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Incremental value of information for discrete-time partially observed stochastic systems*

by

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Abstract: A discrete-time stochastic control problem for general (nonlinear in state, control, observation and noise) models is considered. The same noise can enter into the state and into the observation equations, and the state/observation does not need to be affine with respect to the noise. Under mild assumptions the joint distribution function of the state/observation processes is obtained and used for computing the Gateaux and Fréchet derivatives of the cost function. Under partial observation the control actions are restricted by the measurability requirement and we compute the Lagrange multiplier associated with this "information constraint". The multiplier is called a "dual", or "shadow" price, and in the literature of the subject is interpreted as an incremental value of information . The present and the future are two factors appearing in the multiplier and we study how they are balanced as time goes on. An algorithm for computing extremal controls in the spirit of R. Rishel (1985) is also obtained.

Keywords: stochastic control, filtering, Hausdorff measures, change of variables formula of Federer, Lagrange multipliers, value of information.

1. Introduction

For $F : \mathbb{R}^{n+m} \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{n+m}$ being some measurable function, called sometimes *the dynamic function*, let us define a stochastic control system via the iterative scheme

$$\mathbf{z}_{i+1} = F\left(\mathbf{z}_i, u_i, \xi_{i+1}\right), \quad \mathbf{z}_{i=0} = \mathbf{z}_0, \quad i = \{0, 1, ..., N-1\} \tag{1}$$

where $\mathbf{z}_0, \xi_1, ..., \xi_N$, is a sequence of stochastically independent random elements on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}^{n+m}$, $\mathbb{R}^q$, respectively, with

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densities of distribution functions ρ for $\mathbf{z}_0$, and g for ξ_i, $i = 1, ..., N$. Denote $z = \mathrm{col}(x, y)$, $F = \mathrm{col}(F^1, F^2)$, where $x, F^1 \in \mathbb{R}^n$, and $y, F^2 \in \mathbb{R}^m$. Then (1) takes the form

$$\begin{bmatrix} \mathbf{x}_{i+1} \\ \mathbf{y}_{i+1} \end{bmatrix} = \begin{bmatrix} F^1 \left(\mathbf{x}_i, \mathbf{y}_i, u_i, \xi_{i+1}\right) \\ F^2 \left(\mathbf{x}_i, \mathbf{y}_i, u_i, \xi_{i+1}\right) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x}_{i=0} \\ \mathbf{y}_{i=0} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix}. \tag{2}$$

Next, we assume that the coordinates $(\mathbf{x}_i; i = 0, 1, ...)$ of the state process $(\mathbf{z}_i; i = 0, 1, ...)$, are not observable, while the coordinates $(\mathbf{y}_i; i = 0, 1, ...)$ are. The control action at time i, i.e., u_i, can use only the previous observations $Y_i = (y_0, ..., y_i)$, i.e., $u_i = v_i(Y_i)$, where $v_i : \mathbb{R}^{m(i+1)} \to \mathbb{R}^p$ are measurable mappings. Denote $U_i = (u_0, ..., u_i)$, $V_i = (v_0, ..., v_i)$ and $V = \{V_{N-1}; (v_0(Y_0), ..., v_{N-1}(Y_{N-1}))\}$ and introduce a cost functional

$$J(U_{N-1}) = \mathbb{E}\left[r(\mathbf{z}_N) + \sum_{i=0}^{N-1} r_i(\mathbf{z}_i, u_i)\right] \tag{3}$$

where $\mathbb{E}$ denotes expectation with respect to the measure $\mathbb{P}$. Suppose that $r, r_i, i = 0, ..., N-1$ are measurable and bounded from below. The problem is to find

$$\inf\{J(V_{N-1}); V_{N-1} \in V\}. \tag{4}$$

NOTATION 1 *We use two kinds of symbols: (1) bold (examples are* $\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi$*), to denote random variables, (2) italic (examples are* x, y, u, v*), for deterministic objects such as numbers, variables, mappings, etc. Capital letters are used for sequences; (1) of random variables (for example* $\mathbf{Z}_i = (\mathbf{z}_0, ..., \mathbf{z}_i)$*), (2) deterministic objects (for example* $Z_i = (z_0, ..., z_i)$*).*

REMARK 1 *If* $F^1 \equiv 0$*, then (2) takes the form*

$$\mathbf{y}_{i+1} = F^2\left(\mathbf{x}_0, \mathbf{y}_i, u_i, \xi_{i+1}\right), \quad \mathbf{y}_{i=0} = \mathbf{y}_0$$

and we see that the classical adaptive control problem is a special case of our model.

