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Schur stability of the convex combination of complex polynomials*

by

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Abstract: This paper gives a necessary and sufficient condition for Schur stability of the convex combination of complex polynomials. It is a generalization of the work by Ackerman and Barmish (1988).

Keywords: Schur stability, robust stability, convex combination of the polynomials.

1. Introduction

Stability analysis of many control systems is concerned with the location of zeroes of their characteristic polynomials. In practice, the coefficients of these polynomials are not known exactly. Thus, one of the real problems in stability analysis is to determine Schur stability of the convex combination $C(f_1, f_2, \ldots, f_m)$ of the complex polynomials $f_1(x), f_2(x), \ldots, f_m(x)$. This would mean a generalization of results by Ackerman and Barmish (1988), Białas (2004), Choo and Choi (2007), Jury (1975), and Yang and Hwang (2001). Ackerman and Barmish (1988) gave a necessary and sufficient condition for Schur stability of the convex combination of the real polynomials.

The complex polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where $a_n \neq 0$, is called Schur stable if all its zeros are in the open unit circle. In this paper we will write "stable" instead of "Schur stable". The degree of the polynomial $f(x)$ will be denoted by $deg(f)$. And the field of real (complex) numbers will denoted by $R(C)$.

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If K is a field of real or complex numbers, we will use the following notations:

$$P_n(K) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots$$
$$+ a_1 x + a_0 : a_i \in K (i = 0, 1, \ldots, n), deg(f) = n\},$$
$$S_n(K) = \{f(x) \in P_n(K) : \forall_{x \in C} f(x) = 0 \Rightarrow |x| < 1\}.$$

Let $\overline{f}(x) = \overline{a}_n x^n + \overline{a}_{n-1} x^{n-1} + \cdots + \overline{a}_1 x + \overline{a}_0$ be the conjugate polynomial of the polynomial (1), where $\overline{a}_j$ is the conjugate of the number a_j.

It is clear that:

$$\begin{aligned} f(x) \in S_n(C) &\Leftrightarrow \overline{f}(x) \in S_n(C) \\ f(x) \in S_n(C) &\Leftrightarrow f(x)\overline{f}(x) \in S_{2n}(R). \end{aligned} \tag{2}$$

In this paper we will assume that $deg(f) \geq 1$, $Re(a_n) > 0$ for $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in S_n(K)$.

Let $S(f)$ denote the matrix associated with the polynomial (1), where

$$S(f) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_3 & a_2 - a_0 \\ 0 & a_n & a_{n-1} & \ldots & a_4 - a_0 & a_3 - a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & -a_0 & -a_1 & \ldots & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}. \tag{3}$$

It is easy to notice that the dimensions of the matrix $S(f)$ are $(n-1) \times (n-1)$. Moreover, it is known that if $f(x) \in P_n(R)$, then

$$det(S(f)) = a_n^{n-1} \prod_{1 \leq i < j \leq n} (1 - x_i x_j) \tag{4}$$

where x_i, x_j are zeros of the polynomial $f(x)$.

Consider the complex polynomials

$$a_n^{(i)} x^n + a_{n-1}^{(i)} x^{n-1} + \cdots + a_1^{(i)} x + a_0^{(i)} = f_i(x) \in P_n(C) \tag{5}$$

for $i = 1, 2, \ldots, m$. We will use the notations:

$$V_m = \{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in R^m : \ \alpha_i \geq 0 \ (i = 1, 2, \ldots, m),$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\},$$
$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots$$
$$+ \alpha_m f_m(x) : \ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m\},$$

i.e. $C(f_1, f_2, \ldots, f_m)$ is a convex combination of the polynomials $f_1(x), f_2(x), \ldots,$ $f_m(x)$. The field $Q \subset P_n(C)$ is called stable if $Q \subset S_n(C)$.

From the Viète's formula we have the following corollary:

Corollary 1 *If $f(x)$, $g(x)$ are complex polynomials and $deg(f) \neq deg(g)$, $1 \leq deg(f)$, $1 \leq deg(g)$ then the convex combination $C(f,g)$ is not Schur stable.*

We will now prove the following lemma:

Lemma 1 *If the matrices A, $B \in R^{n\times n}$, $detA \neq 0$, $\lambda \in C$ then λ_0 is a solution of the equation*

$$det(\lambda^2 A + \lambda B + I_n) = 0$$

if and only if $1/\lambda_0$ is the eigenvalue of the matrix

$$Q = \begin{bmatrix} -B & -A \\ I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ unit matrix , 0 is the zero matrix.

