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Hamiltonian trajectories and saddle points in mathematical economics*

by

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Abstract: Infinite-horizon problems of kinds that arise in macroeconomic applications present a challenge in optimal control which has only partially been met. Results from the theory of convex problems of Lagrange can be utilized, to some extent, the most interesting feature being that in these problems the analysis revolves about a rest point of the Hamiltonian, which is at the same time a saddle point of the Hamiltonian in the minimax sense. The prospect is that in this situation the Hamiltonian dynamical system exhibits saddle point behavior in the differential equation sense as well. Some results are provided in this direction and coordinated with notions of asymptotic optimization, which mathematical economists have worked with.

Keywords: infinite horizon optimal control, convex Lagrange problems, Hamiltonian saddle points, overtaking criteria, Ramsey's problem, economic growth.

1. Introduction

Engineering applications were the main stimulus for modern control theory, which developed rapidly in the period starting around 1960. Those applications dictated in many ways the emphasis of the field, for instance in commonly centering on a control set that is compact and independent of the state. In optimal control, important connections were soon recognized with long-standing theory in the calculus of variations. But that subject, despite the refinements it received in the Chicago school, was limited in its traditional setting. In fact, neither basic optimal control nor the calculus of variations responded well in format for applications in areas like operations research and economics, where, for example, control sets might depend on states.

The aim of this paper is to illustrate this in terms of a topic, clearly in the domain of optimal control and the calculus of variations, which economists have had to struggle with in the absence of results oriented toward their interests.

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This has to do with the behavior of optimal trajectories in macroeconomic models of growth over infinite time. The tools we apply are those coming from the "neo-classical" variational theory of the 1970s and beyond, which aimed to place optimal control in a framework resembling that of the calculus of variations while allowing for nonsmoothness and the enforcement of constraints by infinite penalties. That extension first emerged with the advent of convex analysis, its subgradients and dualizations, but eventually proceeded to much broader forms of nonsmooth analysis. Here, convexity will suffice.

The special focus will be on the trajectories of the dynamical systems associated with generalized Hamiltonians that are defined by the Legendre-Fenchel transform. When such a Hamiltonian arises from a Lagrangian problem with full convexity, it is concave in the state and convex in the adjoint state. If the Hamiltonian, or a minor perturbation of it, has a saddle point in the minimax sense, there is reason to imagine that the trajectories may exhibit saddle point behavior there in the dynamical sense. Confirming whether or not this is true is a major challenge, but some results in that direction are available and will be laid out here and applied.

2. Ramsey's problem and its reformulation

Ramsey (1928) posed a fundamental problem in macroeconomics: how can a society achieve an optimal balance between consumption and investment "per household"? A simplified model is the following. There are two real variables that evolve in time:

$c(t)$, the consumption at time t in terms of the rate of spending dollars,
$k(t)$, the nonnegative total amount of dollars currently invested at time t.

An expression $u(c(t))$ gives the utility of consumption while an expression $f(k(t))$ gives the production rate from investment. The problem is to

$$\begin{array}{l}\text{maximize } \int_0^\infty u(c(t))e^{-\rho t}dt \ \text{ subject to}\\ \dot{k}(t) = f(k(t)) - \sigma k(t) - c(t),\ k(0) = k_0.\end{array} \tag{1}$$

Here $\rho > 0$ is a given discount rate and $\sigma > 0$ is the population growth rate. The functions f and u will be assumed to belong to:

$$\begin{array}{rl}\Phi = & \text{the set of all the continuous functions } \varphi : [0,\infty) \to [0,\infty)\\ & \text{such that } \varphi \text{ is twice continuously differentiable on } (0,\infty)\\ & \text{with } \ \varphi''(t) < 0,\ \ \varphi'(t) > 0,\ \ \varphi'(0^+) = \infty \text{ and } \varphi(0) = 0.\end{array} \tag{2}$$

Such functions are, of course, nondecreasing and strictly concave.

This problem is an early example of a host of macroeconomic models dealing with economic growth, not only in consumption and investment but also in the management of natural resource stocks. Efforts at covering such infinite-horizon models through adaptations of existing control theory can be seen, for instance, in the economics text of Seierstad and Sydsaeter (1987).

