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Variational formulation for incompressible Euler flow/ shape-morphing metric and geodesic*

by

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Abstract: Shape variational formulation for Euler flow has already been considered by the author in (1999a, 2007c). We develop here the control approach considering the convection (or mass transport) as the "state equation" while the speed vector field is the control and we introduce the *h-Sobolev curvature* which turns to be shape differentiable. The value function defines a new shape metric; we derive existence of geodesic for a p-pseudo metric, verifying the triangle property with a factor 2^{p-1}, for any $p > 1$. Any geodesic solves the Euler equation for incompressible fluids and, in dimension 3, is not curl free. The classical Euler equation for incompressible fluid (3), coupled with the convection (1) turns to have variational solutions when conditions are imposed on the convected tube ζ while no initial condition has to be imposed on the fluid speed V itself.

Keywords: Sobolev perimeter, Euler flow, morphic metric, geodesic, topological change.

1. Shape and set metrics

The shape optimization and moving domain analysis has been concerned in the last 30 years with topologies and metrics on families of sets. For a bounded open set D in R^N the family of compact subsets $A \subset D$ is compact for the Hausdorff metric. This metric and associated compactness property are very useful for several topics, for example when moving the geometrical domain in a Laplace-Dirichlet boundary value problem. In *morphic* viewpoint the convergence of a sequence Ω_n in *complementary* Hausdorff topology is a very weak concept, it implies the compactivorous property, but if the boundaries $\partial\Omega_n$ have no extra constraints, very few shape functionals have nice l.s.c. property. In this direction we introduce (Delfour and Zolésio, 1994, 2001, 2005) the metrics associated with the oriented distance function b_Ω from which we recover several compactness results associated with the uniform cusp conditions and more general criteria (Delfour and Zolésio, 2007); the concept of geodesic is, nevertheless,

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very difficult. In this direction the image community produced some pragmatic tools, like the so-called *elastic metric* between two sets (which is not a metric) as the norm of the elastic continuous transformation T which would map one set on the other, say $d = ||T - I_d|| + ||T^{-1} - I_d||$. In order to derive a metric and a complete metric space we use the *Courant metric* developed by A.M. Micheletti (1972) for smooth domains and extended it in Delfour and Zolésio (2001). The metric is obtained as an infimum on all transformations T, which are decomposable in $T = T_1 o T_2 o T_3 o ... o T_K$, the infimum being taken on all T and all K. (Of course, we understand that the choice of some specific T, beside any mathematics considerations, would introduce a *prejudice* in the candidate to be a metric. The infimum eliminates any such prejudice in the sense that the resulting metric is just depending on the boundaries). This metric extend for families of submanifolds and geodesic theory can be applied using the Eulerian approach developed in Delfour and Zolésio (2001). In doing so we find that the Courant metric and compactness and geodesic can be directly formulated in the Eulerian framework . As far as we consider only families of measurable subsets in D, the transformation T is then *relaxed* by the convection problem (1) and as we then escape to any flow mapping we are able to enlarge the study to families of sets with possible different topologies. We replace the notion of transformation by connecting tubes and the geodesic will be an optimal tube, solution to a variational problem whose vector field V is solution to Euler equation. In doing so we also have a clean variational approach for the solution of the Euler equation, which seems to extend to the compressible situation.

2. Tube metric

We consider a bounded smooth domain $D \subset R^N$. We designate by χ_Ω (or ζ_Ω, or simply ζ) the characteristic function of a measurable subset $\Omega \subset D \subset R^N$. We consider an admissible family $\mathcal{B}_r$ of measurable subsets with given measure a (see 3.2). For any pair (Ω_0, Ω_1) in this family we consider the set of connecting tubes $\zeta(t,x) = \zeta(t,x)^2 \in C^0([0,1], L^1(D))$ such that $\zeta(i,x) = \zeta_{\Omega_i}(x)$, $i = 0,1$, and verifying $\forall t \in I$, $\int_D \zeta(t,x)dx = a$, where $I = [0,1]$ will designate the time interval (the final time could be any $\tau > 0$, then we choose $\tau = 1$). The *Eulerian approach* consists in considering the connecting tubes ζ as solutions to the weak convection (1) associated to a free divergence speed vector field V: being given $\Omega_i,\ i = 0,1$ subsets in $D \subset R^N$ with $meas(\Omega_i) = a > 0$,

$$\zeta^2 = \zeta,\ \ \frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,\ \ \zeta(i) = \chi_{\Omega_i},\ i = 1,2\ . \tag{1}$$

For any such V the problem (1) may have no solution or several solutions, so the *product space* tool (see Zolésio, 2007b) is to consider the closed *non convex* non empty connecting set:

$$\mathcal{T}(\Omega_0, \Omega_1) = \{(\zeta, V) \in C^0(\bar{I}, L^2(D)) \times L^2_{div},\ verifying\ (1)\ \} \tag{2}$$

where

$$L^2_{div} = \{V \in L^2(I, L^2(D, R^N)),\ \ divV = 0\ \}.$$

2.1. Variational solution for the Euler equation

We minimize a "Tube-Energy" cost functional $\mathcal{E}$, which includes an additive regularizing term, a surface tension like term, needed in a standard setting in order to make use of the parabolic compactness of tubes. In fact, we shall minimize several functionals in the following form, with respect to (ζ, V) in some subset $\mathcal{T}_r(\Omega_0, \Omega_1)$ of $\mathcal{T}(\Omega_0, \Omega_1)$:

$$J(\zeta, V) = \mathcal{E}(\zeta, V) + \sigma \int_0^1 P_{h,r}(\zeta(t))\, dt$$

where

$$\mathcal{E}(\zeta, V) = 1/2 \int_0^1 \int_D (\alpha\, \zeta(t,x) + \beta)\, |V(t,x)|^2 dxdt.$$

We derive existence result for the minimum and as necessary condition the extrema solve the Euler incompressible flow:

$$\alpha \geq 0,\ \beta > 0,\ \frac{\partial}{\partial t}(\,(\alpha\zeta + \beta)\, V) + D(\,(\alpha\zeta + \beta)\, V).V + \nabla \mathbf{P} = \sigma\ \mu_h. \qquad (3)$$

For this we introduce the Sobolev perimeter $P_{h,r}(\Omega_t)$, associated with $L^1(I, \mathcal{H}^r(D))$ norm of the tube $\zeta(t,x)$. The advantage of this Sobolev perimeter is that it turns to be differentiable under smooth transverse field perturbations ζ_s and enable us to define the Sobolev curvature for any domain in this new class of Sobolev sets, so that $\mu_{h,r}$ is the Sobolev curvature of the *interface* associated with the Sobolev perimeter $P_{h,r}(\zeta(t))$. These elements are introduced below. Notice that with the choice of the parameters $\alpha = 0$, $\beta = 1$, equation (3) is the classical Euler equation for incompressible fluids, with no initial (or final) conditions, but with the only condition that the solution V will convect ζ_{Ω_0} onto ζ_{Ω_1} at final time. The tube approach was introduced in Zolésio (2007a,b) for connecting two given domains, whose characteristic functions have some "Sobolev smoothness": $\zeta_i \in \mathcal{H}^r(D)$, for given r such that $0 < r < 1/2$ (this includes the usual finite perimeter sets).