REMARK 2 *The Gaussian noise case with the dynamic function affine in* ξ_{i+1} *and* x_i *was considered by R.S. Liptser and A.N. Shiryiaev (1999). They showed that when the initial conditions are Gaussian, then the conditional law is Gaussian as well.*

REMARK 3 *From the equivalence theorem of J. Zabczyk (1996), Ch. 3, Th. 3.1.1, p. 26) it follows that without loss of generality one can choose as the space* $(\Omega, \mathcal{F}, \mathbb{P})$ *the basic probability space* $\left([0,1), \mathcal{B}([0,1)), \lambda_{[0,1)}\right)$*, and for the noise, a sequence* $\xi_1, ..., \xi_N$ *of independent uniformly distributed random variables on* $[0, 1)$*.*

REMARK 4 *The system in the form (2) can be also viewed as a fully observable dynamic system with the state process* $(\mathbf{y}_i; i = 0, 1, ...)$ *and with two kind of disturbances; colored, represented by* $(\mathbf{x}_i; i = 0, 1, ...)$ *and non-colored, represented by* $(\xi_i; i = 0, 1, ...)$*, respectively.*

The control problem (2)(4), considered in this paper, is a partially observed variation of the fully observed general nonlinear problem considered in Zabczyk (1996). For this version we obtain a recursion for joint measures of the state process $(\mathbf{z}_i; i = 0, 1, ...)$ in Section 2. In Section 3 we compute weak variations and Gateaux derivatives of the cost functional and identify them as Lagrange multipliers in Section 5. The multiplier is called a "dual", or "shadow" price, and is interpreted as an Incremental Value of Information, a concept with a long history. See Remark 7 for more information. The present and the future are two factors appearing in the multiplier and we study how they are balanced as time goes on. Finally, in Section 6 the approach of R. Rishel (1995) is used to compute extremal controls.

2. Filtering

For $\mathbf{Z}_i = (\mathbf{z}_0, ..., \mathbf{z}_i)$, $\mathbf{z}_i = (\mathbf{x}_i, \mathbf{y}_i)$, define the measure

$$\mu_i(A) = \mathbb{P}(\mathbf{Z}_i \in A), \quad A \in \mathcal{B}\left(\mathbb{R}^{(n+m)(i+1)}\right). \tag{5}$$

Let $h : \mathbb{R}^{n(i+2)} \times \mathbb{R}^{m(i+2)} \to \mathbb{R}$, be a bounded Borel function. Then

$$\mathbb{E}h(\mathbf{Z}_{i+1}) = \int h(Z_{i+1})\,\mu_{i+1}(dZ_{i+1}). \tag{6}$$

However, from (2), and stochastic independence assumptions, we also have

$$\begin{aligned}\mathbb{E}h(\mathbf{Z}_{i+1}) &= \mathbb{E}h\left(\mathbf{Z}_i, F\left(\mathbf{z}_i, \mathbf{u}_i, \xi_{i+1}\right)\right)\\ &= \int \left[\int h(Z_i, F(z_i, u_i, \xi))\; g(\xi)\, d\xi\right] \mu_i(dZ_i).\end{aligned} \tag{7}$$

When $q \leq n+m$, the change of variables formula of H. Federer (1996, Th. 3.2.6, p. 245) applies, and

$$\begin{aligned}&\int h(Z_i, F(z_i, u_i, \xi))\, g(\xi)\, d\xi =\\ &= \int \sum_{\xi \in \{G(z_i, u_i, z_{i+1})\}} h(Z_i, F(z_i, u_i, \xi))\, g(\xi)\, JG(z_i, u_i, z_{i+1})\,\mathcal{H}^q(dz_{i+1})\end{aligned} \tag{8}$$

where $\mathcal{H}^q(\mathbb{R}^n)$ is a $q-$ dimensional Hausdorff measure in $\mathbb{R}^n$, and $G(z, u, \cdot)$ is the inverse function of $F(z, u, \cdot)$, $JG(z_i, u_i, z_{i+1})$ is the Jacobian, where

$$JG(a, u, z) = \sqrt{\left[\frac{\partial(G)}{\partial(z)}\right]\left[\frac{\partial(G)}{\partial(z)}\right]^T (a, u, z)} \tag{9}$$

(see Evans and Gariepy, 1992, Ch. 3, Th. 3, p. 88). When for any fixed $(z_i, u_i) \in \mathbb{R}^{n+m} \times \mathbb{R}^p$, the inverse image of the ξ i.e., the set $\{G(z_i, u_i, z_{i+1})\}$ is a singleton, then the sum in the integral above reduces to one single term and the right hand side of (8) now reads

$$= \int h(Z_{i+1}) \mathbb{I}_{F_i}(z_{i+1}) g(G(z_i, u_i, z_{i+1})) JG(z_i, u_i, z_{i+1}) \mathcal{H}^q(dz_{i+1}) \quad (10)$$

where $F_i = \{F(z_i, u_i, \{\text{supp } g\})\}$, and $\mathbb{I}_F$ is the set function of F. Substitute (10) into (7). Since h is arbitrary, comparison the right hand sides of (6) and (7) gives

$$\mu_{i+1}(dZ_{i+1}) = g(G(z_i, u_i, z_{i+1})) \mathbb{I}_{F_i}(z_{i+1}) JG(z_i, u_i, z_{i+1}) \mathcal{H}^q(dz_{i+1}) \mu_i(dZ_i)$$

and by induction we get the joint measure

$$\mu_i(dZ_i) = \prod_{j=0}^{i-1} g(G(z_j, u_j, z_{j+1})) \mathbb{I}_{F_j}(z_{j+1}) JG(z_j, u_j, z_{j+1}) \mathcal{H}^q(dz_{j+1}) \rho(z_0) dz_0. \quad (11)$$

