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Key performance indicators for supply planning of multilevel serial systems with stochastic lead times*

by

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Abstract: This paper examines a supply planning problem for multilevel serial production systems under lead time uncertainties. The techniques used in industry are often based on the assumption that the lead times are known. However, in a supply chain the lead times are often random variables. Therefore, it is necessary to evaluate the influence of the planned lead times on the total cost. An exact performance evaluation technique is developed to calculate the total cost as a function of the planned lead times when the actual lead times are random discrete variables. The sum of the average component holding and tardiness costs at each level, plus the average finished product backlogging cost is considered. Several properties of this function are proven. A numerical example is reported.

Keywords: multilevel serial systems, random lead times, performance evaluation, newsboy model, generalizations.

1. Introduction

Uncertainty in lead times (or delivery times from an external supplier) is a major problem in production systems. These times vary due to many factors, including machine breakdowns, transport delays, poor quality, etc. A late component may delay all subsequent processes and too early availability engenders overstocking. The effects of lead time uncertainty are particularly problematic in multilevel production systems (see Bullwhip effect, Chen et al., 2000). To minimize the influence of these random lead times, firms can implement safety lead times. The

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safety lead time is defined as the difference between the planned and expected lead times. Nevertheless, excess safety lead times create stocks and stocks are expensive. Therefore, the problem is to optimize stock at each level by assigning adequate values of planned lead times.

In this paper, the planned lead time analysis is considered for a multilevel serial supply chain with unlimited number of levels and random component lead times for each level. The demand of the finished product is assumed to be fixed and known. The lead time of each component (delivery time for the next level) is supposed to be a discrete random variable. No restrictive hypothesis is made on such random variables; only that the probability distributions are known.

The holding as well as tardiness costs are considered. Tardiness costs can be due to the cost of revising a schedule. For the first level (level 1), corresponding to the finished product, tardiness means backlog, so for the finished product backlogging cost is introduced. Thus, the problem is to minimize the sum of expected holding, tardiness and backlogging costs. The decision variables are the planned lead times for components at each level. An exact performance evaluation model is proposed.

A similar multilevel production system (supply chain) was already studied by Yano (1987a,b). However, in that case, the lead times of components were continuous random variables. Yano limited the study to two and three stage (level) serial systems due to the difficulties to express the objective function in a closed form when the number of stages exceeds two.

The model suggested in this paper differs from Yano (1987a,b) as follows: we consider a discrete model with no restriction on the number of levels and our model offers the expression of objective function in a closed form.

The rest of the paper is organized as follows. Section 2 presents related publications. Section 3 deals with problem description. Section 4 presents the Key Performance Indicators (KPI). In Section 5, some interesting properties of the problem are presented. A numerical example is reported in Section 6. Finally, some concluding remarks are given in Section 7.

2. Related publications

In literature on supply planning, as far as can be determined, the number of publications on the considered problem with random lead times is modest, at best, in spite of its significance, in contrast with many models for random demand. Mula et al. (2006) have done a review for this domain; an extensive state of the art on the supply planning under uncertainties is also given in Dolgui and Prodhon (2007).

Earlier work includes simulation studies by Whybark and Williams (1976). They suggest that, in a multi-level production-inventory system, when the production and replenishment times are stochastic, safety lead times mechanism may perform better than that of safety stocks. Nevertheless, simulation studies of Grasso and Taylor (1984) led to opposite conclusion and to preference for safety stocks.

In Chen et al. (2000), Chatfield et al. (2004), Kim et al. (2006), simulations are also used for multilevel serial production systems with stochastic lead times. Their main effort dealt with information sharing among levels.

Some analytical models were also suggested. Weeks (1981) developed a one-stage model with tardiness and holding costs, in which processing time is stochastic and demand is deterministic. The author proves that this is equivalent to the standard "newsboy" problem.

Yano (1987a) used an analytic approach to determine optimal planned lead times in serial production systems, in which the actual procurement and processing times may be stochastic, demand is deterministic, and the lot-for-lot policy is used. The distribution of lead times is supposed stationary. The considered cost is the sum of inventory holding and job tardiness costs. The author presents a general solution procedure for two stage serial systems.

A similar problem is studied by Yano (1987b), but another cost is incurred: the rescheduling costs at the intermediate stages. Then, the objective is to minimize the sum of holding costs, rescheduling costs arising from tardiness at intermediate stages of productions, and tardiness of delivery to the customer. The author studied two and three stage serial systems due to the difficulties of the problem and complexity of the model. One of the main difficulties for this model is to express the objective function in a closed form when the number of stages exceeds two.