2.2. The shape morphic metric

We shall consider several energy functionals $\mathcal{E}(V, \zeta)$ at (13) associated with several parameters p, $1 < p < \infty, ...$ The basic idea to derive a shape metric is to consider the value function with $p = 1$. Two main difficulties arise: for existence of godesics (i.e. compactness results) we need $p > 1$ so we shall deal with pseudo-metric and with complete pseudo metric space or simply metric space (without existence of geodesic). Also the perimeter term must be replaced by a time capacity term $\theta_{h,r}$ in order to obtain the first metric axiom. The candidate for the *morphic metric* is then in the form

$$\mathbf{d}(\mathbf{\Omega_0}, \mathbf{\Omega_1}) \;=\; \mathbf{INF}_{(\zeta,\mathbf{V})\in\mathcal{T}(\mathbf{\Omega_0},\mathbf{\Omega_1})} \;\mathcal{F}(\zeta, \mathbf{V})$$

where

$$\mathcal{F}(\zeta, V) = \mathcal{E}(\zeta, V) \,+\, \sigma\, \theta_{h,r}(\zeta).$$

3. Tube-variational principle

For a measurable subset $Q \subset I\times D \subset R^{N+1}$, we shall write ζ_Q for the characteristic function and denote by $\Omega_t,\ a.e.t \in I$, the measurable subset in D (defined up to a subset with zero measure) such that $\zeta_Q(t, .) = \chi_{\Omega_t}$. We say that Q is a tube when we have some continuity, $\zeta \in C^0(\bar{I}, L^1(D))$, more precisely we will consider *Eulerian* description for the tube and introduce a minimal regularity on the *speed* vector field V, $V \in L^p_{div}$ in order to ensure this continuity. This continuity enables us to consider connecting tubes: being given two measurable subsets in D, a tube Q *connects* Ω_0 and Ω_1 if we have $\zeta(i) = \chi_{\Omega_i},\ i = 0, 1$. We shall consider a framework such that the set $\mathcal{T}(\Omega_0, \Omega_1)$ is non empty. For both functionals J and F the optimal solution (ζ, V) will solve a classical Euler equation for incompressible fluid, which will not simplify to a Hamilton-Jacobi equation: the field V will not derive from a potential as its curl will not be zero. Indeed the new curvature term that we shall introduce in relation with the "$\mathcal{H}^r$-perimeter" term will lead to a generalized curvature term on the boundary of the connecting tube, which, in dimension $N = 3$, generates a curl term.

We adopt the convention that for $r = 0$ the space $\mathcal{H}^r(D)$ stands for the Banach space $BV(D)$, so that for $0 \le r < 1/2$, $\mathcal{H}^r(D) \subset L^1(D, R^N)$, with continuous and compact inclusion mapping. Notice that from the results of Luigi Ambrosio (2003), the convection problem is uniquely solved under $L^1(I, BV(D))$ like assumption on the field V. This extra regularity on the vector field V would reduce the set $\mathcal{T}(V, \Omega_0, \Omega_1)$ to a single element, but would imply some viscosity modeling (e.g. some Navier-Stokes like flow in the Eulerian perspective). Here we escape any renormalization benefit, so the solution ζ may be non unique but the regularity $\zeta = \zeta^2 \in L^1(0, 1, \mathcal{H}^r(D))$ will be derived from the variational principle itself (see, e.g., Zolésio, 2001, 2007a,b).

3.1. Speed vector fields

With $1 \le p < \infty$, we introduce

$$L^p_{div} = \{V \in L^p(I \times D, R^N) \;\; s.t.\;\; divV = 0, \;\; V.n_D = 0 \;\}.$$

Proposition 1 *Let $V \in L^p_{div}$ and $\zeta = \zeta^2 \in L^\infty(I \times D)$ be solution to*

$$\frac{\partial}{\partial t}\zeta + \nabla\zeta.V = 0,$$

then $\zeta \in C^0(I, L^1(D))$.

Proof. The convection equation implies that: $\zeta_t = div(\ -\zeta\, V\ ) \in W^{-1,1}(D))$, then

$$\zeta \in C^0(I, W^{-1,1}(D))$$

and as $\zeta = \zeta^2$, the $L^1(D)$ continuity derives from the following

Lemma 1 *Let $\zeta = \zeta^2 \in L^1(I \times D) \cap C^0(I, \mathcal{D}'(D))$, then $\zeta \in C^0(I, L^1(D))$.*

Proof. As

$$||\zeta(t+s) - \zeta(t)||_{L^1(D)} \ = \ ||\zeta(t+s) - \zeta(t)||^2_{L^2(D)}$$

then it is enough to show that $\zeta \in C^0(I, L^2(D))$. We begin by establishing the weak $L^2(D)$ continuity: for any element $f \in L^2(D)$ consider

$$\int_D (\zeta(t+s)(x) - \zeta(t)(x))\, f(x)dx = \int_D (\zeta(t+s,x) - \zeta(t,x)\, \phi(x)dx$$
$$+ \int_D (\zeta(t+s,x) - \zeta(t,x))\,(f(x) - \phi(x))dx.$$

Let there be given $r > 0$, by the choice of $\phi \in \mathcal{D}(D)$ (using here the density of $\mathcal{D}(D)$ in $L^2(D)$), we have

$$|\int_D (\zeta(t+s,x) - \zeta(t,x))\,(f(x) - \phi(x))dx| \ \le\ 2\int_D |f(x) - \phi(x)|dx \le r.$$

So, we derive the continuity for the weak $L^2(D)$ topology. To strong topology it suffices now to consider the continuity of the mapping

$$t \to \int_D |\zeta(t,x)|^2 dx = \int_D \zeta(t,x)dx = ((\zeta(t), 1))_{L^2(D)}.$$

This continuity property enables us to define the connecting concept: given two measurable subsets (defined up to a zero measure subset) $\Omega_i \subset D$, $meas(\Omega_i) = a,\ \ i = 0, 1$, we consider the family of connecting tubes

$$\mathcal{T}_r(\Omega_0, \Omega_1) = \{(\zeta, V) \in L^1(I, \mathcal{H}^r(D)) \times L^p_{div},\ \textit{verifying}\ (1),\ \zeta(i) = \chi_{\Omega_i},\ i = 0, 1\,\}.$$

3.2. Non empty family of connecting tubes

In order to handle non empty tubes, we consider a given measurable subset $\Omega \subset D$, $meas(\Omega) = a$, and its connected family

$$\mathcal{B}_r(\Omega) = \{\omega \subset D \; s.t. \; \exists(\zeta, V) \in C^0(\bar{I}, L^1(D)) \cap L^1(I, \mathcal{H}^r(D)) \times L^p_{div},$$
$$s.t. \; \zeta = \zeta^2, \;\; \zeta_t + \nabla\zeta.V = 0, \quad \zeta(0) = \chi_\Omega, \text{ and } \chi_\omega = \zeta(\mathbf{1}) \}.$$

It is important to notice that if $\Omega \in \mathcal{C}_r$, the continuously moving domain Ω_t such that $\chi_{\Omega_t} = \zeta(t,.)$ is in $\mathcal{C}_r$ for almost every t, but not necessarily for $t = 1$, so that the family $\mathcal{B}_r(\Omega)$ is not a subfamily of $\mathcal{C}_r$. Moreover, as $V \in L^p_{div}$, the moving connecting domain verifies $meas(\Omega_t) = \int_D \zeta(t,x)dx = a > 0 \; a.e.t$, and so it is not empty at $a.e.$ time instant.

By construction we have:

Theorem 1 *For any pair of sets $\Omega_i \in \mathcal{B}_r(\Omega)$, $i = 0, 1$, the connecting tube $\mathcal{T}_r(\Omega_0, \Omega_1)$ is non empty.*

Let $(\zeta^i, V^i) \in \mathcal{T}_r(\Omega, \Omega^i)$, then the piecewisely defined element

$$\begin{aligned}(\zeta(t), V(t)) &= (\zeta^0(1-2t), -2\,V^0(1-2t)), \quad 0 < t < 1/2,\\ &= (\zeta^1(2t-1), 2\,V^1(2t-1)), \quad 1/2 < t < 1\end{aligned} \tag{4}$$

is an element of $\mathcal{T}_r(\Omega_0, \Omega_1)$.