In conclusion we have

Theorem 1
Assume dim $\xi = q \leq n + m = \dim z$. Denote $D_{zu} = \{F(z, u, \{supp\ g\})\}$. Suppose, that for any $(z, u) \in \mathbb{R}^{n+m} \times \mathbb{R}^p$, there exists an inverse to $F(z, u, \cdot)$, denoted by $G(z, u, \cdot)$, i.e., the mapping $G(z, u, \cdot) : D_{zu} \to \mathbb{R}^q$, such that the Jacobian $JG(a, u, b) \in (0, \infty)$ for $(a, u, b) \in \mathbb{R}^{n+m} \times \mathbb{R}^p \times \mathbb{R}^{n+m}$, and that the set $\{G(a, u, b)\}$ is a singleton for any $(a, u, b) \in \mathbb{R}^{n+m} \times \mathbb{R}^p \times D_{au}$. Then, μ_i is given by (11).

Remark 5 *Note that μ_i given by (11) depends on the sequence of control actions $U_i = (u_0, ..., u_i)$. So, we denote this measure by $\mu_i^{(U)}$, instead of μ_i, and $\mathbb{E}^{(U)}$ in place of $\mathbb{E}$ if needed.*

Remark 6 *The joint measure can be used for computing conditional probabilities*

$$P_y(C) \triangleq \mathbb{P}[\mathbf{x}_i \in C \,|\mathbf{Y}_i = y] = \frac{\mu_i(\mathbb{R}^{ni} \times C \times \{y\})}{\mu_i(\mathbb{R}^{n(i+1)} \times \{y\})}, \quad \text{if } \mu_i\left(\mathbb{R}^{n(i+1)} \times \{y\}\right) \neq 0 \quad (12)$$

and in the form convenient for integration in dynamic programming algorithms

$$P_y(da) \triangleq \mathbb{P}[\mathbf{x}_i \in da \,|\mathbf{Y}_i = y] = \frac{\mu_i(\mathbb{R}^{ni} \times da \times \{y\})}{\mu_i(\mathbb{R}^{n(i+1)} \times \{y\})}, \quad \text{if } \mu_i\left(\mathbb{R}^{n(i+1)} \times \{y\}\right) \neq 0$$

$$Q_y(db) \triangleq \mathbb{P}[\mathbf{y}_{i+1} \in db \,|\mathbf{Y}_i = y] = \frac{\mu_{i+1}(\mathbb{R}^{n(i+2)} \times \{y\} \times db)}{\mu_{i+1}(\mathbb{R}^{n(i+2)} \times \{y\} \times \mathbb{R}^m)}, \quad \text{if } \mu_{i+1}\left(\mathbb{R}^{n(i+2)} \times \{y\} \times \mathbb{R}^m\right) \neq 0.$$

The examples below show applications of (12) and (11) for computing conditional expectations. We shall use them in the algorithm given in the last section.

EXAMPLE 1 *For* $h : \mathbb{R}^n \times \mathbb{R}^{m(i+1)} \to \mathbb{R}$

$$\mathbb{E}\left[h\left(\mathbf{x}_i, \mathbf{Y}_i\right)|\mathbf{Y}_i = y\right] = \int_{\mathbb{R}^n} h\left(a, y\right) P_y\left(da\right)$$

$$= \int_{\mathbb{R}^n} h\left(a, y\right) \frac{\mu_i^{(U)}\left(\mathbb{R}^{ni} \times da \times \{y\}\right)}{\mu_i^{(U)}\left(\mathbb{R}^{n(i+1)} \times \{y\}\right)} \triangleq H_i^{(U)}\left(y\right).$$

Hence

$$\mathbb{E}\left[h\left(\mathbf{x}_i, \mathbf{Y}_i\right)|\mathbf{Y}_i\right] = H_i^{(U)}\left(\mathbf{Y}_i\right).$$

EXAMPLE 2 *For* $h : \mathbb{R}^{m(i+2)} \to \mathbb{R}$

$$\mathbb{E}\left[h\left(\mathbf{Y}_i, \mathbf{y}_{i+1}\right)|\mathbf{Y}_i = y\right] = \int h\left(y, b\right) Q_y\left(db\right)$$

$$= \int h\left(y, b\right) \frac{\mu_{i+1}\left(\mathbb{R}^{n(i+2)} \times \{y\} \times db\right)}{\mu_{i+1}\left(\mathbb{R}^{n(i+2)} \times \{y\} \times \mathbb{R}^m\right)} \triangleq G_{i+1}^{(U)}\left(y\right).$$

Hence

$$\mathbb{E}\left[h\left(\mathbf{Y}_i, \mathbf{y}_{i+1}\right)|\mathbf{Y}_i\right] = G_{i+1}^{(U)}\left(\mathbf{Y}_i\right).$$

