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Pontryagin Maximum Principle for coupled slow and fast systems*

by

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Abstract: When slow and fast controlled dynamics are coupled, the variational limit, as the ratio of time scales grows, is best depicted as a trajectory in a probability measures space. The effective control is then an invariant measure on the fast state-control space. The paper presents the form of the Pontryagin Maximum Principle for this variational limit and examines its relation to the Maximum Principle of the perturbed system.

Keywords: optimal control, singular perturbations, maximum principle.

1. Introduction

This paper examines singularly perturbed optimal control problems of the following form:

PROBLEM 1.1

$$\begin{aligned} &\text{minimize} && C(x(b)) \\ &\text{subject to} && \frac{dx}{dt} = f(x,z,u) \\ & && \varepsilon\frac{dz}{dt} = g(x,z,u) \\ & && x(a) = x_0,\ z(a) = z_0, \end{aligned} \tag{1.1}$$

where $[a, b]$ is a fixed time interval. The state variables $x \in R^n$ and $z \in R^m$ belong to the n-dimensional and, respectively, the m-dimensional Euclidean spaces; the control variable u is in a subset U of R^d; the coefficient $\varepsilon > 0$ is fixed and thought of as a small parameter.

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The letter C in the formulation of (1.1) stands for "*cost*". It is clear that an integral cost, say $\int_a^b c(x, z, u)dt$, can be incorporated within the form (1.1) by adding another coordinate to the x-state. Notice, however, that the cost depends only on the final x-state; the reason is that for ε small the z-state can be steered from one state to another on small time intervals, hence with negligible cost. Also, notice that Problem 1.1 is with free end condition. We allude to the reason for this after stating the convergence result.

The optimal (i.e., infimal) possible cost for Problem 1.1 is denoted by val(ε).

We are interested in the case where the parameter ε in (1.1) is small and, in fact, we examine the limit characteristics of the system as $\varepsilon \to 0$. Small ε signifies the state variable z as the "fast state", while x is the "slow state". Indeed, the pace in which the dynamics of the z coordinate evolves is ε^{-1} times the rate at which x evolves. Such singularly perturbed problems arise naturally in many applications, see the classical monograph Kokotovic, Khalil and O'Reilly (1986) and references therein.

An approach to decipher what is the limit behavior as $\varepsilon \to 0$ of (1.1) is to produce a "variational limit" or "nominal", problem; namely, an optimal control system whose optimal value is the limit as $\varepsilon \to 0$ of val(ε), and, if possible, the solution of the limit problem induces, for ε small, approximate solutions to (1.1).

A classical construction of a nominal limit system to the singularly perturbed equation (1.1) is by setting the parameter ε in (1.1) to be equal to 0; this approach was initiated and developed by Petar Kokotovic with students and colleagues; it is an adaptation to the control environment of the Tikhonov method in ordinary differential equations. The Kokotovic method has become a key tool in the theory, and has proven very effective in a variety of concrete applications; see Kokotovic, Khalil and O'Reilly (1986). See also Dontchev (1983) for the perturbation analysis of singularly perturbed optimal control systems.

However, the Kokotovic approach does not cover some interesting problems. Indeed, the approach is based on detecting a manifold of points (x, z, u) that are stationary and asymptotically stable with respect to the fast dynamics, namely, the z-equation in (1.1) when x is held fixed. There are, however, singularly perturbed optimal control problems, for which such a manifold does not exist, or may exist but it does not produce the limit optimal values of (1.1). Indeed, Leizarowitz (2002 a,b) showed that the situation where the Kokotovic approach cannot be applied is, in some sense, generic. In some cases, e.g. when z is a scalar, the Kokotovic approach is guaranteed to be valid, see Artstein and Leizarowitz (2002).

The problems that do not fall under the Kokotovic approach are characterized by the rapid oscillations exhibited on the fast time scale by the near optimal trajectories of (1.1) for small ε. Suggestions of how to cope with such a situation were raised back in the 1980s, see Dontchev and Veliov (1983, 1985) and Gaitsgory (1986, 1991, 1992). The analysis in the present paper makes use of a nominal system that captures the limit distributions of these oscillations. Such a theory within rapidly oscillating parameters was developed by the author, see Artstein (1993) and references therein; the corresponding generalization of Tikhonov's theory was presented in Artstein and Vigodner (1996), see also Artstein (2002a) and its references; the control theory framework, namely generalizing Kokotovic's theory, was initiated in Vigodner (1997) and in Artstein and Gaitsgory (1997). In Section 2 we give a telegraphic introduction to this theory, while displaying the setting for the present paper.

The main results of this paper examine the form that the Pontryagin maximum principle takes when optimal solutions are made of probability measure-valued maps, and the relations between the Pontryagin principle of the limit problem with the corresponding necessary conditions of the singularly perturbed system (1.1).