4. Subsets in D with bounded Sobolev perimeter

We consider families of measurable subsets in D with perimeter-like properties: let $r \in [0, 1/2[$ and denote by $\mathcal{C}_r$ the family of measurable subsets in D with given measure a, $0 < a < meas(D)$, defined as follows:

i) for $r = 0$, $\mathcal{C}_0 = \{\Omega \subset D \; s.t. \; \chi_\Omega \in BV(D), \;\; meas(\Omega) = a\}$

ii) for $0 < r < 1/2$, as we know (Delfour and Zolésio, 2001) that $\{\zeta = \zeta^2 \in BV(D)\} \subset \mathcal{H}^r(D)$, we can replace the space $BV(D)$ by $\mathcal{H}^r(D)$, then we set:

$$\mathcal{C}_r = \{\zeta = \zeta^2 \; \in \mathcal{H}^r(D), \; \int_D \zeta(x)dx = a\}.$$

Theorem 2 *For $0 < r < 1/2$, $\mathcal{C}_r$ is weakly closed in $\mathcal{H}^r(D)$ and any bounded part is relatively compact in $\mathcal{C}_{r'}$ for any r', $0 < r' < r < 1/2$.*

For $r = 0$, $\mathcal{C}_0$ is weakly closed in $BV(D)$ and any bounded part is relatively compact in $L^1(D)$.

For given $h > 0$ we introduce

$$|\Omega|_{loc(h,r)} = \int\!\!\int_{D\times D \cap \{|x-y|<h\}} (1 - \frac{|x-y|^2}{h^2}) \, \frac{|\zeta_\Omega(x) - \zeta_\Omega(y)|}{|x-y|^{N+2r}} \, dxdy \; \leq \; ||\zeta_\Omega||^2_{\mathcal{H}^r(D)}.$$

With $\Omega^c = D \setminus \Omega$ we get:

$$|\Omega|_{loc(h,r)} = 2 \int\int_{\Omega\times\Omega^c\cap\{|x-y|<h\}} (1 - \frac{|x-y|^2}{h^2}) \frac{1}{|x-y|^{N+2r}} \, dxdy.$$

4.1. Sobolev perimeter

In order to define the Sobolev perimeter, we first consider the smooth domain situation: if the boundary $\Gamma = \partial\Omega$ is a C^2 manifold, then with $j_z^3(x) = 1 + zH + z^2 K$ (where H and K are the mean and Gauss curvature of the surface Γ, we assume $N = 3$), we get

$$|\Omega|_{loc(h,r)} = 2\int_\Gamma (\int_{-h}^0 (j_z^N(x)\{\int_{B_h(x+T_z(x))\cap\Omega^c} \frac{(1-(|T_z(x)-y|^2/h^2)^+}{|T_z(x)-y|^{N+2r}} \, dy\})dz)d\Gamma(x).$$

Assuming now that h is small enough compared to the curvatures, we locally approximate in the ball $B_h(x)$ the piece of boundary $\Gamma \cap B(x + T_z(x))$ by a linear space. The term

$$m(h,x,z) = \int_{B_h(x+T_z(x))\cap\Omega^c} \frac{[\,1 - |T_z(x)-y|^2/h^2\,]^+}{|T_z(x)-y|^{N+2r}} dy$$

is no more depending on the point $x \in \Gamma$, so that we set

$$m(h,z) := \int_{B_h(0+T_z(0))\cap\Omega^c} \frac{[\,1 - ((z+y_2)^2 + y_1^2)/h^2\,]^+}{((z+y_2)^2 + y_1^2)^{N/2+r}} dy.$$

We set

$$M(h) = 2 \int_{-h}^0 m(h,z)dz.$$

Then we get

$$|\Omega|_{loc(h,r)} = M(h) \int_\Gamma d\Gamma(x) \; + o(h), \;\; h \to 0.$$

Necessarily, as $||\zeta_\Omega||_{\mathcal{H}^r(D)} < \infty$, this term has a finite limit but this limit is zero:

Proposition 2

$$|\Omega|_{loc(h,r)} \to 0, \;\; h \to 0.$$

Proof. With $E_h = \{|x-y| \le h\}$, $meas(E_h) \to 0$ and $\zeta_{E_h} F \le F$ with

$$F = \frac{|\zeta_\Omega(x) - \zeta_\Omega(y)|}{|x-y|^{N+2r}} \in L^1(D \times D).$$

4.2. Asymptotic analysis when $h \to 0$

Proposition 3 *For any r, $0 < r < 1/2$, there exists a constant $a(r)$ such that*

$$M(h)/h^{1-2r} = a(r) \ + o(1), \quad h \to 0. \tag{5}$$

Proof. For $N = 2$, we get:

$$m(h,z) = 2\int_0^{\sqrt{h^2-z^2}} du(\int_0^{\sqrt{h^2-u^2}} [1-((z+v)^2+u^2)/h^2]^+ \times ((z+v)^2+u^2)^{-(N/2+r)}dv)$$

$$M(h) = 2\int_{-h}^0 dz\{\int_0^{\sqrt{h^2-z^2}} du(\int_0^{\sqrt{h^2-u^2}} [1-((z+v)^2+u^2)/h^2]^+ \times ((z+v)^2+u^2)^{-(1+r)}dv)\}.$$

With $Z = 1/h\, z$, we get

$$M(h) = 2h\int_{-1}^0 dZ\{\int_0^{h\sqrt{1-Z^2}} du(\int_0^{\sqrt{h^2-u^2}} [1-((hZ+v)^2+u^2)/h^2]^+ \times ((hZ+v)^2+u^2)^{-(1+r)}dv)\}.$$

With $U = 1/h\ u$ we get

$$M(h) = 2h^2\int_{-1}^0 dZ\{\int_0^{\sqrt{1-Z^2}} dU(\int_0^{h\sqrt{1-U^2}} [1-((hZ+v)^2+h^2U^2)/h^2]^+ \times ((hZ+v)^2+h^2\,U^2)^{-(1+r)}dv)\}.$$

With $V = 1/h\ v$ we get

$$M(h) = 2h^{1-2r}\int_{-1}^0 dZ\{\int_0^{\sqrt{1-Z^2}} dU(\int_0^{\sqrt{1-U^2}} [1-((Z+V)^2+U^2)]^+ \times ((Z+V)^2+U^2)^{-(1+r)}dV)\}.$$

Notice that as $0 < r < 1/2$ we have $\mu = 1-2r > 0$ and we consider

$$M(h)/h^{1-2r} = a(r) \ + o(1), \tag{6}$$

where the main part $a(r)$, independent of h, $h \to 0$, is given by:

$$a(r) = \int_{-1}^0 dZ\{\int_0^{\sqrt{1-Z^2}} dU(\int_0^{\sqrt{1-U^2}} [1-((Z+V)^2+U^2)]^+ \times ((Z+V)^2+U^2)^{-(1+r)}dV)\}.$$

To get the perimeter, we set

$$P_{h,r}(\Omega) = \frac{1}{2a(r)\ h^{1-2r}} \quad |\Omega|_{loc(h,r)}. \tag{7}$$

Proposition 4 *For all r, $0 < r < 1/2$, and any open set $\Omega \subset D$ with C^2 boundary Γ, $\Gamma \subset \bar{D}$ the following asymptotic holds:*

$$P_{h,r}(\Omega) \to \int_{\Gamma \cap D} d\Gamma, \; h \to 0.$$

4.3. Perimeter estimate

For $r = 0$ we have

$$|\zeta_\Omega|_{BV(D)} = |\Omega| + |\nabla \zeta_\Omega|_{M^1(D,R^N)} \leq |D| + P_D(\Omega).$$

Let $0 < r < 1$, $h > 0$, consider $\rho_h(r) = (1 - r^2/h^2)^+$, so that

$$P_{h,r}(\zeta)(t) = \frac{1}{a(r)h^{1-2r}} \int\int_{\Omega_t \times \Omega_t^c} \frac{\rho_h(|x-y|)}{|x-y|^{N+2r}} \; dxdy.$$

and hence we have

Theorem 3 $\forall (r,p), \; 0 < r < 1/2,$

$$||\zeta_\Omega||^2_{H^r(D)}\, dt \leq \; |D| \, + (\sqrt{2}/h)^{N+2r} |D|^2 \, (\Omega) + \, a(r) h^{1-2r} \, P_{h,r}(\Omega). \tag{8}$$