EXAMPLE 3 *The same quantity as above can be computed in a different way. For* $h : \mathbb{R}^{m(i+2)} \to \mathbb{R}$

$$\mathbb{E}\left[h\left(\mathbf{Y}_i, \mathbf{y}_{i+1}\right)|\mathbf{Y}_i = y\right] = \int \left[\int h\left(y, F^2\left(a, y_i, u_i, \xi\right)\right) g\left(\xi\right) d\xi\right] P_y\left(da\right)$$

$$= \int \left[\int h\left(y, F^2\left(a, y_i, u_i, \xi\right)\right) g\left(\xi\right) d\xi\right] \frac{\mu_i\left(\mathbb{R}^{ni} \times da \times \{y\}\right)}{\mu_i\left(\mathbb{R}^{n(i+1)} \times \{y\}\right)} \triangleq R_i^{(U)}\left(y\right).$$

Hence

$$\mathbb{E}\left[h\left(\mathbf{Y}_i, \mathbf{y}_{i+1}\right)|\mathbf{Y}_i\right] = R_i^{(U)}\left(\mathbf{Y}_i\right).$$

3. Weak variations and Gateaux derivatives

Using results from the previous section we can express the performance criteria in the form

$$J\left(U_{N-1}\right) = \mathbb{E}\left[\sum_{i=0}^{N} r_i\left(\mathbf{z}_i, u_i\right)\right] = \sum_{i=0}^{N} \int r_i\left(a, b, u_i\right) \mu_i^{(U)}\left(\mathbb{R}^{ni} \times da \times \mathbb{R}^{mi} \times db\right)$$

where we put $r_N(a,b,u) = r(a,b)$. Now, in order to obtain the necessary condition for optimality of the control $V^\star_{N-1}$, the common practice (see Rishel, 1985, for instance) is to take the weak variations of $J(U_{N-1})$. Let $j \in \{1,...,N-1\}$, and let $V^{(j)}_{N-1}$ be an admissible control, which satisfies

$$v_i(y_0,...,y_i) = 0 \quad \text{if } i \neq j.$$

Consider the control $V^\star_{N-1} + \epsilon V^{(j)}_{N-1}$. Since $V^\star_{N-1}$ is an optimal control, $J\left(V^\star_{N-1} + \epsilon V^{(j)}_{N-1}\right)$ must have a minimum at $\epsilon = 0$, and if $J\left(V^\star_{N-1} + \epsilon V^{(j)}_{N-1}\right)$ is differentiable with respect to ϵ, its derivative (the Gateaux derivative) must vanish there. If $i < j+1$, then

$$\mu_i^{\left(V^\star + \epsilon V^{(j)}\right)}(dZ_i) = \mu_i^{(V^\star)}(dZ_i)$$

but when $i \geq j+1$, then

$$\begin{aligned} \mu_i^{\left(V^\star + \epsilon V^{(j)}\right)}(dZ_i) &= \\ &= g\left(G\left(z_j, u_j^\star + \epsilon v_j, z_{j+1}\right)\right) \mathbb{I}_{F_j^\epsilon}(z_{j+1})\, JG\left(z_j, u_j^\star + \epsilon v_j, z_{j+1}\right) \mathcal{H}^q(dz_{j+1}) \\ \times \prod_{p=0, p\neq j}^{i-1} & g\left(G\left(z_p, u_p^\star, z_{p+1}\right)\right) \mathbb{I}_{F_p}(z_{p+1})\, JG\left(z_p, u_p^\star, z_{p+1}\right) \mathcal{H}^q(dz_{p+1})\, \rho(z_0)\, dz_0 \end{aligned} \tag{13}$$

where $F_j^\epsilon = F\left(z_j, u_j^\star + \epsilon v_j, \{\text{supp } g\}\right)$. Hence

$$\left[\frac{\partial}{\partial \epsilon} \mu_i^{\left(V^\star + \epsilon V^{(j)}\right)}(dZ_i)\right]_{\epsilon=0} = \tag{14}$$

$$= \begin{cases} 0 \quad \text{if} \quad i < j+1 \\ v_j^T \left\{\nabla_u \left[\ln g\left(G\left(z_j, u_j^\star, z_{j+1}\right)\right) \mathbb{I}_{F_j}(z_{j+1})\, JG\left(z_j, u_j^\star, z_{j+1}\right)\right]\right\} \mu_i^{(V^\star)}(dZ_i) \\ \qquad \text{if} \quad i \geq j+1 \end{cases} .$$