To surmount this difficulty, Elhafsi (2002) developed a recursive scheme evaluating the objective function efficiently for any number of levels without recurring to its expression in a closed form. However, computing time increases relatively quickly with the number of levels. To overcome this problem, the author presented a heuristic. For a special case of this continuous model, where the lead times are distributed exponentially, the author derived the objective function in a closed form.

Kim et al. (2004) suggested a model for constant demand and lead time with Erlang distribution for a single item inventory, and obtained an approximate solution. They launched an interesting conjecture that the behaviour of the analogous single-item inventory control model for the case where both demand and lead time are random can be calculated from the behaviours of the following three models: (i) deterministic demand and lead time, (ii) random demand and deterministic lead time, and (iii) random lead time and deterministic demand.

He, Kim and Hayya (2005) studied the impact of lead time when demand is constant in one level assembly system. In this study the lead time is random and limited, and the economic order quantity (EOQ) policy is used. The authors have shown that the cost varies linearly in the function of the deviation of time.

The problem of planned lead time calculation for one-level assembly systems was already studied in our previous work. In Dolgui and Louly (2002) a Markov model was proposed and in Louly and Dolgui (2002), a new generalization of the Discrete newsboy model was suggested. For a more general case, a branch and bound algorithm was developed in Louly, Dolgui and Hnaien (2008).

3. Problem description

We consider a serial production system with m levels (see Fig. 1). We suppose that the demand D of the finished product is fixed and its due date is known. To satisfy this demand, we need to launch the production processes composed of m serial stages (m levels) for a lot of D items. The level numbers are enumerated as follows: level m corresponds to the first production stage, level $m-1$ to the second stage, and so on. The raw materials are released at level m, semi-finished products are processed at levels $m-1, m-2, \ldots, 2$ and finally, finished product is produced at level 1. After these m levels, the lot of D finished products is delivered to the customer.

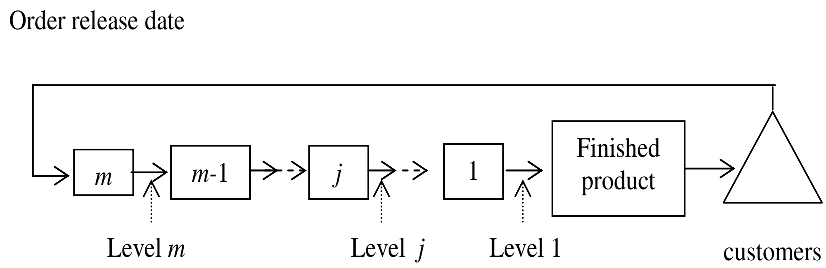

Figure 1. The m-level serial production system

We assume also that the lead time at each level (delivery time for next level) is a discrete random variable. No restrictive hypothesis is made on such random variables; we only suppose that the distribution probabilities are known. The policy is the lot-for-lot for all levels. Level m delivers the semi-finished products to level $m-1$ within a random lead time L_m, level j delivers the semi-finished products to level $j-1$ within a random lead time L_j, $j = 2, ..., m$. When the items arrive at the end of level 1, the customer demand D of finished products is satisfied. There are stocks at each intermediate level (from level m to level 1).

If total lead time exceeds the planned lead time of the component at level j, $j = 1, 2, ..., m$, tardiness is incurred and therefore the corresponding tardiness costs for level m to level 2. For level 1 this is called backlogging cost and corresponds to the finished product backlog. Otherwise, we obtain stocks and so corresponding holding cost for each level (see Fig. 2). Hence, the objective is to minimize the total cost composed of the holding, tardiness and backlogging costs.