A feature of the control system that serves as the variational limit of the singularly perturbed system is that the control set depends on the state. Control theory handles this case by addressing the differential inclusion induced by the controls. We display the resulting maximum principle in Section 3. We also display, in Section 4, the maximum principle for the limit equation under an assumption that the effective control is not state dependent (many systems that do not satisfy this assumption can be transformed into ones that do). The relation with the maximum principle for the perturbed system is analyzed in Section 5. We argue, in particular, that the adjoint equations for the perturbed system are stiff and in general do not converge to the corresponding equations of the variational limit; we pinpoint a condition that implies the convergence; we also include an observation by Asen Dontchev concerning the convergence within the Kokotovic framework. An illustrative example demonstrating, in particular, how the dependence of the effective control on the slow state can be eliminated, is given in the closing section.

2. The variational limit

The ingredients of the variational limit problem are briefly displayed in this section, along with some technical assumptions that will be used throughout. We start up front with the description of the optimal control problem that we suggest to be the limit of (1.1) as $\varepsilon \to 0$. Only then the technical conditions under which the scheme works, and the main consequence of the construction, are displayed.

Recall some useful notions: A *probability measure* μ on a separable metric space M is a countably additive function from the Borel subsets of M into $[0, 1]$ with $\mu(M) = 1$. The *distribution* of a mapping $h(t) : [t_1, t_2] \to M$ is the probability measure $\mu(h(\cdot); t_1, t_2)$ which assigns to a Borel set B of M the value $\mu(h(\cdot); t_1, t_2)(B) = (t_2 - t_1)^{-1}\lambda(\{t : h(t) \in B\})$ where λ is the Lebesgue measure; namely, $\mu(h(\cdot); t_1, t_2)(B)$ is the proportion of time within the interval in which $h(\cdot)$ takes values in B. The *weak convergence of measures* makes the family of probability measures on M into a metric space that we denote by $\mathcal{P}(M)$; a specific metric, which induces the convergence, is the Prohorov metric $\rho(\cdot,\cdot)$, namely, $\rho(\mu,\nu)$ is the smallest η such that $\mu(B) \leq \nu(B^\eta) + \eta$ and $\nu(B) \leq \mu(B^\eta) + \eta$ where B^η is the η-neighborhood of B in M. Let $h(t) : [t_1, \infty) \to M$; if the probability measures $\mu(h(\cdot); t_1, t_2)$ converge as $t_2 \to \infty$, in the weak convergence of measures, to μ_0, then the latter is called the *limit occupational measure* induced by $h(\cdot)$.

Consider now a control system

$$\frac{dz}{ds} = g(z, u), \tag{2.1}$$

defined for $s \in [0, \infty)$, with $z \in R^m$ and $u \in U$. An *admissible dynamics* of (2.1) is a pair $(z(\cdot), u(\cdot))$ of functions from an infinite time interval $[s_1, \infty)$ into $R^m \times U$, that solves the equation, namely, such that $\frac{dz}{ds}(s) = g(z(s), u(s))$ for almost every $s \in [s_1, \infty)$.

DEFINITION 2.1 *A probability measure μ_0 on $R^m \times U$ is called an invariant measure (or a limit occupational measure) of the control system (2.1), if it is the limit in $\mathcal{P}(R^m \times U)$, as $s_2 \to \infty$, of $\mu((z(\cdot), u(\cdot)); s_1, s_2)$, where $(z(\cdot), u(\cdot))$ is an admissible dynamics of (2.1). We say then that $(z(\cdot), u(\cdot))$ generates μ_0.*

Thus, invariant measures of (2.1) are limit occupational measures of admissible trajectories. As mentioned, the notion emerged from the studies reported in Artstein (1993, 2002a,b), Artstein and Gaitsgory (1997), Artstein and Vigodner (1996), and Vigodner (1997), see references therein. Such measures in the control theory framework have been characterized in Gaitsgory (2004), Gaitsgory and Leizarowitz (1999). The justification for the terminology "invariant measure" is that a limit occupational measure is, indeed, an invariant measure in a control-free dynamics; in the control setting a limit occupational measure is a projection of an invariant measure of the flow when the dynamics induced by (2.1) is lifted to the skew product dynamics of shifts; see Artstein (2004b). The invariance, indeed, plays a role in the analysis of such systems.

The following notation is standard; we display it since it plays a key role in the presentation.

NOTATION When $h(m) : M \to V$ is a mapping from a metric space M into a vector space V, and when μ is probability measure on M, we write $h(\mu)$ for $\int_M h(m)\mu(dm)$, namely, $h(\mu)$ is the average of $h(\cdot)$ with respect to μ. For instance, if μ is a probability measure over $R^m \times U$ then

$$f(x, \mu) = \int_{R^m \times U} f(x, z, u)\mu(dz \times du). \tag{2.2}$$

Such averages play a prime role in our analysis.

Now we have all the ingredients needed to formulate the variational control limit, as $\varepsilon \to 0$, of (1.1). Following the formulation we list some technical conditions and the main consequence of the construction.

Consider the control system derived from (1.1) and given as follows.

Problem 2.1

$$\begin{aligned} &\text{minimize} && C(x(b)) \\ &\text{subject to} && \frac{dx}{dt} = f(x,\mu) \\ &&& x(a) = x_0 \\ &&& \mu \in \mathcal{M}(x), \end{aligned} \tag{2.3}$$

with a state variable $x \in R^n$ and where for any given x the control set $\mathcal{M}(x)$ consists of the invariant measures (i.e., limit occupational measures) of the system

$$\frac{dz}{ds} = g(x,z,u) \tag{2.4}$$

with x fixed.