Theorem 2 *Assume: (!)* $F \in C(\mathbb{R}^{n+m} \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^{n+m})$, $r_i \in C_b(\mathbb{R}^{n+m} \times \mathbb{R}^p)$, $\nabla_u r_i \in C_b(\mathbb{R}^{n+m} \times \mathbb{R}^p, \mathbb{R}^p)$, $r \in C_b(\mathbb{R}^{n+m})$, $g \in C^1(\mathbb{R}^q)$, *(!!)* $q = n+m$, *(!!!)* $F(z,u,\cdot)$ *is one-to-one for* $(z,u) \in \mathbb{R}^q \times \mathbb{R}^p$, *its inverse* $G(z,u,\cdot)$ *and the Jacobi determinant* $\mathrm{JG} \in C^{0,1,0}(\mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^q)$, $\mathrm{JG} \in (0,M), M < \infty$, *(!!!!)* $\nabla_u[\ln g(G)\,\mathrm{JG}] \in C_b(\mathbb{R}^{n+m} \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, *then* $J\left(V^\star_{N-1} + \epsilon V^{(j)}_{N-1}\right)$ *is* $\epsilon-$*differentiable, and*

$$\left[\frac{\partial}{\partial \epsilon} J\left(V^\star_{N-1} + \epsilon V^{(j)}_{N-1}\right)\right]_{\epsilon=0} = \tag{15}$$

$$\int v_j^T \nabla_u r_j\left(a,b,u_j^\star\right)\mu_j^{(V^\star)}\left(\mathbb{R}^{nj}\times da\times\mathbb{R}^{mj}\times db\right)$$
$$+\int v_j^T \nabla_u \ln\left[g\left(G\left(a,u_j^\star,b\right)\right)\mathrm{J}G\left(a,u_j^\star,b\right)\right]$$
$$\times\sum_{i=j+1}^{N} r_i\left(a,b,u_i^\star\right)\mu_i^{(V^\star)}\left(\mathbb{R}^{ni}\times da\times\mathbb{R}^{n(N-i)}\times\mathbb{R}^{mj}\times db\times\mathbb{R}^{m(N-i)}\right)$$

which is nothing but the Gateaux derivative. Equivalently

$$\left[\frac{\partial}{\partial\epsilon}J\left(V_{N-1}^\star+\epsilon V_{N-1}^{(j)}\right)\right]_{\epsilon=0}=\mathbb{E}\mathbf{v}_j^T\left\{\nabla_u r_j\left(\mathbf{z}_j,\mathbf{u}_j^\star\right)+\right. \tag{16}$$
$$\left.\left[\sum_{i=j+1}^{N} r_i\left(\mathbf{z}_i,\mathbf{u}_i^\star\right)\right]\nabla_u\ln\left[g\left(G\left(\mathbf{z}_j,\mathbf{u}_j^\star,\mathbf{z}_{j+1}\right)\right)\mathrm{J}G\left(\mathbf{z}_j,\mathbf{u}_j^\star,\mathbf{z}_{j+1}\right)\right]\right\}.$$

Proof. From (!!) and (!!!) follows that $\mathbb{I}_{F_j^\epsilon}(z)\equiv 1$, for $z\in\mathbb{R}^q$, and $i=j,...,N-1$. Now (15), (16) follows from (14), (13) and the product rule for differentiation, i.e., taking into account (14) and $\epsilon-$differentiating the expression

$$J\left(V_{N-1}^\star+\epsilon V_{N-1}^{(j)}\right)=\int r_j\left(a,b,u_j^\star+\epsilon v_j\right)\mu_j^{\left(V_{N-1}^\star\right)}\left(\mathbb{R}^{nj}\times da\times\mathbb{R}^{mj}\times db\right)$$
$$+\sum_{i=j+1}^{N}\int r_i\left(a,b,u_i\right)\mu_i^{\left(V_{N-1}^\star+\epsilon V_{N-1}^{(j)}\right)}\left(\mathbb{R}^{ni}\times da\times\mathbb{R}^{mi}\times db\right)$$

gives the results. ■

4. Subspace constraints and Lagrange multipliers

In this section we select in short some facts from Davis, Dempster and Elliott (1991). Let X be a Banach space with dual space $X^\star$, and let S be a linear subspace of X. We define

$$S^\perp=\{x^\star\in X^\star:\langle x^\star,x\rangle=0,\forall x\in S\}$$

where $\langle x^\star,x\rangle$ denotes the pairing between $x\in X$ and $x^\star\in X^\star$.

Let $\phi: X\to\mathbb{R}$ be a Fréchet differentiable functional and suppose that ϕ achieves its minimum over S at $x_0\in S$. The Fréchet derivative is a map $\phi': X\to X^\star$ such that for $h,x\in X$

$$\phi(x+h)=\phi(x)+\langle\phi'(x),h\rangle+o(\|h\|).$$

LEMMA 1 *If ϕ achieves its minimum over S at $x_0\in S$, then $\phi'(x_0)\in S^\perp$.*

Proof. If $\phi'(x_0) \notin S^{\perp}$ then there exists $h \in S$ such that $\langle \phi'(x_0), h \rangle = \delta > 0$. But then $\phi(x_0 - \epsilon h) = \phi(x_0) - \epsilon(\delta + o(\epsilon)/\epsilon)$, so that $\phi(x_0 - \epsilon h) < \phi(x_0)$ for small ϵ. ■

THEOREM 3 *If $\phi : X \to \mathbb{R}$ is Fréchet differentiable and achieves its minimum over S at $x_0 \in S$, then there exists $\lambda \in S^{\perp}$ such that Lagrange functional*

$$L(x) = \phi(x) + \langle \lambda, x \rangle$$

is stationary at x_0, i.e., $L'(x_0) = 0$.