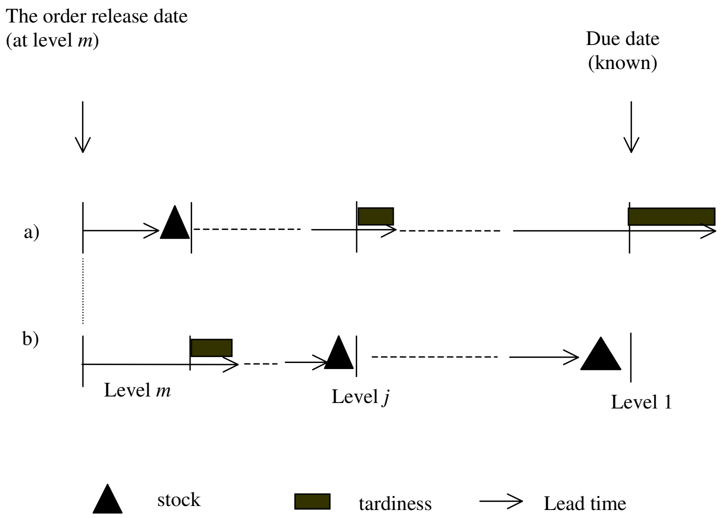

Figure 2. An illustration of the cost incurred

The following notations are introduced:

- c_j components at level j, where $j = 1, 2, ..., m$;
- d_j number of component j needed at level $j-1$;
- b_1 unit backlogging cost for finished product per period;
- b_j unit tardiness cost for component j, $j = 2, \ldots, m$, per period;
- h_j unit holding cost for component j, $j = 1, \ldots, m$, per period;
- D demand of finished products per period (fixed and known);
- L_j actual lead time of the component j (discrete random variable);
- $L'_j(x_{j+1}, x_{j+2}, ..., x_m)$ actual cumulative lead time of level j (proper lead time plus delays due to level $j+1$);
- u_j upper value of L_j;
- $Q_j = d_j D$ lot size for components j;
- x_j planned lead time for component j (integer decision variable);
- $F_j(k) = \Pr(L_j \leq k)$;
- $\Phi_j(k, x_{j+1}, ..., x_m) = \Pr(L'_j \leq k)$;
- $E(.)$ mathematical expectation operator.

Let the lead time for components of level j be a random discrete variable with known distribution: $\Pr(L_j = k)$, $k = 1, 2, ..., u_j$, where u_j is the maximum planned lead time value, for $j = 1, \ldots, m$. The lead time takes into account all processing times at the level j plus transportation time between level j and $j-1$. The actual cumulative lead time of the level j is given in (1):

$$\begin{cases} L'_j(x_{j+1}, x_{j+2}, ..., x_m) = L_j + (L'_{j+1}(x_{j+2}, x_{j+3}, ..., x_m) - x_{j+1})^+, \\ \hfill \text{for } j = 1, 2, ..., m{-}1 \\ L'_m = L_m \end{cases} \tag{1}$$

We present a model of this problem by analytically expressing the criterion to be optimized. This criterion considers the holding, component tardiness, and backlogging costs. For each type of component j, x_j denotes the planned lead time.

Note that in Hnaien et al. (2007) we considered the same problem as in the current article but for the case of a Just in Time (JIT) policy, where there are no holding costs at intermediate levels. For that problem, the objective was to find order release dates at level m (sum of planned lead times for all levels).

4. Key Performance Indicators

PROPOSITION 1 *The total cost is expressed as follows:*

$$\begin{aligned} C(X, L) = &\sum_{j=1}^{m} Q_j \left[h_j(x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m))\right] \\ &+ \sum_{j=1}^{m} Q_j \left[(b_j + h_j)(L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+\right] \end{aligned} \tag{2}$$

where

$$X = (x_1, ..., x_m),$$
$$L = (L'_1, ..., L'_m).$$

Proof. The cost is equal to the sum of the component holding, tardiness as well as finished product backlogging costs. If at a certain level, a job is completed before its planned due date, i.e. $(x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m)) > 0$, then a holding cost is incurring. Thus, the component holding cost is equal to:

$$\sum_{j=1}^{m} h_j Q_j (x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m))^+.$$

There is a tardiness (respectively backlog) of components j (respectively finished product) if the total lead time exceeds the planned lead time of the component at level j, $j = 1, 2, ..., m$, i.e. $(L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j) > 0$.

Thus, the sum of component tardiness cost and the finished product backlogging cost is equal to:

$$\sum_{j=1}^{m} b_j Q_j (L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+.$$

Then, the total cost is equal to:

$$C(X,L) = \sum_{j=1}^{m} b_j Q_j (L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+ \\ + \sum_{j=1}^{m} h_j Q_j (x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m))^+.$$

Note that $(-f)^+ = \max(-f, 0) = f^- = f^+ - f$.