Proof. From the assumption $detA \neq 0$ it follows that $detQ \neq 0$ and $\lambda_0 \neq 0$. Therefore

$$det(\lambda I_{2n} - Q) = \begin{vmatrix} B + I_n\lambda & A \\ -I_n & I_n\lambda \end{vmatrix} = \begin{vmatrix} B + I_n\lambda + \frac{1}{\lambda}A & A \\ 0 & I_n\lambda \end{vmatrix} =$$

$$= \lambda^n det(B + I_n\lambda + \frac{1}{\lambda}A) = \lambda^{2n} det((\frac{1}{\lambda})^2 A + (\frac{1}{\lambda})B + I_n)$$

for every $\lambda \neq 0$. Hence, the thesis of Lemma 1 follows. ■

Let $P_i(t) :< t_0, t_1 > \to C$ $(i = 1, 2, \ldots, m)$ be a continuous function for $t \in < t_0, t_1 >$, where $t_0, t_1 \in R$, $t_0 < t_1$, and

$$T(t) = \{\alpha_1 P_1(t) + \alpha_2 P_2(t) + \cdots + \alpha_m P_m(t) : \ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m\}$$

for $t \in < t_0, t_1 >$.

The following lemma is true.

Lemma 2 *If the functions $P_i(t) :< t_0, t_1 > \to C$ $(i = 1, 2, \ldots, m)$ are continuous for $t \in < t_0, t_1 >$ and $(0,0) \in T(t_0)$, $(0,0) \notin T(t)$ for all $t \in (t_0, t_1 >$, then $(0,0) \in \partial(T(t_0))$, where $\partial(T(t_0))$ is the border of the set $T(t_0)$.*

2. The necessary and sufficient condition for Schur stability of the convex combination of complex polynomials

We will prove the necessary and sufficient condition for Schur stability of the set $C(f_1, f_2, \ldots, f_m)$, where the polynomials $f_1(x), f_2(x), \ldots, f_m(x)$ are determined as in (5).

From the Corollary 1 it follows that, without loss of generality, we can assume that: $deg(f_1) = deg(f_2) = \cdots = deg(f_m) = n$.

At first, we will prove the necessary and sufficient condition for the stability of the convex combination $C(f_1, f_2)$, where

$$f_1(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in P_n(C), \qquad (6)$$
$$f_2(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \in P_n(C)\,.$$

Let

$$W = \begin{bmatrix} S(f_1\overline{f}_2 + \overline{f}_1 f_2)S^{-1}(f_2\overline{f}_2) & S(f_1\overline{f}_1)S^{-1}(f_2\overline{f}_2) \\ -I & 0 \end{bmatrix}, \qquad (7)$$

where $S(\cdot)$ is the matrix defined in (3), I is $(2n-1)\times(2n-1)$ unit matrix, 0 is zero matrix. It is easy to note that matrix W is of dimensions $(4n-2)\times(4n-2)$.

We will prove the following theorem.

THEOREM 1 *If complex polynomials (6) are Schur stable, $Re(a_n) > 0$, $Re(b_n) > 0$, then convex combination $C(f_1, f_2)$ is Schur stable if and only if the eigenvalues $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, 4n-2)$, where the matrix W is defined by (7).*

Proof. We see that $f_1(x)\overline{f}_1(x)$, $f_2(x)\overline{f}_2(x)$, $f_1(x)\overline{f}_2(x)+\overline{f}_1(x)f_2(x)$ are the real polynomials of degree $2n$.