Several questions about (1) immediately come up. Is the problem well formulated in terms of the ordinary differential equation having a solution over $[0,\infty)$ and the integral being well defined? Do optimal $k(\cdot)$ and $c(\cdot)$ exist, and if so, what characterizes them? Will they be unique?

The problem can be interpreted as one of optimal control with $k(t)$ as the state and $c(t)$ as the control. There is a state constraint $k(t)\in[0,\infty)$ combined with a control constraint $c(t)\in[0,\infty)$. A shortcoming is that the dynamics are identified simply as nonlinear, and the concavity of f has no advantageous role to play. An alternative control interpretation, which does put the concavity of f in the foreground, relaxes the equation to $\dot{k}(t)\le f(k(t))-\sigma k(t)-c(t)$, leaving it to the monotonicity of f and u to ensure that equality will hold anyway in optimality. Still, this does not fit a standard formulation in optimal control.

A different approach is to interpret the problem as one in the calculus of variations by substituting $f(k(t))-\sigma k(t)-\dot{k}(t)$ for $c(t)$, thereby suppressing the control entirely:

$$\text{maximize } \int_0^\infty u(f(k(t))-\sigma k(t)-\dot{k}(t))e^{-\rho t}dt \text{ with } k(t)\ge 0,\ k(0)=k_0, \quad (3)$$

which entails extending u from $[0,\infty)$ to $(-\infty,\infty)$ by

$$u(c)=-\infty \text{ for all } c<0. \quad (4)$$

That trick takes us outside the bounds of the classical calculus of variations, of course, as does the assumption that $u'(0^+)=\infty$, and $f'(0^+)=\infty$. Going even farther down that track, however, we arrive at a "neo-classical" formulation as a minimization problem in the calculus of variations with an extended-real-valued Lagrangian and all constraints, other than the initial condition, covered by infinite penalization:

$$\text{minimize } \int_0^\infty L_0(k(t),\dot{k}(t))e^{-\rho t}dt \text{ subject to } k(0)=k_0, \quad (5)$$

where

$$L_0(k,\dot{k})=\begin{cases}-u(f(k)-\sigma k-\dot{k}) & \text{when } k\ge 0 \text{ and } \dot{k}\le f(k)-\sigma k,\\ \infty & \text{otherwise.}\end{cases} \quad (6)$$

Note that $L_0(\cdot,\cdot)$ is a lower semicontinuous convex function on $I\!R\times I\!R$ having as its effective domain (the set where it is less than ∞) the convex set consisting of all $(k,\dot{k})$ such that $k\ge 0$ and $\dot{k}\le f(k)-\sigma k$. The penalization claim is justified as long as the integrand is sure to be bounded from below always by some integrable function, which can be guaranteed by other assumptions on L_0, as will be seen later. Taking one step more, we can introduce

$$L(t,k,\dot{k})=L_0(k,\dot{k})e^{-\rho t} \quad (7)$$

and restate the problem as:

$$\text{minimize} \int_0^\infty L(t,k(t),\dot{k}(t))dt \quad \text{subject to} \quad k(0)=k_0. \tag{8}$$

This formulation ties in with the theory of "generalized problems of Lagrange" of convex type (see Rockafellar, 1970a, 1970b, 1971) as a very particular case, but with the complication of an infinite horizon. The convexity of L with respect to $(k,\dot{k})$ supports a refined analysis with many interesting features.

3. Generalized Lagrange problems and their Hamiltonians

Some basics of the theory of generalized problems of Lagrange in a *finite-horizon* setting need review. We emphasize full convexity, but allow the state now to be n-dimensional, concentrating on the format

$$\text{minimize} \int_0^T L(t,k(t),\dot{k}(t))dt \quad \text{subject to} \quad k(0)=k_0,\ k(T)=k_T, \tag{9}$$

under the assumption that $L(t,\cdot,\cdot)$ is a lower semicontinuous convex function that is proper (never $-\infty$, but, on the other hand, not identically ∞).

What should be assumed with respect to t? The literature has developed a key concept of L being a *normal* integrand. This concept is flexible enough to support measurability properties through all the manipulations and compositions that may be needed. We wiil not go into it here, though, because our eventual aim is not so general; see the papers already cited and Chapter 14 of Rockafellar and Wets (1998). Anyway, the space in which the minimization is to take place is tentatively the space of all absolutely continuous n-dimensional *arcs* on $[0,T]$, by which we mean absolutely continuous functions from $[0,T]$ to $\mathbb{R}^n$. The integral in (9) then has a standardly well defined value in $(-\infty,\infty]$ as long as the integrand is bounded below by some integrable function of t, and otherwise it is assigned the value $-\infty$. However, we will look more closely at this later.