Proof. Notice that

$$P_{h,r}(\zeta)(t) = \frac{1}{a(r)h^{1-2r}} \int\int_{D \times D} \rho_h(|x-y|) \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}} \; dxdy.$$

Moreover,

$$\begin{aligned} ||\zeta(t)||^2_{H^r(D)} &= |\Omega_t|^2 + \int\int_{D \times D} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}} \; dxdy \\ &\leq |D|^p + \int\int_{\{|x-y| > h/\sqrt{2}\}} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+2r}} \; dxdy \\ &+ \int\int_{\{|x-y| \leq h/\sqrt{2}\}} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^{N+Zr}} \; dxdy. \end{aligned}$$

As $\rho_h(r) > 1/2$ for $r < h/\sqrt{2}$ we get

$$\leq \; (\sqrt{2}/h)^{N+r} |D|^2 + \int_\Omega \int_{\Omega^c} \frac{\rho_h(|x-y|)}{|x-y|^{N+2r}} dxdy,$$

that is,

$$||\zeta(t)||^2_{H^r(D)} \leq (\sqrt{2}/h)^{N+r} |D|^2 + \, |\Omega|_{loc(h,r)}.$$

4.4. Sobolev curvature

When $\zeta \in BV(D)$, the perimeter in D is given by

$$P_D(\Omega) = ||\nabla\zeta||_{M^1(D,R^N)}.$$

For a given smooth vector field Z the perimeter $P(\Omega_s)$ of the perturbed domain $\Omega_s = T_s(Z)(\Omega)$ is not differentiable with respect to s. When the boundary Γ is a smooth manifold, then it is differentiable and we have:

$$\frac{\partial}{\partial s} P_D(\Omega_s)_{\{s=0\}} = \int_\Gamma \Delta b_\Omega < Z(0), n > d\Gamma,$$

where $H = \Delta b_\Omega$ is the mean curvature of Γ, so that H appears as the shape gradient of the perimeter (for smooth domains). The BV perimeter being not shape differentiable we introduce slight modification in the previous Sobolev perimeter for shape differentiability and to propose a h-Sobolev curvature. We first analyse the perimeter shape derivative; this term turns to be always differentiable with respect to the transverse perturbations as follows: let us consider some "small" parameter s (perturbation parameter) and any smooth vector field, $\mathcal{Z}(s,x)$, $\mathcal{Z} \in C^0([0,s_0[,\mathcal{D}(D,R^N))$ such that $div_x\mathcal{Z}(s,.) = 0$. As usual, we designate by $T_s(\mathcal{Z})$ its flow mapping and consider the Sobolev perimeter of the s-perturbed set:

$$P_{h,r}(T_s(\mathcal{Z})(\Omega)) = 2\int_{\Omega\times\Omega^c} \frac{[1 - |T_s(\mathcal{Z})(x) - T_s(\mathcal{Z})(y)|^2/h^2]^+}{||T_s(\mathcal{Z})(x) - T_s(\mathcal{Z})(y)||^{N+2r}}\, dxdy.$$

So that

$$\frac{\partial}{\partial s} P_{h,r}(T_s(\mathcal{Z})(\Omega)\,)_{s=0} = \tag{9}$$
$$-2(N+2r)\int_{\Omega\times\Omega^c} \frac{[1 - ||x-y||^2/h^2]^+}{||x-y||^{N+2r}} < \frac{x-y}{||x-y||}, \frac{Z(x)-Z(y)}{||x-y||} > dxdy$$
$$-2\int_{\Omega\times\Omega^c\cap\{|x-y|<h\}} \frac{1}{||x-y||^{N+2r}} < \frac{x-y}{h^2}, Z(x)-Z(y) > dxdy.$$

As $||x-y|| \leq h$ in the previous integrals we have:

$$Z(x) - Z(y) = DZ(x) + \delta(y-x)).(y-x),$$

and there exists a measure $\mu_h(\Gamma(t)$ supported by

$$\Delta_h(\Sigma) = \cup_{0<t<1}\{t\} \times \ (\cup_{x\in\partial\Omega_t} B(x,h)\ )$$

such that

$$< \mu_h,\, Z > = \frac{\partial}{\partial s} P_{h,r}(\zeta^s)_{s=0}.$$

When $h \to 0$, the measure converges to the mean curvature of the moving boundary Γ_t: indeed at (6) the convergence is uniform with respect to the family of smooth domains, whose curvature tensor is uniformely bounded.

5. Transverse field and perturbed tube

Transverse field action preserving tubes and transverse tube analysis has been developed in Zolésio (1998, 2001, 2002), Dziri and Zolésio (1999b, 2007), Moubachir and Zolésio (2006).

Let us consider a perturbation parameter $s \geq 0$ and any smooth horizontal non autonomeous vector field over R^{N+1} (s being the evolution parameter for a dynamic in R^{N+1})

$$\mathcal{Z}(s,\ t,x) = (0,\ z(s,\ t,x)) \in R_t \times\ R^N, \quad div_x z(s,t,.) = 0$$

such that

$$\mathcal{Z}(s,\ 0,x)\ =\ \mathcal{Z}(s,\ 1,x)\ =\ 0. \tag{10}$$

For any element $(\zeta, V) \in \mathcal{T}(\Omega_0, \Omega_1)$, we consider the perturbed tube $(\zeta^s,\ V^s)$ where

$$\begin{aligned} &\zeta^s(t,x) := \zeta o T_s(\ \mathcal{Z})(x))^{-1} \\ &V^s(t,x)\ =\ (D(T_s(\mathcal{Z})^{-1})^{-1}.(\ V(t) o T_s(\mathcal{Z})^{-1}\ -\ \frac{\partial}{\partial t}(T_s(\mathcal{Z})^{-1}\ )\ ). \end{aligned} \tag{11}$$

Notice that $D(T_s(\mathcal{Z})^{-1})^{-1} = D(T_s(\mathcal{Z})) o T_s(\mathcal{Z})^{-1}$.

From classical calculus (Sokolowski and Zolésio, 1991; Zolésio, 1992; DeSaint and Zolésio, 1997; Delfour and Zolésio, 2001; Kawohl et al., 1998) using the strong flow mapping $T_s(\mathcal{Z})$ we get the following stability result for the connecting family:

Theorem 4 *Let there be given* $z \in C^0([0,\ s_1] \times [0,\ 1], C^1(\bar{D}, R^N)),\ z(s,t).n = 0,$ *on* ∂D *and* Ω *a measurable subset in* D. *Consider any pair* $\Omega^i,\ \ i = 0,1$ *in* $\mathcal{B}(\Omega)$, *then, with* $\mathcal{Z} = (0, z)$, *we have:*

$\forall(\zeta, V) \in \mathcal{T}(\Omega_0, \Omega_1)$, *the elements* (ζ^s, V^s) *defined at (11) verify* $(\zeta^s, V^s) \in \mathcal{T}(\Omega_0, \Omega_1)$.

Remark 1 This stability property does not require the function ζ to be a characteristic function. This property still hold true for example for probability measures.

Remark 2 As $V \in H^p_0$, the moving domain verifies $meas(\Omega_t) = \int_D \zeta(t,x)dx = a$ and the s-perturbed moving domain Ω^s_t such that $\chi_{\Omega^s_t} = \zeta(t) o T_s(\mathcal{Z}(t))^{-1}$ (or equivalently $\Omega^s_t = T_s(\mathcal{Z}(t))(\Omega_t)$), verifies $meas(\Omega^s_t) = a > 0$ if $div_x z(s,t,.) = 0$ in D.

