn}$, we derive

$$|(C + C')x^0| - |(C + C')x| = (|p| - 1)|(C + C')x| \leq 0,$$

i.e.

$$\forall x \in L_1^m(C) \ \ \forall C' \in \mathbf{R}^{mn} \quad (f(x, C + C') \,\overline{\prec}\, f(x^0, C + C')).$$

Case 2: $x \in X \setminus L_1^m(C)$. Then

$$|C_1 x^0| < |C_1 x|.$$

Therefore, there exists $\varepsilon = \varepsilon(x) > 0$ such that for any $C' \in \Omega(\varepsilon)$ the following inequality is true:

$$|(C_1 + C_1')x^0| < |(C_1 + C_1')x|,$$

which implies $f(x^0, C + C') \prec f(x, C + C')$. Thus, we have

$$\exists \varepsilon = \varepsilon(x) > 0 \;\; \forall C' \in \Omega(\varepsilon) \qquad (f(x, C + C') \,\overline{\prec}\, f(x^0, C + C')).$$

From this we derive

$$\exists \varepsilon^0 > 0 \;\; \forall x \in X \backslash L_1^m(C) \;\; \forall C' \in \Omega(\varepsilon^0) \quad (f(x, C + C') \,\overline{\prec}\, f(x^0, C + C')),$$

where $\varepsilon^0 = \min\{\varepsilon(x) : \; x \in X \setminus L_1^m(C)\}$.

Resuming the considered cases, we conclude that formula (27) holds.

Theorem 3 is proved. ■

The following corollaries follow directly from Theorems 1, 2 and 3.

Corollary 11 *Problem $Z^m(C)$ is strongly quasi-stable if at least one of the following conditions holds:*

(i) $\mathbf{0}_{(n)} \in X$,

(ii) $|L_1^m(C)| = 1$,

(iii) *elements of $L_1^m(C)$ are pairwise collinear.*

Corollary 12 *If $Z^m(C)$ is strongly quasi-stable, then it is stable.*

Corollary 13 *(Emelichev and Gurevsky, 2007b) Let $\mathbf{0}_{(n)} \notin X$. Then, for the Boolean problem $Z_B^m(C)$, $m \geq 1$, the following statements are equivalent:*

(i) *$Z_B^m(C)$ is quasi-stable,*

(ii) *$Z_B^m(C)$ is strongly quasi-stable,*

(iii) $|L^m(C)| = |L_1^m(C)| = 1$.

Corollary 14 *(Emelichev and Gurevsky, 2007b) If $\mathbf{0}_{(n)} \in X$, then the Boolean problem $Z_B^m(C)$, $m \geq 1$, is strongly quasi-stable for any $C \in \mathbf{R}^{mn}$.*

6. Conclusions

Resuming the results obtained in Theorems 1, 2 and 3, we conclude that the relations between various types of stability of the problem $Z^m(C)$ can be described by the following scheme (Fig. 1):

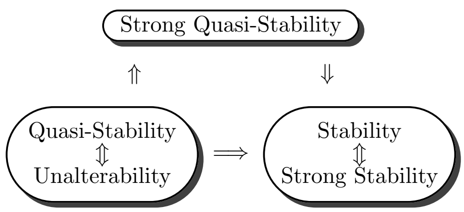

Figure 1. Scheme of relations between various types of stability of $Z^m(C)$

REMARK 1 *In view of the equivalence of any two norms in a finite-dimensional linear space (see, e.g., Suhubi, 2003), all the propositions formulated in this paper (Theorems 1, 2, 3 and Corollaries 1–4, 5–10, 11–14) hold for any norm in the space of matrices* $\mathbf{R}^{mn}$.

7. Examples

The following examples illustrate that the inverse implications of the scheme are not true.

Example 1 shows that a stable (strongly stable) problem is not strongly quasi-stable, and consequently, not quasi-stable (unalterable).

EXAMPLE 1 *Let* $m = n = 2$, $X = \{x^1, x^2, x^3\}$, $x^1 = (2,1)^T$, $x^2 = (-4,0)^T$, $x^3 = (0,2)^T$,

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

Then, $L^2(C) = L^2_1(C) = \{x^1, x^2, x^3\}$, $W^2(C) = \emptyset$. *According to Corollary 1, problem* $Z^2(C)$ *is stable, but in view of Theorem 3 it is not strongly quasi-stable.*

The following example shows that a problem can be strongly quasi-stable not being quasi-stable (unalterable).

EXAMPLE 2 *Let* $m = 2$, $n = 3$, $X = \{x^1, x^2, x^3\}$, $x^1 = \mathbf{0}_{(3)}$, $x^2 = (1,1,-1)^T$, $x^3 = (-2,-2,2)^T$,

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 4 & 7 \end{pmatrix}.$$

Then, $L^2(C) = X$. *Therefore,* $|L^2(C)| = 3$. *It follows from Corollary 7 that* $Z^2(C)$ *is not quasi-stable. On the other hand, taking into account Corollary 11 and* $\mathbf{0}_{(3)} \in X$, *we obtain that* $Z^2(C)$ *is strongly quasi-stable.*

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