The *Hamiltonian*, associated with the Lagrangian L, is defined to be the function

$$H(t,k,p) = \sup_{\dot{k}} \left\{ p{\cdot}\dot{k} - L(t,k,\dot{k}) \right\}, \tag{10}$$

which by virtue of the convexity of $L(t,\cdot,\cdot)$ is *concave in k and convex in p*. The set of points (k,p) where $H(t,k,p)$ is finite has almost a product structure. Specifically, in terms of the set $K(t) \subset \mathbb{R}^n$, consisting of all k for which there exists p with $L(t,k,p) < \infty$, which is convex, one has $H(t,k,p) = -\infty$ for all p if $k \notin K(t)$, whereas $H(t,k,p) > \infty$ for all p if $k \in K(t)$. Moreover, the convex set $\left\{ p \,\middle|\, H(t,k,p) < \infty \right\}$ is the same set $P(t) \subset \mathbb{R}^n$ for all k in the relative interior of $K(t)$, and for other $k \in K(t)$ must at least lie between $P(t)$ and its closure. Later, we shall see how this plays out in Ramsey's problem.

The Hamiltonian *dynamical system* in this setting requires that

$$\dot{k}(t) \in \partial_p H(t, k(t), p(t)), \qquad -\dot{p}(t) \in \partial_k H(t, k(t), p(t)), \tag{11}$$

for almost every t, where $\partial_p H(t, k(t), p(t))$ refers to the set of subgradients at $p(t)$ of the convex function $H(t, k(t), \cdot)$, and $\partial_k H(t, k(t), p(t))$ refers to the set of subgradients at $k(t)$ of the concave function $H(t, \cdot, p(t))$ (i.e., the negatives of the subgradients of the convex function $-H(t, \cdot, p(t))$). These subgradient sets are empty, unless $k(t) \in K(t)$ and $p(t) \in P(t)$, in particular.

THEOREM 1 (SUFFICIENCY) *If an arc $k(\cdot)$ satisfies the endpoint constraints in (9) and obeys the Hamiltonian dynamics in (11) together with an arc $p(\cdot)$, then $k(\cdot)$ is optimal in problem (9).*

This result comes from Rockafellar (1970b). Since both endpoints of $k(\cdot)$ are fixed, no transversality requirement has to be imposed on $p(0)$ or $p(T)$.

Necessary conditions for optimality are more subtle but amount to the endpoint and Hamiltonian conditions holding in the presence of some *constraint qualification*. Results on that were derived in Rockafellar (1971). For dealing with state constraints, i.e., when $K(t) \neq I\!R^n$, there are additional refinements affecting both suffiency and necessity, see Rockafellar (1972). If adjoint state constraints enter, i.e., when $P(t) \neq I\!R^n$, an enlargement of the problem from absolutely continuous arcs to arcs of bounded variation may be called for, see Rockafellar (1976a).

How does the picture look when the Lagrangian has the discounting structure in (7)? The Hamiltonian comes out then in the form

$$H(t, k, p) = e^{-\rho t} H_0(k, e^{\rho t} p), \qquad H_0(k, p) = \sup_{\dot{k}} \left\{ p{\cdot}\dot{k} - L_0(k, \dot{k}) \right\}. \tag{12}$$

In terms of H_0, which is again concave-convex, the system (11) translates to

$$\dot{k}(t) \in \partial_q H_0(k(t), e^{\rho t} p(t)), \qquad -\dot{p}(t) \in e^{-\rho t} \partial_k H_0(k(t), e^{\rho t} p(t)). \tag{13}$$

But one does not't have to stop there. The substitution $q(t) = e^{\rho t} p(t)$ converts these dynamics into an equivalent *autonomous* system:

$$\dot{k}(t) \in \partial_q H_0(k(t), q(t)), \qquad -\dot{q}(t) \in \partial_k H_0(k(t), q(t)) - \rho q(t). \tag{14}$$

That formulation in *discounted* Hamiltonian dynamics opens up many things.