6. Tube energy

We shall make use of the following compactness result, see Zolésio (2002, 2007b), Moubachir and Zolésio (2006):

THEOREM 5 *Let $p > 1$ and $0 \leq r < 1/2$. Consider a sequence $\zeta_n \in C_r$, bounded in $L^1(I, \mathcal{H}^r(D))$ together with $\frac{\partial}{\partial t}\zeta_n$ bounded in $L^p(I, W^{-1,1}(D))$, then there exists a subsequence and an element*

$$\zeta \in C_r \cap L^1(I, B_r(D)) \cap W^{1,1}(I, W^{-1,1}(D)) \subset C^0(I, L^1(D))$$

such that ζ_n strongly converges to ζ in $L^1(I, L^1(D))$ with $\frac{\partial}{\partial t}\zeta \in L^p(I, M^1\ (D, R))$ verifying

$$||\zeta||_{L^1(I,\mathcal{H}^r(D))} \leq lim\,inf\ ||\zeta_n||_{L^1(I,\mathcal{H}^r(D))}$$

and

$$||\frac{\partial}{\partial t}\zeta||_{L^p(I,W^{-1,1}(D))} \leq lim\,inf\ ||\frac{\partial}{\partial t}\zeta_n||_{L^p(I,W^{-1,1}(D))}.$$

Moreover, if we define the "r-perimeters" as

$$P_0(\zeta)(t) := ||\nabla_x \zeta(t)||_{M^1(D,R^N)},$$

$$r > 0,\ P_{h,r}(\zeta)(t) = \int\int_{D\times D} \rho_h(|x-y|)|\zeta(x) - \zeta(y))|/|x-y|^{(N+2r)}\ \ dxdy$$

then $\zeta(t,x) = \zeta^2(t,x),\ a.e.(t,x) \in I \times D$ and $\zeta \in C^0(I,\ L^1(D))$ imply that the mapping

$$t \in \bar{I}\ \rightarrow\ P_{h,r}(\zeta)(t)\ \ is\ l.s.c. \tag{12}$$

6.1. Existence of minimizing tube

Being given $\alpha \geq 0$, $\beta > 0$, $\sigma > 0$, we consider the following Tube-Energy functional:

$$\mathcal{E}_r^p(\zeta, V) = 1/2 \int_0^1 \int_D (\alpha\,\zeta(t,x) + \beta)\,|V(t,x)|^p dxdt + \sigma \int_0^1 p_r(\zeta)(t)\,dt. \tag{13}$$

THEOREM 6 *Let $0 \leq r < 1$. For any $\Omega \in \mathcal{BP}$ and any pair of sets $\Omega_i \in \mathcal{B}(\Omega)$, $i = 0, 1$, the functional $\mathcal{E}_r^p$ reaches its minimum on $\mathcal{T}_r(\Omega_0, \Omega_1)$.*

Proof. We consider a minimizing sequence $(\zeta_n, V_n) \in \mathcal{T}(\Omega_0, \Omega_1)$. There exist subsequences such that $V_n \rightharpoonup V$, weakly in $L^p(I \times D)$ and $\zeta_n \to \zeta$ strongly in $L^1(I \times D)$. In fact, as $(\zeta_n)_t = div(\ -\zeta_n\ V_n\ )$, we have $p > 1$ and:

$$||\zeta_n||_{L^1(I,B_r(D))}\ \leq\ M_1, \quad ||(\zeta_n)_t||_{L^p(I,W^{-1,1}(D))}\ \leq\ M_2,$$

the conclusion derives from the compacity result. From this strong L^1 convergence we derive that $\zeta^2 = \zeta$. We consider the weak formulation for the convection problem (1):

$$\forall \psi \in C^1(I \times \bar{D},\ R^N),\ \psi(0,.) = 0,$$

$$\int_0^1 \int_D \zeta_n\,(\ -\psi_t\ - \nabla\psi.V_n\ )\,dxdt = -\int_{\Omega_1} \psi(0,x)dx$$

in which we can pass to the limit and we conclude that $(\zeta, V) \in \mathcal{T}(\Omega_0, \Omega_1)$. Moreover the element (ζ, V) is a classic minimizer as the two terms are weakly lower semi continuous, respectively, for each of the weak topologies, as we have

$$\int_0^1 \int_D \zeta(t,x)\, |V(t,x)|^p dxdt = \int_0^1 \int_D |\zeta(t,x)\, V(t,x)|^p dxdt$$

and

$\zeta_n V_n$ weakly converges in $L^p(I \times D)$ to ζV.

(Indeed, for any $\phi \in L^{p^*}(I \times D)$ we have $|\phi\,(\zeta_n - \zeta)|^{p^*} \leq 2^{p^*}\,|\phi|^{p^*} \in L^1(I \times D)$ while $\phi\,(\zeta_n(t,x) - \zeta(t,x)) \to 0,\ a.e.(t,x),$ so that $\phi\,\zeta_n \to \phi\,\zeta$ *strongly in* $L^{p^*}(I \times D)$). Now, as V_n weakly converges to V, we get

$$\int_0^1 \int_D \phi\ \ V_n\, \zeta_n\, dxdt \ \to\ \int_0^1 \int_D \phi\ \ V \zeta\, dxdt;$$

so that $V_n\, \zeta_n$ weakly converges in $L^p(I \times D)$ to $V\, \zeta$.

6.2. Euler equation solved by the minimizer

In order to analyse the necessary conditions associated with any minimizer of $\mathcal{E}^p$ over the set $\mathcal{T}(\Omega_0, \Omega_1)$ we introduce transverse transformations of the tube. Without any loss of generality and in order to simplify the calculus we consider here the specific quadratic situation:

Transverse derivative, quadratic case ($p = 2$)

Assume that $div_x \mathcal{Z}^t = 0$, then

$$\int_D (\alpha \zeta^s(t,x) + \beta)\ |V^s(t,x)|^2\, dx = \int_D (\alpha \zeta(t,x) + \beta)\ |V^s(t) o T_s(\mathcal{Z}^t)(x)|^2\, dx,$$

so that the optimality of the element (ζ, V) writes:

$$1/s\ (\ \mathcal{E}(\zeta^s,\, V^s o T_s) - \mathcal{E}(\zeta,\, V)\ )\ \geq\ 0.$$

Now the following quotient has a strong limit in $L^2(I \times D)$:

$$\begin{aligned}
&\frac{V^s o T_s - V}{s} = \frac{d}{ds}[V^s o T_s(\mathcal{Z}^t)\,]_{s=0}\\
&= \frac{d}{ds}[\,(D(T_s(\mathcal{Z}^t)^{-1})^{-1}.(\ V(t)\ -\ \frac{\partial}{\partial t}(T_s(\mathcal{Z}^t)^{-1})o T_s(\mathcal{Z}^t)\ \ )]_{s=0}\\
&= \frac{d}{ds}[\,(D(T_s(\mathcal{Z}^t) o T_s(\mathcal{Z}^t)^{-1}.(\ V(t)\ -\ \frac{\partial}{\partial t}(T_s(\mathcal{Z}^t)^{-1}\,)o T_s(\mathcal{Z}^t)\ \ )]_{s=0}\\
&= \frac{\partial}{\partial t} Z(t)\ +\ DZ(t).V(t)\ \in L^2(I \times D, R^N),
\end{aligned}$$

where we always denote $Z(t)(x) = Z(t,x) := \mathcal{Z}^t(0,x)$ (that is, at $s=0$). Indeed, we know that if V were smoother, say $V \in L^2(H^1(\Omega))$, we would have:

$$\frac{\partial}{\partial s}[V^s]_{s=0} = Z_t + [Z(t), V(t)] := H_V.Z,$$

where the Lie bracket is $[Z,V] = DZ.V - DV.Z$, so we would get the previous expression for the derivative of $V^s oT_s(\mathcal{Z}^t)$, as $(V^s oT_s)_s = (V^s)_s + DV^s.DZ(t)$. This analysis in strong form is used in the non cylindrical shape analysis (or dynamical domains analysis) in several previous works, see, e.g. Dziri and Zolésio (1999a,2007), Cannarsa, Da Prato and Zolésio (1990), Da Prato and Zolésio (1988a,b).