Proof. We have only to set $\lambda = -\phi'(x_0)$. ■

5. Application to stochastic control

To apply the above results to our problem, we take X to be the space $L_p^{\theta}(N \times (\Omega, \mathcal{F}, \mathbb{P}))$, $\theta > 1$, of all controls $W = \{W_{N-1}; (w_0(Z_0), ..., w_{N-1}(Z_{N-1}))\}$ satisfying

$$\mathbb{E} \sum_{i=0}^{N-1} \|w_i(Z_i)\|^{\theta} < \infty$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^p$ and S a subspace of X of all controls $V = \{V_{N-1}; (v_0(Y_0), ..., v_{N-1}(Y_{N-1}))\}$. It is clear that S is a linear subspace of X. Then, $X^{\star}$ is the space $L_p^{\eta}(N \times (\Omega, \mathcal{F}, \mathbb{P}))$, where $\eta = \theta / (\theta - 1)$ and

$$S^{\perp} = \{\lambda \in X^{\star} : \mathbb{E} \langle \lambda, V \rangle = 0, \forall V \in S\}.$$

The relationship between Gateaux and Fréchet derivative of ϕ is that if the Gateaux derivative takes the form

$$\mathbb{E} \sum \lambda_j \mathbf{v}_j \tag{17}$$

for some $\lambda = \mathrm{col}(\lambda_0, ..., \lambda_{N-1}) \in X^{\star} = L_p^{\eta}(N \times (\Omega, \mathcal{F}, \mathbb{P}))$, then ϕ is Fréchet differentiable and $\phi'(\mathbf{u}) = \lambda$. Hence, from (16)(17) we obtain

THEOREM 4 *Under the notations and assumptions of Theorems 1 and 2*

$$\begin{aligned} \lambda_j = & \nabla_u r_j\left(\mathbf{z}_j, \mathbf{u}_j^{\star}\right) + \\ & \left[\sum_{i=j+1}^{N} r_i\left(\mathbf{z}_i, \mathbf{u}_i^{\star}\right)\right] \nabla_u \ln\left[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^{\star}, \mathbf{z}_{j+1}\right)\right) JG\left(\mathbf{z}_j, \mathbf{u}_j^{\star}, \mathbf{z}_{j+1}\right)\right] \end{aligned} \tag{18}$$

for $j = 0, 1, ..., N-1$.

Proof. The RHS of (18) is bounded, hence it belongs to $L^{\eta}(\Omega, \mathcal{F}, \mathbb{P})$ for any $\eta \geq 0$. ■

Remark 7 *(Incremental value of information) In stochastic optimization problems one can encounter at least two approaches to defining the value of information. The first, leading to the so called* Incremental Value of Information *was initiated by M.H.A. Davis, M.A.H. Dempster and R.J. Elliott (1991) and it uses an idea of R.J.-B. Wets (1975). The second, initiated by T. Banek and R. Kulikowski (2003) and independently by M. Schweizer and D. Becherer (2003) is based on the idea that information can be the object of trade and its value for a particular agent is a consequence of its utility. In this paper we follow the former way. First, by introducing the Lagrange multiplier, we turn the optimization problem into the one global minimization over all controls from* $W \subset L_p^{\theta}(N \times (\Omega, \mathcal{F}, \mathbb{P}))$ *which are* $\mathbf{Z} = \mathrm{col}(\mathbf{z}_0, ..., \mathbf{z}_{N-1})-$*measurable, i.e., take the form* $W = \{W_{N-1}; (w_0(Z_0), ..., w_{N-1}(Z_{N-1}))\}$*. Second, the Lagrange multiplier has an interpretation as a price system for small violations of the constraint, in our case, small* $\mathbf{Z}_i-$*measurable perturbations of the controls. To understand this approach the best way is perhaps to recall the beautiful idea of Joseph Louis de Lagrange in classical mechanics. In order to extend the Newtonian dynamics of free particles to the general case where the particles are allowed to move on some surfaces only, Lagrange introduced a new force, a "reaction" of the surface. If there is no friction, then the reaction must be orthogonal to the surface. Thus, to determine the reaction it is enough to find its length. It appears that this can be defined uniquely so that a free particle keeps moving on the surface only if it is affected by this reaction. Hence, the problem with constraints was reduced to the known problem with forces (but without constraints). To find analogy with our problem replace the free particles by the* $\mathbf{Z}-$*measurable control actions, the surface of permissible movement by the linear space* $V = \{V_{N-1}; (v_0(Y_0), ..., v_{N-1}(Y_{N-1}))\}$ *and the reaction by* λ*. Under action of* λ*, the* $\mathbf{Z}-$*measurable controls (the elements of* W*) will stay in the space* V*, exactly as the particles in the Lagrangian mechanics do. For economic interpretation, used here, one has to remember the form of the Lagrange functional appearing in Theorem 3. Additional term* $\langle \lambda, x \rangle$*, the second term in the sum, can be viewed as an extra cost, a penalty for small violation of the constraint by the unfair control action. Due to the linearity of the term, the multiplier is a cost "per capita", i.e., the "dual" price. It is worth mentioning that the violations considered here are in contrast with anticipative (allowed to know the future) perturbations considered by Davis, Dempster and Elliott (1991). Finally, our price system perhaps may have some practical value for a controller who has an extra option, for instance he can buy the observations* $\mathbf{X} = \mathrm{col}(\mathbf{x}_0, ..., \mathbf{x}_{N-1})$*, or for creating a technical device, an observation system able to produce* $\mathbf{X}$*. The question interesting for the controller is to know what is the right price for buying the observations* $\mathbf{X} = \mathrm{col}(\mathbf{x}_0, ..., \mathbf{x}_{N-1})$*? Our price system tells only how much a small violation of the constraint costs and thus can serve as a linear approximation.*