Notice that an admissible trajectory of (2.3) has the form $(x(\cdot),\mu(\cdot))$, namely, its second coordinate is measure-valued. We denote the optimal cost of Problem 2.1 by val(0).

Now we display the conditions under which our results are derived. In order not to blur the main message no attempt is done here to identify the most general conditions (some conditions can be easily lifted, relaxation of other, however, is a challenge). We say that a state z_1 can be steered (in regard to a given control problem) to z_2 within a time period $[0,\tau_0]$ if for some feasible control and initial condition $z(0)=z_1$ the trajectory satisfies $z(\tau)=z_2$ for some $\tau \le \tau_0$.

Assumption 2.1

1. The function $f(x,z,u)$ is continuously differentiable in the x-variable; both $f(x,z,u)$ and $g(x,z,u)$ are Lipschitz in (x,z) and continuous in u.
2. The constraint set U is compact.
3. The cost function $C(x)$ is continuously differentiable.
4. Boundedness: There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$, and there exists a bounded set B in $R^n \times R^m$ such that whenever $(x_\varepsilon(\cdot), z_\varepsilon(\cdot), u_\varepsilon(\cdot))$ is admissible (namely, solve (1.1)) for $\varepsilon \le \varepsilon_0$ and the cost it induces, say c_ε, satisfies $|c_\varepsilon - \text{val}(\varepsilon)| \le \delta_0$, then $(x_\varepsilon(t), z_\varepsilon(t)) \in B$ for all t.
5. Controllability: Within the set B specified in the previous assumption, the fast system is totally controllable uniformly in x, namely, there exists a bound τ_0 such that whenever (x,z_1) and (x,z_2) are in B, the state z_1 can be steered to z_2 within the time interval $[0,\tau_0]$, this with respect to the fast equation with x fixed.
6. Lipschitz Continuity: Denote by $\mathcal{M}_B(x)$ those invariant measures in $\mathcal{M}(x)$ that are generated by an admissible trajectory $(x, z(\cdot), u(\cdot))$ of (2.4) with x fixed, that is included in the set B. Let $F(x) = \{f(x,\mu) : \mu \in \mathcal{M}_B(x)\}$. Then, $F(x)$ is a Lipschitz set-valued map with respect to the Hausdorff metric.

REMARK 2.1 It is easy to see that the controllability assumption, item (5) of Assumption 2.4, implies that for any $x \in B$ the set $\mathcal{M}_B(x)$ defined in item (6) of the assumption is convex and compact in the space of probability measures; hence the set $F(x)$ is compact and convex in R^n.

In order to verify that Problem 2.1 is a variational limit as $\varepsilon \to 0$ for Problem 1.1, we need to understand in what sense admissible trajectories $(x_\varepsilon(\cdot), z_\varepsilon(\cdot), u_\varepsilon(\cdot))$ of (1.1) converge to an admissible trajectory $(x_0(\cdot), \mu_0(\cdot))$ of (2.3). The convergence we consider for the x-component is the uniform convergence of functions on $[a, b]$. The convergence in the measure-valued coordinates is taken in the sense of Young measures (similar to the convergence in the sense of relaxed controls). Namely, $(z_j(\cdot), u_j(\cdot))$ converge to $\mu(\cdot)$ if for every continuous and bounded function $h(z, u) : R^m \times U \to R$ the integrals $\int_a^b h(z_j(t), u_j(t))dt$ converge to $\int_a^b h(z, \mu)dt$ (for elaborations see, e.g., Artstein, 1993, 2002a).

We are ready to state the key result on the desired variational limit. The following standard terminology is used: The family of feasible trajectories $(x_\varepsilon(\cdot), z_\varepsilon(\cdot), u_\varepsilon(\cdot))$ of (1.1) is *near optimal* if the costs they generate, say c_ε, satisfy $c_\varepsilon - \mathrm{val}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

THEOREM 2.1 *Under Assumption 2.4 the optimal values val(ε) converge to val(0) as $\varepsilon \to 0$. Furthermore, any sequence of near optimal feasible trajectories $(x_{\varepsilon_j}(\cdot), z_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot))$ of (1.1), with $\varepsilon_j \to 0$, has a subsequence which converges to a feasible optimal trajectory $(x_0(\cdot), \mu_0(\cdot))$ of (2.3), and for any feasible optimal trajectory $(x_0(\cdot), \mu_0(\cdot))$ of (2.3) there are feasible near optimal trajectories $(x_\varepsilon(\cdot), z_\varepsilon(\cdot), u_\varepsilon(\cdot))$ of (1.1) that converge to it.*