4. Infinite-horizon extension

The infinite-horizon problem we wish to investigate in general, as an extension of the finite-horizon problem (9), has the form

$$\text{minimize } \int_0^\infty L_0(k(t), \dot{k}(t)) e^{-\rho t} dt \ \text{ subject to } k(0) = k_0. \tag{15}$$

Do the preceding results for (9) have some counterpart for (15)? A particular trouble-spot is what to make of the absence of a terminal constraint. Should one just allow $k(T)$ to do anything as $T \to \infty$, or should a sort of infinite-horizon terminal constraint be imposed? Mathematical economists have had to contend with these issues unaided by much technical literature, and various difficulties have not been resolved to satisfaction.

An immediate question, of course, is whether the infinite-horizon integral in (15) can be managed with any ease. Although it has a well defined value always, the value might in principle be $-\infty$ for some arcs $k(\cdot)$, and that could be a source of trouble. There is no difficulty for instance if L_0 is bounded from below on $\mathbb{R}^n \times \mathbb{R}^n$, but that need not always be the situation. Another question in need of an answer is whether it is right to count on having

$$\int_0^\infty L_0(k(t), \dot{k}(t)) e^{-\rho t} dt \;=\; \lim_{T\to\infty} \int_0^T L_0(k(t), \dot{k}(t)) e^{-\rho t} dt. \tag{16}$$

Perhaps one has to step back from the formulation in (15) and develop a notion of optimality concerned only with how the integral over $[0, T]$ behaves as $T \to \infty$, rather than the limit itself. Such a notion is available as the following *overtaking criterion*, for instance: an arc $k(\cdot)$ is optimal under this criterion if for all other arcs $k'(\cdot) \neq k(\cdot)$ likewise starting from k_0, one has

$$\liminf_{T\to\infty} \Big[\int_0^T L_0(k'(t), \dot{k}'(t)) e^{-\rho t} dt - \int_0^T L_0(k(t), \dot{k}(t)) e^{-\rho t} dt \Big] \geq 0. \tag{17}$$

This is an *asymptotic* minimization property. Even so, nuisances may arise.

The property in (17) is "weak" overtaking, more specically; "strong" overtaking would have a strict inequality. Results utilizing an overtaking criterion can be seen for instance in Leizarowitz (1985) and the book of Carlson, Haurie and Leizarowitz (1991). However, our approach below will be to identify a saddle point property which, through an associated change of variables, will reduce such asymptotic minimization to true minimization.

The *autonomous* representation of the discounted Hamiltonian dynamics by (14) produces a major clue. In an autonomous system, the question of "rest points" naturally comes up. For (14), a *rest point* is by definition a pair $(\bar{k}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that a solution is furnished by the constant trajectory $(k(t), q(t)) \equiv (\bar{k}, \bar{q})$. Obviously this corresponds to having

$$0 \in \partial_q H_0(\bar{k}, \bar{q}), \qquad \rho \bar{q} \in \partial_k H_0(\bar{k}, \bar{q}). \tag{18}$$

Then, through Theorem 1, the arc $k(t) \equiv \bar{k}$ is optimal over any time interval $[0, T]$ for minimizing $\int_0^T L_0(k(t), \dot{k}(t)) e^{-\rho t} dt$ with respect to arcs that start and end at $\bar{k}$. Plausibly then, it might be optimal over $[0, \infty)$ with respect to arcs that start at $\bar{k}$, nothing being required as $t \to \infty$. Anyway, this suggests a *steady-state* optimality of a kind economists can be especially attracted to because $\bar{k}$

would constitute an equilibrium, toward which one could hope that optimal arcs in (15) might tend regardless of their starting points.

Around a rest point $(\bar{k}, \bar{q})$ for (14), there is still further simplification to be made through the change of variables

$$x = k - \bar{k}, \qquad r = q - \bar{q}.$$

Define the *reduced* Hamiltonian by

$$\overline{H}_0(x, r) = H_0(\bar{k} + x, \bar{q} + r) - H_0(\bar{k}, \bar{q}) - \rho\bar{q}{\cdot}x. \tag{19}$$

Concavity-convexity is inherited by $\bar{H}_0$ from H_0, and the rest point relations (18) translate to

$$0 \in \partial_x \overline{H}_0(0, 0) \text{ and } 0 \in \partial_p \overline{H}_0(0, 0), \text{ with } \overline{H}_0(0, 0) = 0. \tag{20}$$

The autonomous system (14) is shifted to the system

$$\dot{x}(t) \in \partial_r \overline{H}_0(x(t), r(t)), \qquad -\dot{r}(t) \in \partial_x \overline{H}_0(x(t), r(t)) - \rho r(t), \tag{21}$$

and then has a rest point at $(0, 0)$.