Conclusion 1 *The necessary condition of optimality given in Theorem 2 now* *takes the form*

$$\mathbb{E}\left\{\lambda_j \left|\mathbf{Y}_j\right.\right\} = 0$$

for $j = 0, 1, ..., N-1$. *Indeed, since* $\mathbf{v}_j$ *in Theorem 2 is an arbitrary* $\mathbf{Y}_j$-*measurable function, a standard argument using the definition of conditional expectation implies the last expression follows from (16)(18).*

REMARK 8 *The RHS of (18) expresses what can be understood as the right balance in the information pricing process between the present and the future. To see this just observe that the ratio* $\nabla_u r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right)/r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right)$ *is equal to* $\nabla_u \ln r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right)$, *which suggests the following representation*

$$\lambda_j = w_j r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + w_{j+1} r_{j+1}\left(\mathbf{z}_{j+1}, \mathbf{u}_{j+1}^\star\right) + ... + w_N r_N\left(\mathbf{z}_N, \mathbf{u}_N^\star\right)$$

of λ_j, *where weights are*

$$w_j = \nabla_u \ln r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) \quad \text{the present}$$
$$w_{j+1} = ... = w_N = \nabla_u \ln\left[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right) \mathrm{J}G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right] \quad \text{the future.}$$

Hence, there are two different weights; at present time j, *the weight* w_j, *which appears to depend on the present cost* r_j, *but not depend on the system dynamics function* F, *and the weights responsible for the future* $w_{j+1}, ..., w_N$, *which (1) appear to be equal i.e.,* $w_{j+1} = ... = w_N$, *(2) depend on the system function* F *and the variables* $\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}$, *but does not depend on the costs* $r_{j+1}, ..., r_N$.

CONCLUSION 2 *Let*

$$\eta_j\left(\mathbf{Z}_{jN}, \mathbf{U}_{jN}^\star\right) = \frac{r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right)}{\sum_{i=j}^N r_i\left(\mathbf{z}_i, \mathbf{u}_i^\star\right)}$$

where $\mathbf{Z}_{jN} = (\mathbf{z}_j, ..., \mathbf{z}_N)$, $\mathbf{U}_{jN} = (\mathbf{u}_j, ..., \mathbf{u}_N)$, *then*

$$\frac{\lambda_j}{\sum_{i=j}^N r_i\left(\mathbf{Z}_{jN}, \mathbf{U}_{jN}^\star\right)} = \eta_j\left(\mathbf{Z}_{jN}, \mathbf{U}_{jN}^\star\right) \nabla_u \ln\left(r_j, \mathbf{z}_j, \mathbf{u}_j^\star\right)$$
$$+ \left[1 - \eta_j\left(\mathbf{Z}_{jN}, \mathbf{U}_{jN}^\star\right)\right] \nabla_u \ln\left[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right) \mathrm{J}G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right]$$

which shows that the normalized Lagrange multipliers are convex linear combinations of the present and the future weights. Moreover, the combination coefficients $(\eta_j, 1-\eta_j)$ *are the ratios of the present and remaining costs per the total cost, respectively. Since the game played between the present and the future is expressed via the combination coefficients* $(\eta_j, 1-\eta_j)$, $j = 0, 1, ..., N$, *we may observe how dominant is the role of the future (here it means how valuable is the missing information) at the beginning of control actions and how this role decreases as time goes on.*

6. An algorithm for computing extremal controls

We call a control extremal if it satisfies the necessary condition for optimality expressed in Conclusion 1, that is, if for each $j = 0, 1, ..., N-1$,

$$\mathbb{E}\Bigg\{\nabla_u r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + \tag{19}$$

$$+\left[\sum_{i=j+1}^{N} r_i\left(\mathbf{z}_i, \mathbf{u}_i^\star\right)\right] \nabla_u \ln\left[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right) JG\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right] |\mathbf{Y}_j \Bigg\} = 0.$$

Following Rishel (1985) we use this condition to compute u. To begin, define

$$V_j\left(\mathbf{Z}_j\right) = \mathbb{E}\left\{\sum_{i=j}^{N} r_i\left(\mathbf{z}_i, \mathbf{u}_i^\star\right) |\mathbf{Z}_j\right\}.$$