6.3. Necessary condition

Quadratic term $\mathcal{E}^2$ ($p=2$)

As

$$\begin{aligned}
&\int_0^1 \int_D ((\alpha\zeta^s + \beta)\,|V^s|^2 - (\alpha\zeta + \beta)|V|^2)/s\; dxdt \\
&= \int_0^1 \int_D ((\alpha\zeta + \beta)\,(\,|V^s oT_s|^2 - |V|^2)/s\; dxdt \\
&= \int_0^1 \int_D ((\alpha\zeta + \beta)\;(V^s oT_s + V)\,(V^s oT_s - V)/s\; dxdt \\
&\to 2\int_0^1 \int_D ((\alpha\zeta + \beta)\,V.\,(\frac{\partial}{\partial t}Z(t) + DZ(t).V(t)\,)\; dxdt \\
&= -2 < \frac{\partial}{\partial t}((\alpha\zeta + \beta)\,V\;) + \text{"}D(\,(\alpha\zeta + \beta)\,V).V\text{"}\;,\; Z\; >_{\mathcal{D}'\times\mathcal{D}}
\end{aligned}$$

where

$$\text{"}D(\,(\alpha\zeta + \beta)\,V).V\text{"}_i = \partial_j(\;(\alpha\zeta + \beta)\;V_i\,V_j\;) \in W^{-1,1}(D).$$

6.4. h-perimeter in $\mathcal{E}$

In the interesting case, where $\mathcal{H}^r(D) = H^r(D)$, we consider, for any given "small" $h > 0$ the $L^1(I)$ norm of the perimeter:

$$p_{h,r}(\zeta) := \int_0^1 (\int_{D\times D} \rho_h(||x-y||)\;\; \frac{|\,\zeta(x) - \zeta(y)\,|}{||x-y||^{N+2r}}\, dxdy)\, dt \tag{14}$$

so that is is enough to choose the surface tension term in the form $\sigma\, p_h(\zeta)$. This term turns to be always differentiable with respect to the transverse perturbations as follows:

$$
\begin{aligned}
& p_{h,r}(\zeta o T_s(\mathcal{Z})^{-1}) \\
= & \int_0^1 \int_{D\times D} \rho_h(||T_s(\mathcal{Z})(x) - T_s(\mathcal{Z})(y)||) \, \frac{|\zeta(x)-\zeta(y)|}{||T_s(\mathcal{Z})(x) - T_s(\mathcal{Z})(y)||^{N+2r}} \, dxdydt.
\end{aligned}
$$

So, for a.e. t in I we have

$$
\begin{aligned}
& \frac{\partial}{\partial s} p_{h,r}(\zeta^s(t))_{s=0} = \qquad (15) \\
& \int_{D\times D} \rho_h(||x-y||) \, \frac{|\zeta(x)-\zeta(y)|}{||x-y||^{N+2r}} < \frac{x-y}{||x-y||}, \frac{Z(t,x)-Z(t,y)}{||x-y||} > dxdy \\
& + \int_{D\times D} \rho_h'(||x-y||) \, \frac{|\zeta(x)-\zeta(y)|}{||x-y||^{N+2r}} < x-y, Z(t,x)-Z(t,y) > dxdy.
\end{aligned}
$$

As $||x-y|| \leq h$ in the previous integrals, we have:

$$
Z(t,x) - Z(t,y) = \; DZ(t,x) + \delta(t)(y-x)).(y-x)
$$

and there exists a measure $\mu_h(\Gamma(t)$ supported by

$$
\Delta_h(\Sigma) = \cup_{0<t<1} \{t\} \times \; (\cup_{x\in\partial\Omega_t} B(x,h) \;)
$$

such that

$$
< \mu_h, \, Z > = \frac{\partial}{\partial s} P_{h,r}(\zeta^s(t))_{s=0}.
$$

In some sense, when $h \to 0$, the measure converges to the mean curvature of the moving boundary Γ_t.

7. Variational solution to incompressible Euler-convection problem

We have the

THEOREM 7 *Let Ω be any given element in $\mathcal{B}$. Then any minimizer (ζ, V) to the functional $\mathcal{E}^2$ over the family of tubes $\mathcal{T}(\Omega_0, \Omega_1)$ solves the following problem:*

$$
\frac{\partial}{\partial t}\zeta \, + \, \nabla\zeta.V = \, 0, \;\; \zeta(0) = \chi_{\Omega_0}, \;\;\; \zeta(1) = \chi_{\Omega_1} \qquad (16)
$$

$$
divV = 0, \;\;\; \zeta = \zeta^2 \qquad (17)
$$

$$
\exists \mathbf{P} \; s.t. \; \frac{\partial}{\partial t}(\,(\alpha\zeta + \beta)\,V) \, + \, D(\,(\alpha\zeta+\beta)\,V).V + \nabla\mathbf{P} = \, \mu_h. \qquad (18)
$$

Note, see Zolésio (1999a), that equation (18) writes

$$(\alpha\zeta + \beta)\ (\frac{\partial}{\partial t}V\ +\ DV.V\ )\ +\ \nabla\mathbf{P} = 1/2\,\mu_h. \tag{19}$$

More generally, we have:

THEOREM 8 *Let Ω be any given element in $\mathcal{B}$. Then any minimizer (ζ, V) to the functional $\mathcal{E}^p$ over the family of tubes $\mathcal{T}(\Omega_0, \Omega_1)$ solves the following problem:*

$$\frac{\partial}{\partial t}\zeta\ +\ \nabla\zeta.V\ =\ 0,\ \ \zeta(0) = \chi_{\Omega_0},\ \ \ \zeta(1) = \chi_{\Omega_1} \tag{20}$$

$$divV = 0,\ \ \ \zeta = \zeta^2 \tag{21}$$

$$\exists\mathbf{P}\ s.t.\ \frac{\partial}{\partial t}((\alpha\zeta{+}\beta)||V||^{p-2}V) + D((\alpha\zeta + \beta)||V||^{p-2}V).V + \nabla\mathbf{P} = 1/p\mu_h. \tag{22}$$

We could also consider

$$\tilde{\mathcal{E}}^p(\zeta, V) = \int_0^1 \int_D ||V(t,x)||^p dxdt\ +\ \int_0^1 ||\zeta(t)||_{B(D)}dt$$

and we would have got the

THEOREM 9 *Let Ω be any given element in $\mathcal{B}$ and $p > 1$. Then any minimizer (ζ, V) to the functional $\tilde{\mathcal{E}}^p$ over the family of tubes $\mathcal{T}(\Omega_0, \Omega_1)$ solves the following problem:*

$$\frac{\partial}{\partial t}\zeta\ +\ \nabla\zeta.V\ =\ 0,\ \ \zeta(0) = \chi_{\Omega_0},\ \ \ \zeta(1) = \chi_{\Omega_1} \tag{23}$$

$$divV = 0,\ \ \ \zeta = \zeta^2 \tag{24}$$

$$\exists\mathbf{P}\ s.t.\ \frac{\partial}{\partial t}(||V||^{p-2}\,V)\ +\ D(||V||^{p-2}\,V).V + \nabla\mathbf{P} = 1/p\,\mu_h. \tag{25}$$

The Euler equation does not reduce to Hamilton-Jacobi equation for a scalar potential

It is an important point that the right hand side in the previous Euler flow equation is not curl free, so it does not derive from a potential, and the geodesic field V does not reduce to a gradient term as in an incompressible perfect fluid. Indeed, the support of $curlV$ is included in the boundary of the moving set Ω_t. In the very simple situation of $B(D) = BV(D)$ and Γ_t being a smooth surface we would get

$$< curl(\mu_h), Z >=< \mu_h, curlZ >= \int_0^1 \int_{\Gamma_t} H_t n_t.curlZ(t)\, d\Gamma_t(x)dt$$

$$= \int_0^1 \int_{\Gamma_t} H_t div_{\Gamma_t}(n_t \times Z(t)\,)\ d\Gamma_t(x)dt$$

$$= -\int_0^1 \int_{\Gamma_t} (\nabla_{\Gamma_t} H_t \times n_t).Z(t)\ \ d\Gamma_t(x)dt,$$

and γ_t being the trace operator on the manifold Γ_t:

$$curl\mu(t) = \gamma^*_{\Gamma_t}.(\nabla_{\Gamma_t} H_t \times n_t)$$

which is zero if and only if the surface Γ_t has a constant mean curvature. Still assuming the interface Γ_t to be a smooth manifold we would get the restrictions of V to the open domains Ω_t and Ω_t^c as gradients so that would be in the following form: $V = \chi_{\Omega_t}\nabla\phi_1(t) + (1-\chi_{\Omega_t})\nabla\phi_2(t)$.