The subgradient relations in (20) say, moreover, that the rest point $(0, 0)$ of (21) is a *saddle point* of $\overline{H}_0$ in the *minimax* sense:

$$\overline{H}_0(x, 0) \leq \overline{H}_0(0, 0) \leq \overline{H}_0(0, r) \text{ for all } x \text{ and } r. \tag{22}$$

This is intriguing, because it suggests the possibility of an associated saddle point behavior of the Hamiltonian system in the dynamical sense. When there is no discounting ($\rho = 0$), for instance, $\overline{H}_0(x(t), r(t))$ has to be constant along trajectories of (21); see Rockafellar (1970b). In the one-dimensional case ($n = 1$) with $\overline{H}_0$ strictly concave in x and strictly convex in p, say, the level set where $\overline{H}_0 = 0$ typically appears to be the union of two curves that cross through the origin. Trajectories that start in that set have to remain in it, and it seems one must have the kind of pattern often seen with ODEs, where one of curves is comprised of trajectories that tend toward the origin, and the other is comprised of trajectories that tend away from it. Could this somehow carry over to n dimensions and in perturbation to positive discounting, at least if ρ is not too high?

5. Passage to a reduced Lagrangian

Before proceeding to the exploration of such possibilities, we need to understand better the infinite-horizon integral we are contemplating in (15). The reduced Hamiltonian $\overline{H}_0$ helps with that as well. Under the Legendre-Fenchel transform, $\overline{H}_0$ corresponds to the *reduced Lagrangian*

$$\overline{L}_0(x, \dot{x}) = \sup_r \Big\{ x{\cdot}r - \overline{H}_0(x, r) \Big\},$$

for which the formula works out through (19) to

$$\overline{L}_0(x,\dot{x}) = L_0(\bar{k}+x,\dot{x}) - L_0(\bar{k},0) - \bar{q}\cdot[\dot{x}-\rho x]. \tag{23}$$

An important observation is that the relations (22) for $\overline{H}_0$ correspond under the Legendre-Fenchel transform to having

$$\overline{L}_0(x,\dot{x}) \geq \overline{L}_0(0,0) = 0 \quad \text{for all } (x,\dot{x}). \tag{24}$$

There is nothing troublesome, then, about infinite-horizon integrals involving $\overline{L}_0$, in contrast, perhaps, to those involving L_0. They are sure to be nonnegative, although possibly ∞, and to obey the rule that

$$\int_0^\infty \overline{L}_0(x(t),\dot{x}(t))e^{-\rho t}dt = \lim_{T\to\infty}\int_0^T \overline{L}_0(x(t),\dot{x}(t))e^{-\rho t}dt. \tag{25}$$

Furthermore, we have from (23) that, in terms of $x(t) = k(t) - \bar{k}$,

$$\begin{aligned}\textstyle\int_0^T L_0(k(t),\dot{k}(t))e^{-\rho t}dt = \int_0^T \overline{L}_0(x(t),\dot{x}(t))e^{-\rho t}dt \\ +e^{-\rho T}\bar{q}\cdot[k(T)-k(0)] + \tfrac{1-e^{-\rho T}}{\rho}L_0(\bar{k},0).\end{aligned} \tag{26}$$

We can employ this relation along with (25) to gain an understanding of the infinite-horizon objective in (15) as the limit in (16).