So, if we consider $f = (x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m))$, $C(X,L)$ can be rewritten as follows:

$$C(X,L) = \sum_{j=1}^{m} (b_j + h_j) Q_j (L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+ \\ + \sum_{j=1}^{m} h_j Q_j (x_j - L'_j(x_{j+1}, x_{j+2}, ..., x_m)). \qquad \blacksquare$$

The cost $C(X,L)$ is a random variable. Therefore, we will calculate the mathematical expectation of $C(X,L)$ noted $EC(X)$.

Proposition 2 *The expected cost can be expressed as follows:*

$$EC(X) = \sum_{j=1}^{m} Q_j \left[h_j (x_j - E\left(L'_j(x_{j+1}, x_{j+2}, ..., x_m)\right)\right] \\ + \sum_{j=1}^{m} Q_j \left[(b_j + h_j) \sum_{k \geq 0} (1 - \Phi_j(x_j + k, x_{j+1} ..., x_m)) \right] \tag{3}$$

where

$$\begin{cases} \Phi_j(k, x_{j+1}, ..., x_m) = \sum_{s=1}^{k} \Pr(L_j = s)\, \Phi_{j+1}(x_{j+1} + k - s, x_{j+2}, ..., x_m), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } j = 1, ..., m-1 \\ \Phi_m(k) = F_m(k) \end{cases} \tag{4}$$

Proof. From relation (2), we derive the total expected cost:

$$EC(X) = E(C(X,L)) = \\ = \sum_{j=1}^{m} Q_j \left[h_j \left(x_j - E\left(L'_j(x_{j+1}, x_{j+2}, ..., x_m) \right) \right) + (b_j + h_j) E(Z) \right]$$

where:

$$Z = (L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+,$$

Z is a positive discrete random variable with a finite number of possible values, and its mathematical expectation is:

$$E(Z) = \sum_{i\geq 0} i\,\Pr(Z\!=\!i) = \sum_{i\geq 0}\sum_{k=0}^{i-1} \Pr(Z\!=\!i) = \sum_{k\geq 0}\sum_{i>k}\Pr(Z\!=\!i) = \sum_{k\geq 0}\Pr(Z\!>\!k).$$

Thus, we obtain:

$$E(Z) = \sum_{k\geq 0} \Pr\left((L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+ > k\right).$$

Given that

$$\begin{aligned}
&\Pr\left((L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+ > k\right) = \\
&= 1 - \Pr\left((L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j)^+ \leq k\right) \\
&= 1 - \left(\Pr\left((L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j) \leq k\right) \times \Pr\left(k \geq 0\right)\right).
\end{aligned}$$

Therefore:

$$E(Z) = \sum_{k\geq 0}\left(1 - \left(\Pr(L'_j(x_{j+1}, x_{j+2}, ..., x_m) - x_j \leq k) \times \Pr\left(k \geq 0\right)\right)\right).$$

In the previous expression the sum is computed for $k \geq 0$, thus:

$$\begin{aligned}
E(Z) &= \sum_{k\geq 0}\left(1 - \Pr(L'_j(x_{j+1}, x_{j+2}, ..., x_m) \leq x_j + k)\right) \\
&= \sum_{k\geq 0}\left(1 - \Phi_j(x_j + k, x_{j+1}, ..., x_m)\right)
\end{aligned}$$

where,

$$\begin{aligned}
&\Phi_j(k, x_{j+1}..., x_m) = \Pr(L'_j(x_{j+1}, x_{j+2}, ..., x_m) \leq k) \\
&= \Pr(L_j + (L'_{j+1}(x_{j+2}, ..., x_m) - x_{j+1})^+ \leq k) \\
&= \sum_{s=1}^{k} \Pr(L_j = s) \times \Pr\left((L'_{j+1}(x_{j+2}, ..., x_m) - x_{j+1})^+ \leq k - s\right) \\
&= \sum_{s=1}^{k} \Pr(L_j\!=\!s) \times \Pr\left(L'_{j+1}(x_{j+2}, ..., x_m) - x_{j+1} \leq k - s\right) \times \Pr\left(k - s \geq 0\right).
\end{aligned}$$

But, $k - s \geq 0$, thus:

$$\Phi_j(k, x_{j+1}..., x_m) = \sum_{s=1}^{k} \Pr(L_j = s)\Phi_{j+1}(x_{j+1} + k - s, x_{j+2}, ..., x_m).$$