Proof. Variants of the result, under similar conditions, were established in various places; see, e.g., Artstein and Gaitsgory (2000) and Artstein (2004a). We therefore present here only a telegraphic overview. From the assumptions we can deduce that the trajectories $(x_{\varepsilon_j}(t), z_{\varepsilon_j}(t), u_{\varepsilon_j}(t))$ are in the compact set $B \times U$. Compactness of the convergence in the sense of Young measures implies that a convergence of a subsequence exists, and boundedness of $f(\cdot,\cdot,\cdot)$ on $B \times U$ implies that the x-coordinate converges uniformly. Denote the limit by $(x_0(\cdot), \mu_0(\cdot))$. On a small t-interval the slow variable hardly changes, hence, when ε is small, the dynamics of (1.1) on a small interval is very close to the dynamics of (2.4) with a fixed x; in the limit, as $\varepsilon \to \infty$, the measure $\mu_0(t)$ is, for almost every t, an invariant measure in $\mathcal{M}(x_0(t))$ and, clearly, belongs to $\mathcal{M}_B(x_0(t))$. The definition of convergence and the continuity of the cost function imply that the cost induced by the limit measure is the limit of the respective costs. This establishes the first claimed convergence and that $\mathrm{val}(0) \le \liminf \mathrm{val}(\varepsilon)$. To verify the converse inequality and the second approximation claim consider an admissible trajectory $(x_0(\cdot), \mu_0(\cdot))$ of (2.3). By the definition of invariant measures the measure $\mu_0(t)$, for a given t, can be approximated by the distribution of an admissible trajectory of the fast equation (2.4) with $x = x_0(t)$. For ε small this trajectory can be considered on a small interval around t, on which the slow variable x changes only a little. The Lipschitz condition on $F(\cdot)$ allows, with the aid of the total controllability assumption, to compose out of these local approximations an approximation on the entire interval $[a, b]$ of the Young measure $\mu_0(\cdot)$. The definition of convergence and the continuity of the cost function imply then that the cost induced by the approximation is close to the one induced by the measure-valued control. This implies both the second approximation claim and that $\text{val}(0) \geq \limsup \text{val}(\varepsilon)$; this completes the proof. ∎

REMARK 2.2 The condition that $F(x)$ is a Lipschitz set-valued map cannot be removed or, say, replaced by continuity. A nice counterexample was given in Alvarez and Bardi (2009), see a variant of it in Artstein (2004a).

REMARK 2.3 The proof of Theorem 2.1 indicates how to employ an optimal solution of the variational limit Problem 2.1 in order to generate near optimal solutions to the perturbed systems (1.1). We refer to such possible constructions in the sequel, in the context of the necessary conditions.

REMARK 2.4 From the proof of Theorem 2.1 it is clear why we do not introduce an end condition into Problem 1.1. Indeed, with an end condition only the convergence in the first part of the proof is justified; in particular, the approximation of an optimal solution to the limit problem may not satisfy the end condition.

3. Pontryagin principle for the limit problem

In this section we display the differential inclusions version of the Pontryagin Maximum Principle for Problem 2.1. The classical version under, however, an additional condition is displayed in the next section. Relations with the corresponding principle of the perturbed system are explored in Section 5.

The reason we resort to the differential inclusions version of the Pontryagin maximum principle is that the control set in Problem 2.1 is inherently state-dependent. To this end consider the system

PROBLEM 3.1

$$\begin{aligned} &\text{minimize} && C(x(b)) \\ &\text{subject to} && \frac{dx}{dt} \in F(x) \\ &&& x(a) = x_0, \end{aligned} \tag{3.1}$$

with $F(x)$ given in item 6 of Assumption 2.1.

With the formulation of the previous control problem and the framework of the maximum principle for differential inclusions, a Pontryagin maximum principle for the variational limit Problem 2.1 can be formulated. The differential inclusion in (3.1) coincides with the one suggested by Gaitsgory (1992) and used by Gaitsgory and Grammel (1997) to produce a Pontryagin Principle for such systems. In our case the parameterization of the right hand side of the inclusion by the invariant measures will play a major role. In regard to a maximum principle for differential inclusions see Clarke (2005a,b) for an overview of classical and for the recent developments of the theory; see also Vinter (2000) for Hamiltonian formulations. Our assumptions, however, allow us to resort to more classical results as follows; recall that these assumptions were introduced in order to assure that Problem 2.1 is indeed a variational limit of Problem 1.1. Here and throughout an expression, e.g. pv, denotes the scalar product of p and v and ∂ denotes the generalized gradient, possibly the gradient if it exists.

THEOREM 3.1 *Under Assumption 2.1 let $(x^*(\cdot), \mu^*(\cdot))$ be an optimal trajectory for Problem 2.1. Then an n-dimensional vector function $p(\cdot)$ exists, that is absolutely continuous and satisfies almost everywhere the inclusions*

$$\begin{aligned} \frac{d}{dt}p(t) &\in -\partial_x H(x^*(t), p(t)) \\ \frac{d}{dt}x^*(t) &\in \partial_p H(x^*(t), p(t)) \end{aligned} \tag{3.1}$$

where the Hamiltonian is defined by

$$H(x,p) = \max\{pv : v \in F(x)\}; \tag{3.2}$$

furthermore, the transversality condition $-p(b) = \lambda \partial C(x^*(b))$ is satisfied for some $\lambda \geq 0$.