8. Shape-morphing pseudo-metric on $\mathcal{B}(\Omega)$

The minimum of any of the previous energy terms cannot be a metric as it would violate the first axiom: indeed if the two domains are equal, $\Omega_1 = \Omega_2 = \Omega$, the term
$\sigma\int_0^1 p(\zeta(t))\ dt = ||\nabla\chi_\Omega||_{M^1(D,R^n)}$ or $\int\int_{\Omega\times\Omega^c} \rho_h(||x-y||)||x-y||^{-N-2r}dxdy$
is not zero.

The idea *would be* to consider the following expression for the shape metric:

$$\bar{d}^p(\Omega_0,\Omega_1) = \ INF_{\{\,(\zeta,V)\in\mathcal{T}_r(\Omega_0,\Omega_1)\}}$$
$$\int_0^1\int_D(\alpha+\beta\zeta)||V(t,x)||^p dxdt \ + \ \text{“}\int_0^1 |P_{h,r}(\zeta)'(t)|^p\ dt\text{''}. \tag{26}$$

Indeed, the last term is not finite in general as it would imply $P_{h,r}(\zeta)(t)$ to be time continuous, which is known to be false (the perimeter is *l.s.c.* as in the celebrate “*camembert entamé*” example: take a circular cheese camembert and subtract a radial triangular part with angle α, if the perimeter is $p(\alpha)$ then $p(0) = 2\pi < liminf_{\alpha\to 0}p(\alpha) = 2\pi\ +\ 2R$).

We relax this term by introducing (see Zolésio, 2007a) the “time capacity” term

$$\theta^p(\zeta) = INF_{\{\mu\in K^p(\zeta)\ \}}\ \int_0^1\ |\mu'(t)|^p\ dt \tag{27}$$

with the closed convex set

$$K^p(\zeta) = \{\ \mu\in W^{1,p}(I)\ s.t.\ ||\nabla_x\zeta(t)||_{M^1(D,R^N)} \leq\ \mu(t)\ a.e.t\in I\,\}. \tag{28}$$

Then the metric is

$$d^p(\Omega_0,\Omega_1) := INF_{\{(\zeta,V)\in\mathcal{T}^p(\Omega_0,\Omega_1)\}}\int_0^1\int_D(\alpha+\beta\zeta)||V(t,x)||^p dxdt + \theta^p(\zeta). \tag{29}$$

THEOREM 10 *For $p\geq 1$, $\sigma\geq 0$, $\alpha>0$, $\beta\geq 0$, d^p is a p-quasi metric on $\mathcal{B}(\Omega)$: $\forall(\Omega_0,\Omega_1,\Omega_1)\in B^p(\Omega)^3$,*

$$d^p(\Omega_0,\Omega_1) = 0\ \ \textit{iff}\ \ \Omega_0=\Omega_1,\quad d^p(\Omega_0,\Omega_2) = d^p(\Omega_0,\Omega_2)$$
$$d^p(\Omega_0,\Omega_2) \leq\ 2^{p-1}\,(\,d^p(\Omega_0,\Omega_1) + \,d^p(\Omega_1,\Omega_2)\ \ ).$$

Notice that with $p=1$, $\sigma\geq 0$, d^1 is a metric on $B(\Omega)$.

THEOREM 11 *Let $p > 1$, $\sigma > 0$, $\alpha > 0$, $\beta \geq 0$, then, equipped with d^p, the family $\mathcal{B}(\Omega)$ is a complete quasi-metric space.*

8.1. Metric with geodesic

In order to get differentiable energy term we first choose $0 < r < 1/2$ and $\mathcal{H}^r = H^r(D)$ but again we correct the energy term $p_h(\zeta)$ as we need to reach zero when the two sets Ω_i are equal. Again we relax this term by introducing:

$$\theta^p_{h,r}(\zeta) = INF_{\{\nu \in K^p_{h,r}(\zeta)\ \}} \int_0^1 |\nu'(t)|^p \, dt \tag{30}$$

with the closed convex set, with $r < 1$,

$$K^p_{h,r}(\zeta) = \{\nu \in W^{1,p}(I), \int\int_{D\times D} \rho_h(|x-y|) \frac{|\zeta(t,x)-\zeta(t,y)|}{|x-y|^{N+2r}} dxdy \\ \leq \nu(t) \ a.e.t \in I\}. \tag{31}$$

Then the metric is

$$d^p_{h,r}(\Omega_0,\Omega_1) := \ INF_{\{\ (\zeta,V)\in \mathcal{T}^p_h(\Omega_0,\Omega_1)\ \}} \int_0^1 \int_D (\alpha+\beta\zeta)||V(t,x)||^p dxdt \\ + \ \theta^p_{h,r}(\zeta), \tag{32}$$

and a similar definition holds for the family of connecting tubes $\mathcal{T}_r(\Omega_0,\Omega_1)$.

THEOREM 12 *For $p \geq 1$, $0 < r < 1/2$, $\sigma \geq 0$, $d^p_{h,r}$ is a p-quasi metric on $\mathcal{B}_r(\Omega)$: $\forall(\Omega_0,\Omega_1,\Omega_1) \in B^p(\Omega)^3$,*

$$d^p_{h,r}(\Omega_0,\Omega_1) = 0 \ \ iff \ \ \Omega_0 = \Omega_1, \quad d^{h,r}_p(\Omega_0,\Omega_2) = d^p_{h,r}(\Omega_0,\Omega_2) \\ d^p_{h,r}(\Omega_0,\Omega_2) \leq \ 2^{p-1}\,(\ d^p_{h,r}(\Omega_0,\Omega_1) + \ d^p_{h,r}(\Omega_1,\Omega_2)\ \ ).$$

Notice that with $p = 1$, $\sigma \geq 0$, $d^1_{h,r}$ is a metric on $\mathcal{B}(\Omega)$.

THEOREM 13 *Let $p > 1$, $\sigma > 0$, then equipped with $d^{p,r}_h$ the family $\mathcal{B}(\Omega)$ is a complete quasi-metric space. Moreover, the geodesic (ζ,V) between to elements Ω_i, $i = 0,1$ solves the following Euler problem for some "pressure" term $\mathbf{P}$ (where the measure μ_h is as defined previously)*

$$\frac{\partial}{\partial t}\zeta \ + \ \nabla\zeta.V \ = \ 0, \ \ \zeta(0) = \chi_{\Omega_0}, \ \ \ \zeta(1) = \chi_{\Omega_1} \tag{33}$$

$$divV = 0, \ \ \ \zeta = \zeta^2 \tag{34}$$

$$\frac{\partial}{\partial t}(\,(\alpha\zeta+\beta)||V||^{p-2}\,V) \ + \ D(\,(\alpha\zeta+\beta)||V||^{p-2}\,V).V + \nabla\mathbf{P} \\ = \ |\nu'(t)|^{p-2}\,\nu'(t) \ \ \mu_h. \tag{35}$$