The necessary condition. We will prove that if $C(f_1, f_2) \subset S_n(C)$, then $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, 4n-2)$. From the assumption $C(f_1, f_2) \subset S_n(C)$ we have

$$f(x)\overline{f}(x) = [\alpha f_1(x) + (1-\alpha) f_2(x)][\alpha \overline{f}_1(x) + (1-\alpha)\overline{f}_2(x)] \in S_{2n}(R)$$

for all $\alpha \in (0, 1)$. Moreover

$$S(f\overline{f}) = \alpha^2 S(f_1\overline{f}_1) + (1-\alpha)\alpha S(f_1\overline{f}_2 + \overline{f}_1 f_2) + (1-\alpha)^2 S(f_2\overline{f}_2).$$

From the Jury's theorem (Jury, 1975) and the expression (4) it follows that there exist an inverse matrix $S^{-1}(f_2\overline{f}_2)$ and

$$det\, S(f\overline{f}) = det[\alpha^2 S(f_1\overline{f}_2) + (1-\alpha)\alpha S(f_1\overline{f}_2 + \overline{f}_1 f_2) + (1-\alpha)^2 S(f_2\overline{f}_2)] \neq 0,$$
$$det\Big[\Big(\frac{\alpha}{1-\alpha}\Big)^2 S(f_1\overline{f}_1)S^{-1}(f_2\overline{f}_2) + \Big(\frac{\alpha}{1-\alpha}\Big) S(f_1\overline{f}_2 + \overline{f}_1 f_2)S^{-1}(f_2\overline{f}_2) + I\Big] \neq 0$$

for all $\alpha \in (0, 1)$. Therefore

$$det[\lambda^2 S(f_1\overline{f}_1)S^{-1}(f_2\overline{f}_2) + \lambda S(f_1\overline{f}_2 + \overline{f}_1 f_2)S^{-1}(f_2\overline{f}_2) + I] \neq 0$$

for all $\lambda \in (0, \infty)$. Hence and from Lemma 1 it follows that $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, 4n-2)$.

The sufficient condition. Now we will prove that if $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, 4n-2)$ then $C(f_1 f_2) \subset S_n(C)$.

For the proof by reductio ad absurdum, we assume that there exists a polynomial $\alpha_1 f_1(x) + (1-\alpha_1)f_2(x) \in C(f_1, f_2)$, which is not stable. Therefore, the polynomials

$$[\alpha_1 f_1(x) + (1-\alpha_1)f_2(x)][\alpha_1 \overline{f}_1(x) + (1-\alpha_1)\overline{f}_2(x)] \notin S_{2n}(R).$$

Hence, it follows that there exist $\alpha_0 \in (0,1)$ and $z_0 \in \{z \in C : |z| = 1\}$ such that

$$[\alpha_0 f_1(z_0) + (1-\alpha_0)f_2(z_0)][\alpha_0 \overline{f}_1(\overline{z}_0) + (1-\alpha_0)\overline{f}_2(\overline{z}_0)] = 0.$$

Hence, and from expression (4), we have

$$det\, S[(\alpha_0 f_1 + (1-\alpha_0)f_2)(\alpha_0 \overline{f}_1 + (1-\alpha_0)\overline{f}_2)] = 0.$$

Therefore

$$det[\left(\frac{\alpha_0}{1-\alpha_0}\right)^2 S(f_1\overline{f}_1)S^{-1}(f_2\overline{f}_2) + \frac{\alpha_0}{1-\alpha_0} S(f_1\overline{f}_2 + \overline{f}_1 f_2)S^{-1}(f_2\overline{f}_2) + I] = 0$$
$$det[\lambda_0^2 S(f_1\overline{f}_1)S^{-1}(f_2\overline{f}_2) + \lambda_0 S(f_1\overline{f}_2 + \overline{f}_1 f_2)S^{-1}(f_2\overline{f}_2) + I] = 0$$

for $\lambda_0 = \frac{\alpha_0}{1-\alpha_0} \in (0, \infty)$.

From the above and Lemma 1 we have the contradiction to the assumption $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, 4n-2)$. ■

In a particular case, when the polynomials $f_1(x)$, $f_2(x)$ are real, Theorem 2 is true.

THEOREM 2 *(Białas, 2004) If the polynomials (6) are real and Schur stable, $a_n > 0$, $b_n > 0$, then the convex combination $C(f_1, f_2)$ is Schur stable if and only if the eigenvalues $\lambda_i(W) \in (-\infty, 0)$ $(i = 1, 2, \ldots, n-1)$, where $W = S(f_1)S^{-1}(f_2)$.*

Now we will prove the necessary and sufficient condition for Schur stability of the convex combination of polynomials (5).