Proof. In terms of an optimal trajectory $x^*(\cdot)$ of Problem 3.1 the result follows from Loewen and Vinter (1987) together with the derivations on differential inclusions displayed in Clarke (1983, Chapter 3). Indeed, it is easy to see that our assumptions imply that the conditions for the Pontryagin principle as stated in Loewen and Vinter (1987) (see also Vinter, 2000, Chapter 7) are valid within the framework of Problem 3.1. The identification of an optimal solution of Problem 3.1 with a solution of the form $(x^*(\cdot), \mu^*(\cdot))$ of Problem 2.1, namely, the translation of $v \in F(x)$ into $f(x, \mu)$ for $\mu \in \mathcal{M}(x)$, follows from standard, classical, selection arguments. This completes the proof. ∎

The second inclusion in (3.1) amounts to

$$f(x^*(t), \mu^*(t)) \in \partial_p H(x^*(t), p(t)), \tag{3.3}$$

namely, reflecting the maximum principle within the necessary condition. This can be characterized in terms of optimization of a semi-infinite control problem. We provide such a characterization in Proposition 4.1 below, in the context of the Pontryagin principle under an additional condition.

4. Pontryagin principle for the limit problem in a special case

We present in this section the Pontryagin maximum principle for Problem 2.1 under the following additional condition.

ASSUMPTION 4.1 The control sets $\mathcal{M}_B(x)$ in Problem 2.3 do not vary with x; we denote the common set by $\mathcal{M}_B$.

REMARK 4.1 The previous assumption need not be satisfied in general. It is satisfied, for instance, when the function $g(x, z, u)$ in (1.1) does not depend on x (this situation is, in fact, frequently examined in the literature). In many cases, however, it is possible to re-parameterize the optimal control problem and make the control set independent of the state in a way that the Pontryagin principle for state-independent control sets can be applied. One such example, within the context of the present paper, is displayed in the closing section.

THEOREM 4.1 *Under Assumptions 2.1 and 4.1 let* $(x^*(\cdot), \mu^*(\cdot))$ *be an optimal trajectory for Problem 2.1. Then, an* n*-vector function* $p(\cdot)$ *exists satisfying the equation*

$$\frac{d}{dt}p(t) = -p(t)D_x f(x^*(t), \mu^*(t)) \tag{4.1}$$

and such that for almost every t *the expression*

$$p(t)f(x^*(t), \mu) \leq p(t)f(x^*(t), \mu^*(t)) \tag{4.2}$$

is satisfied for all invariant measures $\mu \in \mathcal{M}_B$*, and the transversality condition* $-p(b) = \lambda \partial C(x^*(b))$ *is satisfied for some* $\lambda \geq 0$ *(here* D_x *is the Jacobian operator and* ∂ *is the gradient).*

Proof. The conditions fit the standard Pontryagin maximum principle, see Clarke (2005b). ■

As in the standard manner the Pontryagin principle is applied, the maximization in (4.2) (which corresponds to (3.3)) may be employed in the computations. It is of interest then to formulate it in terms of the original data (rather than in terms of invariant measures), as follows:

PROPOSITION 4.1 *Within the statement of Theorem 4.1, suppose that* $\mu^*(t)$ *is generated by an admissible trajectory* $(x^*(t), z(\cdot), v(\cdot))$ *of (2.4); then* $(z(\cdot), v(\cdot))$ *is a solution to the optimal control problem*

$$\textit{maximize} \lim_{S\to\infty} \frac{1}{S}\int_0^S p(t)f(x^*(t), z, u)ds \tag{4.3}$$

subject to $(x^*(t), z(s), u(s))$ *being a feasible trajectory of*

$$\frac{dz}{ds} = g(x^*(t), z, u) \tag{4.4}$$

with $(x^*(t), z(s))$ *(here* $x^*(t)$ *is fixed) constrained to belong to* B. *Conversely, if* $(z(\cdot), u(\cdot))$ *solves (4.3)-(4.4) and generates a measure* μ^* *then the latter satisfies (4.2).*

Proof. The necessity follows easily from the fact that if a measure μ is the limit occupational measure of a mapping $(z(\cdot), u(\cdot)) : [t_1, \infty) \to R^m \times U$ then for a fixed p the weak convergence of measures implies that

$$pf(x, \mu) = \lim_{S \to \infty} \frac{1}{S} \int_0^S pf(x, z(s), u(s)) ds. \tag{4.5}$$

Conversely, suppose that an admissible pair $(z(\cdot), u(\cdot))$ that solves (4.3)-(4.4) generates μ. If μ does not satisfy (4.2) then a measure that establishes the contradiction to (4.2) is generated by an admissible pair that contradicts the assumption that $(z(\cdot), u(\cdot))$ solves (4.3). This completes the proof. ■

Discussion 4.1 The preceding result allows for employing the maximum principle in the generation of near optimal solutions to the perturbed problem with small ε, mimicking the manner the Pontryagin maximum principle is employed in computational schemes. Indeed, the end condition $p(b)$ for the adjoint equation (4.1) can, in many cases, be identified via the transversality condition (e.g., when there is only an integral cost that is augmented into the slow state by an additional coordinate, say, x_1, then the support vector of a global minimum would be $p(b) = (-1, 0, \ldots, 0)$). Then one solves (4.3)-(4.4) for the end point $p(b)$. The solution $(z(\cdot), u(\cdot))$ may not generate an invariant measure, but with the aid of the controllability guaranteed in Assumption 2.1, a solution that generates an invariant measure can be constructed. Then, in successive steps, backward values, say $x((b - \Delta)$ and $p(b - \Delta)$ can be computed when plugging the invariant measure into the differential equations in (2.3) and (4.1) respectively, and an estimate for the invariant measure at $t - \Delta$ can be constructed, etc. The combination of slow solution and the computed fast solution is an approximation to a candidate optimal solution for ε small.