Finally:

$$EC(X) = \sum_{j=1}^{m} Q_j \left[h_j(x_j - E\left(L'_j(x_{j+1}, x_{j+2}, ..., x_m)\right)\right]$$
$$+ \sum_{j=1}^{m} Q_j \left[(b_j + h_j) \sum_{k \geq 0} (1 - \Phi_j(x_j + k, x_{j+1} ..., x_m)) \right]. \qquad \blacksquare$$

Note that for the particular case of only one level, the total cost (2) can be rewritten as follows:

$$C(x_1, L_1) = Q_1[h_1(x_1 - L_1) + (b_1 + h_1)(L_1 - x_1)^+]. \tag{5}$$

The corresponding expected cost (3) is as follows:

$$EC(X) = Q_1 \left[h_1(x_1 - E\left(L_1\right)) + (b_1 + h_1) \sum_{k \geq 0} (1 - F_1(x_1 + k)) \right]. \tag{6}$$

Hence, the optimal solution for one level system is given by the well known *newsboy* model:

$$F_1(x_1 - 1) \leq \left(\frac{b_1}{b_1 + h_1} \right) \leq F_1(x_1). \tag{7}$$

5. Problem properties

Using the previous evaluation model, we present in this section some interesting properties for this problem.

5.1. Partial increments of cost functions

We will use the following partial increment functions (Louly, Dolgui and Hnaien, 2008):

$$G_j^+(X) = EC(x_1, ..., x_j + 1, ..., x_m) - EC(x_1, ..., x_j, ..., x_m), \tag{8}$$
$$G_j^-(X) = EC(x_1, ..., x_j - 1, ..., x_m) - EC(x_1, ..., x_j, ..., x_m). \tag{9}$$

These partial increments represent the evolution of the objective function due to increment or decrement of a decision variable. An optimal solution X must satisfy the requirements (10) and (11), otherwise there is a neighboring solution better than X.

$$G_j^+(X) \geq 0, \text{ for } j = 1, \ldots, m, \tag{10}$$
$$G_j^-(X) \geq 0 \text{ for } j = 1, \ldots, m. \tag{11}$$

PROPOSITION 3 *The function $G_j^+(X)$ can be rewritten as follows:*

$$G_j^+(X) \leq Q_j\left[-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)\right] + \sum_{s=1}^{j-1} Q_s h_s \tag{12}$$

$$G_j^+(X) \geq Q_j\left[-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)\right] - \sum_{s=1}^{j-1} Q_s (b_s + h_s). \tag{13}$$

Proof.

$$\begin{aligned} G_j^+(X) = & \sum_{s=1}^{m} Q_s\big[h_s(x_s - E\left(L'_s(x_{s+1}, ..., x_j + 1, ..., x_m)\right))\big] \\ & + \sum_{s=1}^{m} Q_s\big[(b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j + 1, ..., x_m))\big] \\ & - \sum_{s=1}^{m} Q_s\big[h_s(x_s - E\left(L'_s(x_{s+1}, ..., x_j, ..., x_m)\right)\big] \\ & - \sum_{s=1}^{m} Q_s\big[(b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j, ..., x_m))\big]. \end{aligned}$$

This difference between these two costs can be calculated term by term according to the value of the number s. The terms can be separated to facilitate the calculation. Let us note $G_j^+(X) = A + B + C$.

The first term is for the values of s larger than j. The difference equals zero for this group:

$$\begin{aligned} A = & \sum_{s=j+1}^{m} Q_s\big[h_s(x_s - E\left(L'_s(x_{s+1}, ..., x_m)\right)) + (b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_m))\big] \\ - & \sum_{s=j+1}^{m} Q_s\big[h_s(x_s - E\left(L'_s(x_{s+1}, ..., x_m)\right)) + (b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, , ..., x_m))\big] \\ = & \ 0. \end{aligned}$$

The second term is for $s = j$. The difference is as follows:

$$\begin{aligned} B = & \ Q_j\big[\, h_j\big(\, x_j + 1 - E\left(L'_j(x_{j+1}, x_{j+2}, ..., x_m)\right)\big)\,\big] \\ & + Q_j\big[(b_j + h_j)\sum_{k\geq 0}(1 - \Phi_j(x_j + 1 + k, x_{j+1}..., x_m))\big] \\ & - Q_j\big[\, h_j\left(x_j - E\left(L'_j(x_{j+1}, x_{j+2}, ..., x_m)\right)\right)\big] \\ & - Q_j\big[(b_j + h_j)\sum_{k\geq 0}(1 - \Phi_j(x_j + k, x_{j+1}..., x_m)\big] \end{aligned}$$

$$= Q_j[\,h_j - (b_j + h_j)(1 - \Phi_j(x_j, x_{j+1}..., x_m))\,]$$
$$= Q_j[\,-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)\,].$$