From the law of iterated conditional expectation

$$\begin{aligned} V_j\left(\mathbf{Z}_j\right) &= r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + \mathbb{E}\left\{\sum_{i=j+1}^{N} r_i\left(\mathbf{z}_i, \mathbf{u}_i^\star\right) |\mathbf{Z}_j\right\} \\ &= r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + \mathbb{E}\left\{\mathbb{E}\left[\sum_{i=j+1}^{N} r_i\left(\mathbf{z}_i, \mathbf{u}_i^\star\right) |\mathbf{Z}_{j+1}\right] |\mathbf{Z}_j\right\} \\ &= r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + \mathbb{E}\left\{V_{j+1}\left(\mathbf{Z}_{j+1}\right) |\mathbf{Z}_j\right\} \end{aligned} \tag{20}$$

with the terminal condition

$$V_N\left(\mathbf{Z}_N\right) = r\left(\mathbf{z}_N\right). \tag{21}$$

Now, the law of iterated conditional expectation and (1) imply that the LHS of (19) is given by

$$\mathbb{E}\Bigg\{\nabla_u r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) + \tag{22}$$

$$+V_{j+1}\Big(\mathbf{Z}_j, F\left(\mathbf{z}_j, \mathbf{u}_\mathbf{j}^\star, \xi_{j+1}\right)\Big) \nabla_u \ln\left[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right) JG\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right] |\mathbf{Y}_j \Bigg\}.$$

By using the Examples 1 and 2, given in Section 2 we have

$$\mathbb{E}\left\{\nabla_u r_j\left(\mathbf{z}_j, \mathbf{u}_j^\star\right) |\mathbf{Y}_j\right\} = \int \nabla_u r_j\left(a, \mathbf{y}_j, u_j^\star\right) \frac{\mu_j^{(U)}\left(\mathbb{R}^{nj} \times da \times \{\mathbf{Y}_j\}\right)}{\mu_j^{(U)}\left(\mathbb{R}^{n(i+1)} \times \{\mathbf{Y}_j\}\right)} \tag{23}$$

and

$$\mathbb{E}\{V_{j+1}\Big(\mathbf{Z}_j, F\Big(\mathbf{z}_j, \mathbf{u_j^\star}, \xi_{j+1}\Big)\Big)\nabla_u \ln\big[g\left(G\left(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\right)\right) JG\big(\mathbf{z}_j, \mathbf{u}_j^\star, \mathbf{z}_{j+1}\big)\big] \,|\mathbf{Y}_j\,\}=$$

$$\int \left[\int V_{j+1}\left(X_j, \mathbf{Y}_j, F\left(x_j, \mathbf{y}_j, u_j^\star, \xi\right)\right) g'\left(\xi\right) d\xi\right]$$

$$\times \nabla_u G\left(x_j, \mathbf{y}_j, u_j^\star, x_{j+1}, y_{j+1}\right) \frac{\mu_{j+1}^{(U)}\left(dX_{j+1} \times \{\mathbf{Y}_j\} \times dy_{j+1}\right)}{\mu_{j+1}^{(U)}\left(dX_{j+1} \times \{\mathbf{Y}_j\} \times dy_{j+1}\right)} +$$

$$\int \left[\int V_{j+1}\left(X_j, \mathbf{Y}_j, F\left(x_j, \mathbf{y}_j, u_j^\star, \xi\right)\right) g\left(\xi\right) d\xi\right]$$

$$\times \nabla_u \ln JG\left(x_j, \mathbf{y}_j, u_j^\star, x_{j+1}, y_{j+1}\right) \frac{\mu_{j+1}^{(U)}\left(dX_{j+1} \times \{\mathbf{Y}_j\} \times dy_{j+1}\right)}{\mu_{j+1}^{(U)}\left(dX_{j+1} \times \{\mathbf{Y}_j\} \times dy_{j+1}\right)}, \tag{24}$$

hence, substitution of (24) into (22) gives an integral expression for the necessary condition, moreover, it suggest the following algorithm, which uses backward induction to compute $\mathbf{u_j^\star}$ and $V_j\left(\mathbf{Z}_j\right)$ such that $\mathbf{u_j^\star}$ satisfies the optimality condition (14).

Step 0:
Set $V_N\left(Z_N\right) = r\left(z_N\right)$ and $j = N - 1$.

Step 1:
Given $V_{j+1}\left(Z_{j+1}\right)$, put

$$\widetilde{V}_{j+1}\left(Z_j, u_j, \xi_{j+1}\right) = V_{j+1}\left(Z_j, F\left(x_j, y_j, u_j, \xi_{j+1}\right)\right).$$

Step 2:
Evaluate (24) and denote the sum (23) + (24) by $Z\left(Y_j, u_j\right)$.

Step 3:
Solve

$$Z\left(Y_j, u_j\right) = 0 \quad \text{for} \quad u_j^\star.$$

Step 4:
Compute

$$V_j\left(Z_j\right) = r_j\left(z_j, u_j^\star\right) + \int \widetilde{V}_{j+1}\left(Z_j, u_j^\star, \xi\right) g\left(\xi\right) d\xi\,.$$

Step 5:
If $j = 0$, stop. Otherwise, decrease j by 1 and go to **Step 1.**

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