5. Pontryagin principle for the perturbed problem

We relate in this section the maximum principle of the variational limit as displayed in Theorem 4.1, to the maximum principle of the perturbed system (1.1). The latter reads as follows.

Theorem 5.1 *In addition to Assumption 2.1 suppose that both* $f(x, z, u)$ *and* $g(x, z, u)$ *are continuously differentiable in* (x, z). *Let* $(x^*_\varepsilon(\cdot), z^*_\varepsilon(\cdot), u^*_\varepsilon(\cdot))$ *be an* *optimal trajectory of Problem 1.1. Then an $(n+m)$-vector function $(p(\cdot), q(\cdot))$ exists, satisfying the equations*

$$\begin{aligned} \frac{d}{dt}p(t) &= -p(t)D_x f(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t)) - q(t)\frac{1}{\varepsilon}D_x g(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t) \\ \frac{d}{dt}q(t) &= -p(t)D_z f(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t)) - q(t)\frac{1}{\varepsilon}D_z g(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t)) \end{aligned} \tag{5.1}$$

and such that for almost every t the control $u^*_\varepsilon(t)$ maximizes the expression

$$p(t)f(x^*_\varepsilon(t), z^*_\varepsilon(t), u) + q(t)\frac{1}{\varepsilon}g(x^*_\varepsilon(t), z^*_\varepsilon(t), u) \tag{5.2}$$

for all control elements $u \in U$; in addition, the transversality condition $-p(b) = \lambda\partial C(x^*(b))$ is satisfied for some $\lambda \geq 0$.

Proof. The conditions fit the standard smooth version of the Pontryagin maximum principle; the displayed condition is the form it takes in our case. ∎

A simple inspection reveals that the Pontryagin principle for the perturbed equation with small ε utilizes a stiff system that is hard to examine and to compute. We may, however, inquire about the limit as $\varepsilon \to 0$ of these equations. Since the q-coordinates are dual variables for the fast state variable and since the latter may exhibit rapid oscillations, it is unlikely that a meaningful limit of these variables can be detected. A limit of the slow-state adjoint variables exists under some conditions as follows.

THEOREM 5.2 *Within the setting of Theorem 5.1 let $(x^*_\varepsilon(\cdot), z^*_\varepsilon(\cdot), u^*_\varepsilon(\cdot))$ be optimal solutions to Problem 1.1 that converge as $\varepsilon \to 0$ to a solution $(x^*(\cdot), \mu^*(\cdot))$ of Problem 2.1. Let $(p_\varepsilon(\cdot), q_\varepsilon(\cdot))$ and $p_0(\cdot)$ be the respective solutions of (5.1)-(5.2) and (4.1)-(4.2) with a common end condition $p(b)$. Suppose further that the expression $q_\varepsilon(t)\frac{1}{\varepsilon}D_x g(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t))$ converges weakly (say in the space L_2) to zero. Then, $p_\varepsilon(t)$ converges uniformly on $[a,b]$ to $p_0(t)$.*

Proof. The uniform convergence of $x^*_\varepsilon(\cdot)$ to $x^*(\cdot)$ together with the convergence in the sense of Young measures of $(z^*_\varepsilon(\cdot), u^*_\varepsilon(\cdot))$ to $\mu^*(\cdot)$ and the continuity of $D_x f(x,z,u)$ in x imply that the time-varying linear coefficient of the p-equation in (5.1), namely, $D_x f(x^*_\varepsilon(\cdot), z^*_\varepsilon(\cdot), u^*_\varepsilon(\cdot))$, converges weakly to the corresponding coefficient on (4.1), namely $D_x f(x^*(\cdot), \mu^*(\cdot))$. With the additional assumption of the weak convergence of the inhomogeneous term to zero, the claim follows from a standard continuous dependence argument. ∎

REMARK 5.1 It is not clear what general conditions would imply the weak convergence to zero of $q_\varepsilon(t)\frac{1}{\varepsilon}D_x g(x^*_\varepsilon(t), z^*_\varepsilon(t), u^*_\varepsilon(t))$. A simple example, that was already mentioned earlier in connection with the Pontryagin principle (see Remark 4.1), is that $g(x,z,u)$ is independent of x.