Let

$$R(C(f_1, f_2, \ldots, f_m)) = \{z \in C : \bigvee_{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m} \alpha_1 f_1(z) + \alpha_2 f_2(z) + \cdots + \alpha_m f_m(z) = 0\}.$$

THEOREM 3 *If the complex polynomials*

$$f_i(x) = a_n^{(i)} x^n + a_{n-1}^{(i)} x^{n-1} + \cdots + a_1^{(i)} x + a_0^{(i)},$$

where $Re(a_n^{(i)}) > 0$ $(i = 1, 2, \ldots, m)$, are Schur stable, then the convex combination

$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : (\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m\}$$

is Schur stable if and only if the convex combinations $C(f_i, f_j)$ $(i, j = 1, 2, \ldots, m;$ $i < j)$ *are Schur stable.*

Proof. From the assumption, $Re(a_n^{(i)} > 0)$ $(i = 1, 2, \ldots, m)$ and also from the fact that the roots of a polynomial are continuous functions of their coefficients it follows that there exists

$$\max_{z \in R(C(f_1, f_2, \ldots, f_m))} |z| = |z_0| = r \tag{8}$$

where $z_0 = \alpha_0 + i\beta_0 = re^{i\phi_0}$.

The necessary condition follows from the relation

$$C(f_i, f_j) \subset C(f_1, f_2, \ldots, f_m).$$

The sufficient condition. We will prove that from the assumption $C(f_i, f_j) \subset S_n(C)$ $(i, j = 1, 2, \ldots, m; i < j)$ it follows that $C(f_1, f_2, \ldots, f_m) \subset S_n(C)$.

For the proof by reductio ad absurdum, we suppose that there exist number $z_0 = \alpha_0 + i\beta_0 \in C$ and polynomial $f(z) \in C(f_1, f_2, \ldots, f_m)$ such that

$$f(z_0) = 0 \quad \text{and} \quad |z_0| \geq 1. \tag{9}$$

Without loss of generality we can assume that $z_0 = \alpha_0 + i\beta_0$ is defined by (8).

Let $z(\epsilon) = (r + \epsilon)e^{i\phi_0}$ for $\epsilon \geq 0$.

Consider the sets:

$$\begin{aligned} T(z(\epsilon)) = T((r+\epsilon)e^{i\phi_0}) = \{&\alpha_1 f_1(z(\epsilon)) + \alpha_2 f_2(z(\epsilon)) + \cdots \\ &+\alpha_m f_m(z(\epsilon)) : (\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m\}, \\ A_{ij}(z_0) = \{&\alpha_1 f_i(z_0) + \alpha_2 f_j(z_0) : (\alpha_1, \alpha_2) \in V_2\}, \end{aligned}$$

where $\epsilon \geq 0$, $i, j = 1, 2, \ldots, m$; $i < j$.

We can see that $T(z(\epsilon)) \in C$, $(0, 0) \in T(z_0)$ and

$$(0,0) \notin T(z(\epsilon)) \quad \text{for every} \quad \epsilon > 0,$$

$$\partial(T(z_0)) \subset \bigcup_{1 \leq i < j \leq n} A_{ij}(z_0),$$

where $\partial(T(z_0))$ denotes the border of the set $T(z_0)$.

Hence and from Lemma 2 it follows that $(0, 0) \in \partial(T(z_0))$.

Therefore, there exist $i_0, j_0 \in \{1, 2, \ldots, m\}$ such that

$$(0,0) \in \{\alpha_1 f_{i_0}(z_0) + \alpha_2 f_{j_0}(z_0) : \ (\alpha_1, \alpha_2) \in V_2\},$$

where $|z_0| \geq 1$. This is the contradiction to the assumption $C(f_{i_0}, f_{j_0}) \subset S_n(C)$. ∎

Theorem 3 is a generalization of the work by Ackerman and Barmish (1988), where the necessary and sufficient condition for Schur stability of the convex combination of real polynomials was provided.