The third term is calculated for $s < j$, it is as follows:

$$\begin{aligned}
C &= \sum_{s=1}^{j-1} Q_s\big[h_s(x_s - E\,(L'_s(x_{s+1}, ..., x_j + 1, ..., x_m)))\big]\\
&+ \sum_{s=1}^{j-1} Q_s\big[(b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j + 1, ..., x_m))\big]\\
&- \sum_{s=1}^{j-1} Q_s\big[h_s(x_s - E\,(L'_s(x_{s+1}, ..., x_j, ..., x_m)))\big]\\
&- \sum_{s=1}^{j-1} Q_s\big[(b_s + h_s)\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j, ..., x_m))\big]\\
&= \sum_{s=1}^{j-1} Q_s h_s\big[E\,(L'_s(x_{s+1}, ..., x_j, ..., x_m)) - E\,(L'_s(x_{s+1}, ..., x_j + 1, ..., x_m))\big]\\
&+ \sum_{s=1}^{j-1} Q_s(b_s + h_s)\big[\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j + 1, ..., x_m))\big]\\
&- \sum_{s=1}^{j-1} Q_s(b_s + h_s)\big[\sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j, ..., x_m))\big].
\end{aligned}$$

Or, $0 \leq E\,(L'_s(x_{s+1}, ..., x_j, ..., x_m)) - E\,(L'_s(x_{s+1}, ..., x_j + 1, ..., x_m)) \leq 1$ and $0 \leq \sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j, ..., x_m)) - \sum_{k\geq 0}(1 - \Phi_s(x_s + k, ..., x_j + 1, ..., x_m)) \leq 1$.

Then, this last term C satisfies the following inequalities:

$$-\sum_{s=1}^{j-1} Q_s(b_s + h_s) \leq C \leq \sum_{s=1}^{j-1} Q_s h_s.$$

Finally, we conclude:

$$G_j^+(X) \leq Q_j\,[-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)] + \sum_{s=1}^{j-1} Q_s h_s$$
$$G_j^+(X) \geq Q_j\,[-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)] - \sum_{s=1}^{j-1} Q_s(b_s + h_s) \qquad \blacksquare$$

5.2. Properties of decisions variables

PROPOSITION 4 *The following properties are valid:*

$$\Phi_j(x_j, x_{j+1}..., x_m) \geq \alpha_j \text{ for } j = 1, \dots, m, \tag{14}$$
$$\Phi_j(x_j - 1, x_{j+1}..., x_m) \leq \beta_j \text{ for } j = 1, \dots, m, \tag{15}$$
$$F_m(x_m) \geq \alpha_m, \tag{16}$$
$$F_m(x_m - 1) \leq \beta_m, \tag{17}$$
$$F_j(x_j) \geq \alpha_j \text{ for } j = 1, \dots, m, \tag{18}$$

where $\alpha_j = \dfrac{Q_j b_j - \sum_{s=1}^{j-1} Q_s h_s}{Q_j(b_j + h_j)}$ *and* $\beta_j = \dfrac{Q_j b_j + \sum_{s=1}^{j-1} Q_s(b_s + h_s)}{Q_j(b_j + h_j)}$, for $j = 1, ..., m$.

Proof. According to (12):

$$G_j^+(X) \leq Q_j\left[-b_j + (b_j + h_j)\Phi_j(x_j, x_{j+1}..., x_m)\right] + \sum_{s=1}^{j-1} Q_s h_s.$$

Thus:

$$\frac{Q_j b_j - \sum_{s=1}^{j-1} Q_s h_s}{Q_j(b_j + h_j)} \leq \Phi_j(x_j, x_{j+1}..., x_m), \text{ for } j = 1, \dots, m.$$

Thus, property (14) is proved.

As $G_j^-(X) = -G_j^+(x_1, ..., x_j - 1, ..., x_m)$, we can derive the following inequality from (11):

$$0 \leq Q_j\left[b_j - (b_j + h_j)\Phi_j(x_j - 1, x_{j+1}..., x_m)\right] + \sum_{s=1}^{j-1} Q_s(b_s + h_s).$$

Thus,

$$\Phi_j(x_j - 1, x_{j+1}..., x_m) \leq \frac{Q_j b_j + \sum_{s=1}^{j-1} Q_s(b_s + h_s)}{Q_j(b_j + h_j)}.$$

Thus, property (15) is proved.