REMARK 5.2 Asen Dontchev has pointed out to me that when the Kokotovic framework is applicable, the convergence of the slow adjoint variable is guaranteed, however, to a limit different than the one proposed in this paper. Suppose we know that the optimal solutions $(x_\varepsilon^*(t), z_\varepsilon^*(t), u_\varepsilon^*(t))$ of (1.1) converge uniformly to a solution $(x_0^*(t), z_0^*(t), u_0^*(t))$ of the Kokotovic limit problem (namely, the differential-algebraic system obtained when $\varepsilon = 0$ is used in (1.1)); further, we suppose that $u_0^*(t)$ is such that the fast dynamics in (1.1) when $x_0^*(t)$ and $u_0^*(t)$ are frozen, is exponentially stable (the asymptotic stability of this fast dynamics is a pivotal property needed in the Tikhonov-Kokotovic scheme). This is reflected in the property that $D_z g(x_0^*(t), z_0^*(t), u_0^*(t))$ is a stable matrix (i.e., all its eigenvalues have negative real part). We apply now to the necessary conditions (5.1) a change of variables $\varepsilon r(t) = q(t)$ and $\tau = -t$ (the latter means that we solve the equation starting from the end point). Then, (5.1) is transformed into

$$\begin{aligned} \frac{d}{d\tau}p(\tau) &= p(\tau)D_x f(x_\varepsilon^*(\tau), z_\varepsilon^*(\tau), u_\varepsilon^*(\tau)) + r(\tau)D_x g(x_\varepsilon^*(\tau), z_\varepsilon^*(\tau), u_\varepsilon^*(\tau)) \\ \varepsilon\frac{d}{d\tau}r(\tau) &= p(\tau)D_z f(x_\varepsilon^*(\tau), z_\varepsilon^*(\tau), u_\varepsilon^*(\tau)) + r(\tau)D_z g(x_\varepsilon^*(\tau), z_\varepsilon^*(\tau), u_\varepsilon^*(\tau)). \end{aligned} \tag{5.3}$$

The aforementioned stability of the coefficients implies that if the data $(x_\varepsilon^*(\tau), z_\varepsilon^*(\tau), u_\varepsilon^*(\tau))$ in (5.3) are fixed, the system becomes a classical Tikhonov system. In particular, with the fixed data and fixing the initial condition $r(b)$, the solutions converge as $\varepsilon \to 0$ to the solution of the corresponding differential-algebraic equation (see Dontchev, 1983, Lemma 3.1, for explicit estimates for the convergence rate). The uniform convergence of the data to $(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau))$ together with the assumptions employed in Theorem 5.1 imply that the corresponding solutions converge to the solution of (5.3) when the data are replaced by $(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau))$, namely, the system

$$\begin{aligned} \frac{d}{d\tau}p(\tau) &= p(\tau)D_x f(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)) + r(\tau)D_x g(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)) \\ \varepsilon\frac{d}{d\tau}r(\tau) &= p(\tau)D_z f(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)) + r(\tau)D_z g(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)). \end{aligned} \tag{5.4}$$

Here the stationary limit, say $r_0(\tau)$, of the solutions $r_\varepsilon(\tau)$, is

$$r_0(\tau) = -p(\tau)D_z f(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau))(D_z g(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)))^{-1}. \tag{5.5}$$

Plugging the $r_0(\tau)$ in the p-equation of (5.4) gives rise to an equation for the limit of the slow adjoint variable, namely,

$$\begin{aligned} \frac{d}{d\tau}p(\tau) = p(\tau) \quad &(D_x f(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)) + \\ &D_z f(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau))(D_z g(x_0^*(\tau), z_0^*(\tau), u_0^*(\tau)))^{-1}). \end{aligned} \tag{5.6}$$

Notice that this limit equation differs from equation (4.1) exhibited in Theorem 4.1, which governs the adjoint variable of our variational limit; indeed, (4.1) is developed within a different framework. Also note that since $r_\varepsilon(\cdot)$ converges to a finite value, the fast adjoint variable $q(t)$ in the Kokotovic framework converges to zero.

6. An illustrative example

The purpose of this section is to demonstrate the form the abstract conditions take on a concrete example and examine the power of working with the limit. We also demonstrate the possibility to re-parameterize the system in order to arrive at a state independent control set. The example is a variant of an example previously suggested by Veliov (1996) and examined in Artstein (2002b).

Example 6.1

$$\begin{aligned} \text{minimize} \quad & \int_0^1 \quad -|z_1(t) - 2z_2(t)| \; dt \\ \text{subject to} \quad \frac{dx_1}{dt} & = u \\ \varepsilon\frac{dz_1}{dt} & = -z_1 + ux_1 \\ \varepsilon\frac{dz_2}{dt} & = -2z_2 + ux_1 \\ x_1(0) & = 1, \;\; z_1(0) = z_2(0) = 0, \end{aligned} \tag{6.1}$$

with x_1, z_1 and z_2 scalars, and $u \in [-1, 1]$. The problem can be set in the form (1.1) by augmenting it with the addition of a coordinate, say x_2, to the slow variable, that satisfies

$$\frac{dx_2}{dt} = -|z_1(t) - 2z_2(t)|, \quad x_2(0) = 0; \tag{6.2}$$

then replacing the cost in (6.1) by $C((x_1, x_2)) = x_2$.