Proof. We consider the derivative of the capacity term $\theta^p_{h,r}(\zeta^s)$, where $\zeta^s = \zeta(t)oT_s(Z)$. The idea is to use the derivative of a minimum in the form $f(s) = Min_{\{\lambda\in K\}}F(s,\lambda)$ where the set K is compact for some topology $\mathcal{T}$ and not depending on the parameter s while F is differentiable with derivatives l.s.c. with respect to $\mathcal{T}$, see Cuer and Zolésio (1988), Delfour and Zolésio (2001). To obtain this setting we write

$$\theta^p_{h,r}(\zeta^s) = Min_{\{\lambda\in K^p_{h,r}(\zeta)\}}\int_0^1 |\frac{\partial}{\partial t}(\lambda + a^s(t))|^p dt$$

where

$$a^s(t) = \int\int_{D\times D}(\frac{\rho_h(|T_s(x)-T_s(y)|}{|T_s(x)-T_s(y)|^{N+r}} - \frac{\rho_h(|x-y|}{|x-y|^{N+r}})\ \ |\zeta(t,x)-\zeta(t,y)|\ dxdy,$$

which turns out to be differentiable with respect to s, and at $s = 0$ we have

$$\frac{\partial}{\partial s}(||T_s(x)-T_s(y)||)_{s=0} =< \frac{x-y}{||x-y||},\ Z(x)-Z(y) >$$

so that we get

$$\dot{a}(t) := \frac{\partial}{\partial s}a^s(t)_{s=0} = \int\int_{D\times D}\{\ \nabla\rho_h(|x-y|) < \frac{x-y}{|x-y|}, Z(x)-Z(y) >$$
$$-\alpha\rho_h(|x-y|) < \frac{x-y}{|x-y|}, \frac{Z(x)-Z(y)}{|x-y|} > \ \ \}\ \ \frac{|\zeta(t,x)-\zeta(t,y)|}{|x-y|^{N+\alpha}}\ \ dxdy.$$

Then, we get:

$$\frac{\partial}{\partial s}\theta^p_{h,r}(\zeta^s)_{s=0} = \int_0^1 p|\nu'(t)|^{p-2}\,\nu'(t)\,\dot{a}(t)\ dt.$$

Now, the point is to prove the convergence of any Cauchy sequence such that $d^p_{h,r}(\Omega_p,\Omega_q) \to 0$ as $p,q\to\infty$. To begin with we obtain the existence of a minimizing element $(\bar{\zeta}_n,\bar{V}_n)$ in $\mathcal{T}_r(\Omega_0,\Omega_1)$.

As $p_{h,r}(\zeta_n(0)) = p_{h,r}(\zeta_{\Omega_0})$, we have $p_{h,r}(\zeta_n(.))$ bounded in $W^{1,p}(0,1)\subset C^0([0,1])$, then $\theta(\zeta_n)\le M_1$ and then from (8) we get $||\zeta_n||_{L^p(I,W^{r,p}(D))}\le M_2$ and the existence of the minimizing element derives as in the previous energy minimization, except for the specific care for the term

$$p_{h,r}(\zeta_n) = \int_0^1 |\frac{\partial}{\partial t}[\int_{D\times D}\rho_h(||x-y||)\ \ \frac{|\,\zeta_n(x)-\zeta_n(y)\,|}{||x-y||^{N+2r}}\,dxdy]|^p\,dt.$$

The analysis is as follows: $p_h(\zeta_n(.))$ weakly converges in $W^{1,p}(0,1)$ to some element $\mu\in W^{1,p}(0,1)$, ζ_n strongly converges in $L^p(I\times D)\cap C^0(\bar{I},L^p(D))$ to an element ζ. From Fatou Lemma (applied at each time t) we have $\theta(\zeta(t))\le\mu(t)$, but we wonder if $\mu = p_{h,r}(\zeta(.))$.

Let $\psi \in \mathcal{D}(0,1)$, then we have:

$$\int_0^1 \frac{\partial}{\partial t} p_h(\zeta_n(t))\psi dt = -\int_0^1 (\int_{D\times D} \rho_h(||x-y||)\frac{|\,\zeta_n(t,x)-\zeta_n(t,y)\,|}{||x-y||^{N+2r}}\,dxdy)\frac{\partial}{\partial t}\psi dt$$

$$= \int_0^1 ||\zeta_n(t)||^p_{W^{r,p}(D)}\frac{\partial}{\partial t}\psi dt$$

$$-\int_0^1 (1-\rho_h(||x-y||)\ )\ \frac{|\,\zeta_n(t,x)-\zeta_n(t,y)\,|}{||x-y||^{N+2r}}\,dxdy)\frac{\partial}{\partial t}\psi dt,$$

the second term converges (as $n \to \infty$) from the Lebesgue Theorem as the kernel is not singular and is dominated by a constant together with the pointwise convergence of subsequence of ζ_n. For any $\eta > 0$ we consider

$$\theta_{h,\eta}(\zeta(t)) = \int\int_{\{(x,y)\in D\times D,\ ||x-y|>\eta\}} \rho_h(||x-y||)\frac{|\zeta(t,x)-\zeta(t,y)|}{|x-y|^{N+2r}}dxdy$$

$$\zeta^n \to \zeta,\ p_h(\zeta^n) \to \mu,\ \sigma - W^{1,p}(0,1)$$

then $p_h(\zeta^n)$ is bounded in $L^p(I, W^{1,p}(0,1))$. This morphic metric can be handled numerically. In this direction we developed several Galerkin approaches based on level set parametrization for the moving domain (Zolésio, 2007a, 2009). In several experiments the geodesic turns out to be numerically stable (Blanchard and Zolésio, 2008, 2009; Toniolo and Zolésio, 2009).

9. Asymptotic analysis

An important issue is the asymptotic analysis when $\alpha + \beta \to 0$, (see Zolésio, 2009): for $p = 1$ the vector field just appears through the speed boundary element $v(t) =< V(t), n_t >$ on $\partial\Omega_t$ so that

$$||\frac{\partial}{\partial t}\zeta||_{L^1(I,M^1(D))} = \int_0^1 \int_{\Omega_t} |v(t,x)|\,d\Gamma_t(x)dt$$

and the metric takes the following intrinsic form: the Eulerian vector field is no more necessary (in the limit it would solve, formally, some eikonal equation). We simply consider the set of characteristic functions

$$\mathcal{C} = \{\ \zeta = \zeta^2 \in L^1(I\times D)\ \},\ \ \mathcal{C}^0 = \mathcal{C} \cap C^0(I, L^1(D)), \tag{36}$$

the family of *connecting tubes*

$$\mathcal{T}^0(\Omega_0,\Omega_1) = \{\zeta \in \mathcal{C}^0\ s.t.\ \zeta(i) = \chi_{\Omega_i},\ i = 0,1\ \ \}, \tag{37}$$

considering the Banach space of bounded measure $M^1(D)$ we set

$$p \geq 1,\ \ \mathcal{C}^p\ =\ \{\ \zeta \in \mathcal{C}\ s.t.\ \frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D))\ \} \tag{38}$$

that is

$$\mathcal{C}^p = \mathcal{C} \cap L^p(I, BV(D)) \subset C^0(I, L^1(D)) \tag{39}$$

$$p \geq 1,\ \mathcal{C}^p = \{\ \zeta \in \mathcal{C}^0\ s.t.\ \frac{\partial}{\partial t}\zeta \in L^p(I, M^1(D))\ \}. \tag{40}$$

COROLLARY 1 *Let $p \geq 1$, then*

$$d_p(\Omega_0, \Omega_1) = Inf_{\{\zeta \in \mathcal{C}^p,\, \zeta(i) = \chi_{\Omega_i}\ \}} \int_0^1 ||\frac{\partial}{\partial t}\zeta(t)||^p_{M^1(D)} dt \tag{41}$$

is a quasi metric. When $p = 1$, d_1 is a metric.

In level set representation, let $\Omega_i = \{x \in D,\ \phi_i(x) > 0\ \}$, then the moving domain Ω_t is sought in the form $\Omega_t = \{x \in D,\ \phi(t,x) > 0\ \}$ for some smooth function ϕ verifying the connection property: $\phi(i,x) = \phi_i(x),\ i = 1,2$, and it turns out that

$$||\frac{\partial}{\partial t}\zeta(t)||_{M^1(D)} = \int_{\{x \in D,\ \phi(t,x) = t\}} \frac{\partial}{\partial t}\phi(t,x)|\ ||\nabla\phi(t,x)||^{-1} d\Gamma_t(x).$$

Using an "ad hoc" Galerkin approximation we obtain geodesic connecting domains with different topologies.

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