Using (4), property (16) is immediately derived from (14) and property (17) is immediately derived from (15).

Using (4):

$$\Phi_j(x_j, x_{j+1}..., x_m) \leq F_j(x_j), \text{ for } j = 1, \dots, m.$$

Finally,

$$\frac{Q_j b_j - \sum_{s=1}^{j-1} Q_s h_s}{Q_j(b_j + h_j)} \leq F_j(x_j), \text{ for } j = 1, \ldots, m,$$

i.e. we obtain property (18). ■

Note: for $m = 1$, properties (15)-(18) are equivalent to the well known newsboy model.

6. Numerical example

We give an illustrative example with 2 levels ($m = 2$). The lead time of each type of component is a discrete random variable, which takes values from 1 to 5 ($u_1 = u_2 = 5$), i.e., $1 \leq L_j \leq 5$. The unit holding costs are given in Table 1 and only one type of each component j is needed to produce the finished product, i.e., $Q_j = 1$. The distribution probabilities of all lead times are given in Table 2.

Table 1. Unit holding costs

j	1	2
h_j	10	10

Table 2. Probability distributions of the lead times

w	1	2	3	4	5
$\Pr(L_1 = w)$	0.50	0.30	0.10	0.05	0.05
$\Pr(L_2 = w)$	0.20	0.20	0.30	0.10	0.20

In the following tables (Tables 3, 4, and 5), the expected costs for different values of tardiness costs ($b_1 = 10$ and $b_2 = 5$; $b_1 = 20$ and $b_2 = 10$; $b_1 = 40$ and $b_2 = 20$) are reported.

We can see that the optimal solution of the first instance with $b_1 = 10$ and $b_2 = 5$ is (3, 2). The expected cost is 21.05.

These results show that it is necessary to set big values for the planned lead times when the unit backlogging costs are quite large. As we can see, when the backlogging (tardiness) costs increase, the optimal solutions for planned lead times increase also up to the upper values of lead times, (5, 5) in this example.

Table 3. Expected costs for different values of x_1 and x_2 where b_1=10 and b_2=5

	x_1				
x_2	1	2	3	4	5
1	36.00	27.00	22.20	22.00	25.20
2	29.65	21.65	**21.05**	24.05	30.65
3	32.90	26.40	28.60	34.80	33.10
4	44.50	38.50	43.30	40.90	33.70
5	59.00	54.00	50.00	42.00	33.00

Table 4. Expected costs for b_1=20 and b_2=10

	x_1				
x_2	1	2	3	4	5
1	73.00	54.00	41.80	36.50	**36.30**
2	59.30	43.30	37.40	36.90	41.80
3	61.80	48.80	47.10	51.40	43.85
4	78.00	66.00	68.20	59.60	43.80
5	99.00	89.00	78.00	61.00	42.50

Table 5. Expected costs for b_1=40 and b_2=20

	x_1				
x_2	1	2	3	4	5
1	144.00	108.00	81.00	65.50	58.50
2	118.60	86.60	70.10	62.60	64.10
3	119.60	93.60	84.10	84.60	65.35
4	145.00	121.00	118.00	97.00	64.00
5	179.00	159.00	134.00	99.00	**61.50**

7. Conclusions

The problem of planned lead time evaluation has not been sufficiently studied, especially for multilevel production systems with random actual lead times. That was the motivation of this paper.

Here, multilevel supply planning was studied under lead time uncertainties for the case where the actual lead times are independent random discrete variables. The cost function was the sum of finished product backlogging, component holding and tardiness costs for all levels. A mathematical model for performance evaluation was suggested.

The proposed model takes into account the dependence among level inventories and is a generalization of the well-known discrete newsboy model.

Further research will be dedicated to the development of efficient optimization algorithms using this evaluation model. It is also interesting to study the extensions of this approach for multilevel assembly systems. The main difficulty will be to represent in a treatable form the dependence among component inventories necessary to assemble a semi-finished product. In this perspective, the models of this paper may be useful for an approximate approach which consists in cutting the bill of material (BOM) tree of the finished product into multi-level linear parts (branches).

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