Had x_1 been fixed, the problem solved in Artstein (2002b) could be modified as to cover the present version. In particular, the optimal solution of the limit problem involves an invariant measure supported on a periodic trajectory in the (z_1, z_2) space. The average control on this periodic solution is zero, hence the x_1 coordinate would not change. A direct inspection, however, reveals that the larger $|x_1|$ the better is the optimal value. Therefore, it may make sense to have a positive average of the control in order to increase this coordinate. Thus, there is a tradeoff between the advantage in increasing x_1 and the instantaneous contribution to the value. The Pontryagin Principle for the limit system reveals the right balance between the two.

As displayed, (6.1) does not satisfy Assumption 4.1; indeed, the invariant measures clearly depend on x_1. Consider, however, the change of variables

$$\begin{aligned} y_1 &= xz_1 \\ y_2 &= xz_2, \end{aligned} \tag{6.3}$$

this for, say, $x > 0$ (which is the domain we are interested in any way since $x_1(0) > 0$). With this change of variables (6.1) becomes, in its augmented version,

$$\begin{array}{lll} \text{minimize} & x_2 & \\ \text{subject to} & \dfrac{dx_1}{dt} & = u \\ & \dfrac{dx_2}{dt} & = -x_1|y_1(t) - 2y_2(t)| \\ & \varepsilon\dfrac{dy_1}{dt} & = -y_1 + u \\ & \varepsilon\dfrac{dy_2}{dt} & = -2y_2 + u \\ & x_1(0) & = 1,\ x_2(0) = 0,\ y_1(0) = y_2(0) = 0. \end{array} \tag{6.4}$$

The fast equation now is free of the slow variable and, in particular, the invariant measures of the fast system do not depend on the slow variable. It is also easy to see that all the other conditions needed for Theorem 4.1 are satisfied hence the theorem can be implemented. The resulting necessary conditions for the optimization of the variational limit are as follows.

The nature of the problem (in particular since x_2 does not appear in the right hand side of the state equation) yields that the second coordinate, namely $p_2(t)$, of the adjoint vector is fixed and can be set as -1 (compare, e.g., with Lee and Markus, 1967, Chapter 4). Given an optimal pair $(x^*(t), \mu^*(t))$, the first coordinate $p_1(t)$ of the support vector then satisfies (applying (4.1) with $p_2(t) = -1$)

$$\frac{d}{dt}p_1(t) = \int_{R^2} |y_1 - 2y_2|\ \mu^*(t)(dy_1, dy_2) \tag{6.5}$$

and with terminal condition $p_1(1) = 0$. In turn, the invariant measure $\mu^*(t)$ is generated by a solution to (see (4.3)-(4.4))

$$\text{maximize} \quad \lim_{S\to\infty} \frac{1}{S}\int_0^S (p_1(t)u(s) + x_1(t)|y_1(s) - 2y_2(s)|)\ ds \tag{6.6}$$

where $p_1(t)$ and $x_1(t)$ are fixed (recall that $x_1(t) > 0$) and $(u(s), y_1(s), y_2(s))$ solves the two fast equations, namely,

$$\begin{aligned} \frac{dy_1}{ds} &= -y_1 + u \\ \frac{dy_2}{ds} &= -2y_2 + u. \end{aligned} \tag{6.7}$$

A variant of the infinite horizon problem (6.6)-(6.7) was solved in Artstein (2002b), namely without the term $p_1(t)u(s)$ and with $x_1(t) = 1$ in the integrand. Much of the analysis in Artstein (2002b) is valid here as well. In particular, the fast dynamics are identical and the bang-bang arguments in Artstein (2002b, Sections 3) are valid; they imply that the optimal invariant measure for (6.6)-(6.7) is determined by two numbers, say η_1 and η_2, with $-1 < \eta_1 < \eta_2 < 1$ as follows. Consider the periodic solution of (6.7) that consists of two pieces: One that goes through the point $(\eta_2, \frac{1}{2}\eta_2)$ is R^2 with $u = -1$ and the second, that goes through the point $(\eta_1, \frac{1}{2}\eta_1)$ is R^2 with $u = 1$. The intersection of these two trajectories determine the switching points between the two bang-bang values $u = -1$ and $u = 1$ of the optimal solution. The two points η_1 and η_2 can now be determined by minimizing (6.6), which becomes now periodic (in Artstein, 2002b, the case with $p_1(t) = 0$, then $\eta_1 = \eta_2$, is solved explicitly). The optimal invariant measure and the optimal value depend, of course, on the adjoint variable $p_1(t)$ and the state $x_1(t)$. In turn, the optimal value determines in an obvious way the right hand side of the adjoint equation (6.5) for $p_1(\cdot)$. All in all, we get an explicit system for the adjoint equation and the optimal value, from which a candidate for an optimal solution (which is indeed the optimal solution, due to existence and uniqueness of solutions) of the variational limit can be determined (possibly numerically, which is beyond the scope of the present paper). Once η_1 and η_2 above are determined, a feedback solution to the problem can easily be constructed, see Artstein (2002b); this feedback solution is a near optimal solution to the perturbed equation with ε small.

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