3. Example

It is easy to verify that the complex polynomials:

$$f_1(x) = 6x^2 - ix + 1,\ f_2(x) = 4x^2 - 2x + (1-i),\ f_3(x) = 20x^2 - ix + 1$$

are Schur stable.
Moreover

$$\begin{aligned}
f_1(x)\overline{f_1}(x) &= 36x^4 + 13x^2 + 1,\ f_2(x)\overline{f_2}(x) \\
&= 16x^4 - 16x^3 + 12x^2 - 4x + 2, \\
f_3(x)\overline{f_3}(x) &= 400x^4 + 41x^2 + 1,\ f_1(x)\overline{f_2}(x) \\
&= 24x^4 - (12+4i)x^3 + (10+8i)x^2 + (-1-i)x + (1+i), \\
f_1(x)\overline{f_3}(x) &= 120x^4 + 14ix^3 + 27x^2 + 1, \\
f_2(x)\overline{f_3}(x) &= 80x^4 + (-40+4i)x^3 + (24-22i)x^2 + (3+i)x + (1-i), \\
f_1(x)\overline{f_2}(x) + \overline{f_1}(x)f_2(x) &= 48x^4 - 24x^3 + 20x^2 - 2x + 2, \\
f_1(x)\overline{f_3}(x) + \overline{f_1}(x)f_3(x) &= 240x^4 + 54x^2 + 2, \\
f_2(x)\overline{f_3}(x) + \overline{f_2}(x)f_3(x) &= 160x^4 - 80x^3 + 48x^2 + 6x + 2.
\end{aligned}$$

We will verify the stability of the convex combination $C(f_1, f_2, f_3)$ by using Theorems 1 and 3.

From (3) and (7) we can compute the matrices $W(f_1f_2)$, $W(f_1f_3)$, $W(f_2, f_3)$. For example

$$W(f_1f_2) = \begin{bmatrix} S(f_1\overline{f_2} + \overline{f_1}f_2)S^{-1}(f_2\overline{f_2}) & S(f_1\overline{f_1})S^{-1}(f_2\overline{f_2}) \\ -I & 0 \end{bmatrix} =$$

$$= \frac{1}{1856}\begin{bmatrix} 5376 & 2400 & 1536 & 4416 & 3360 & 4224 \\ 448 & 4736 & 3584 & 1248 & 2640 & 9984 \\ 720 & -936 & 7488 & 624 & -840 & 5856 \\ -1856 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1856 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1856 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we can compute the matrices $W(f_1f_3)$ and $W(f_2f_3)$.
Next, we can compute (by using MATHEMATICA) the eigenvalues of these matrices. As the result we obtain:

$$\begin{aligned}
&\lambda_1(W(f_1f_2)) = 2.97396,\ \lambda_2(W(f_1f_2)) = 1.75818 + i0.112615, \\
&\lambda_3(W(f_1f_2)) = 1.75818 - i0.112615,\ \lambda_4(W(f_1f_2)) = 0.780743, \\
&\lambda_5(W(f_1f_2)) = 1.10585 + i0.858365,\ \lambda_6(W(f_1f_2)) = 1.10585 - i0.858365,
\end{aligned}$$

$$\lambda_1(W(f_1f_3)) = 0.333334,\ \lambda_2(W(f_1f_3)) = 0.3 + i5.9 * 10^{-9},$$
$$\lambda_3(W(f_1f_3)) = 0.3 - i5.9 * 10^{-9},\ \lambda_4(W(f_1f_3)) = 0.300001,$$
$$\lambda_5(W(f_1f_3)) = 0.263157,\ \lambda_6(W(f_1f_3)) = 0.222221,$$
$$\lambda_1(W(f_2f_3)) = 0.379688,\ \lambda_2(W(f_2f_3)) = 0.149927,$$
$$\lambda_3(W(f_2f_3)) = 0.19961,\ \lambda_4(W(f_2f_3)) = 0.0857401,$$
$$\lambda_5(W(f_2f_3)) = 0.148595 + i0.0939353,$$
$$\lambda_6(W(f_2f_3)) = 0.148595 - i0.0939353.$$

We can see that

$$\lambda_i(W(f_1f_2)) \notin (-\infty, 0),\ \lambda_i(W(f_1f_3)) \notin (-\infty, 0),\ \lambda_i(W(f_2f_3)) \notin (-\infty, 0)$$

for $i = 1, 2, \dots, 6$.

Hence, from Theorems 1 and 3 it follows that the convex combinations $C(f_1, f_2)$, $C(f_1, f_3)$, $C(f_2, f_3)$ and $C(f_1, f_2, f_3)$ are Schur stable.

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