Theorem 2 (reduced minimization) *Over the class of arcs $k(\cdot)$ on $[0,\infty)$ having $e^{-\rho T}k(T)\to 0$ as $T\to\infty$ and their counterparts $x(\cdot)$ with $x(t) = k(t)-\bar{k}$, the limit formula (16) for the infinite-horizon integral in (15) holds with the integrand over $[0,\infty)$ being bounded from below by an integrable function on $[0,\infty)$, and moreover*

$$\int_0^\infty L_0(k(t),\dot{k}(t))e^{-\rho t}dt = \int_0^\infty \overline{L}_0(k(t),\dot{k}(t))e^{-\rho t}dt + \gamma(\rho) \tag{26}$$

where

$$\gamma(\rho) = \begin{cases}(1-e^{-\rho t})/\rho & \text{for } \rho > 0,\\ 0 & \text{for } \rho = 0.\end{cases}$$

In this way, the restricted problem

$$\begin{aligned}&\text{minimize } \textstyle\int_0^\infty L_0(k(t),\dot{k}(t))e^{-\rho t}dt \ \text{ subject to } k(0)=k_0\\ &\text{under the growth constraint that } \lim_{T\to\infty} e^{-\rho T}k(T) = 0\end{aligned} \tag{28}$$

is equivalent to the problem

$$\begin{aligned}&\text{minimize } \textstyle\int_0^\infty \overline{L}_0(x(t),\dot{x}(t))e^{-\rho t}dt \ \text{ subject to } x(0)=x_0\\ &\text{under the growth constraint that } \lim_{T\to\infty} e^{-\rho T}x(T) = 0\,.\end{aligned} \tag{29}$$

We conclude that difficulties over the class of all absolutely continuous arcs possibly being too broad in problem (15) can be cured by imposing a natural growth condition on arcs that fits with the given discount rate ρ. The Hamiltonian analysis of optimality can then be carried out equivalently in terms of the reduced Hamiltonian.

6. Saddle point analysis of solutions

Having reached this stage, we can concentrate on the reduced minimization problem (29), or, for that matter, the relaxed version of it without the terminal growth condition, to the extent that a solution $x(\cdot)$ to (29) might automatically entail having $e^{-\rho T}x(T) \to 0$. Results from Rockafellar (1973, 1976b) can then be applied by virtue of $(0,0)$ being a saddle point of $\overline{H}_0$ in the minimax sense (22).

For optimality in (29) and its expression in the reduced dynamics (21), there are two situations in which much can be said. The first has $\rho > 0$ and requires an assumption of strong concavity-convexity of H_0 around the rest point $(\bar{k}, \bar{q})$, or equivalently that assumption on $\overline{H}_0$ around (0,0), and it puts a corresponding upper bound on ρ. The second has $\rho = 0$ and only asks for strict concavity-convexity locally.

Recall that a finite convex function f on a convex set $C \subset I\!R^n$ is *strictly convex* if for all x_0, $x_1 \in C$ and $\tau \in (0,1)$ one has

$$f((1-\tau)x_0 + \tau x_1) < (1-\tau)f(x_0) + \tau f(x_1).$$

It is *strongly convex* with modulus $\mu > 0$ if actually

$$f((1-\tau)x_0 + \tau x_1) \leq (1-\tau)f(x_0) + \tau f(x_1) - \frac{\mu}{2}\tau(1-\tau)|x_0 - x_1|^2,$$

where $|\cdot|$ denotes the Euclidean norm. Such strong convexity implies strict convexity and is equivalent to the function $f - \frac{\mu}{2}|\cdot|^2$ being convex on C. Strict and strong concavity are defined analogously.

We will be interested in two special subsets of $I\!R^n \times I\!R^n$ in connection with the Hamiltonian system (21):

$$\begin{aligned} M^+ = {}& \text{the set of } (x_0, r_0) \text{ from which a trajectory } (x(\cdot), r(\cdot)) \\ & \text{for (21) on } [0,\infty) \text{ starts and tends to } (0,0) \text{ as } t \to \infty, \\ M^- = {}& \text{the set of } (x_0, r_0) \text{ from which a trajectory } (x(\cdot), r(\cdot)) \\ & \text{for (21) on } (-\infty, 0] \text{ starts and tends to } (0,0) \text{ as } t \to -\infty. \end{aligned} \tag{30}$$

The trajectories described will be called M^+-trajectories and M^--trajectories, respectively.

Theorem 3 (positive discounting) *Suppose there is a neighorhood of* $(0,0)$ *in which* $\overline{H}_0$ *is* α*-strongly concave in its first argument and* β*-strongly convex in its second argument, with*

$$\alpha > 0, \qquad \beta > 0, \qquad 0 < \rho < 2\sqrt{\alpha\beta}. \tag{31}$$

(a) Any arc $x(\cdot)$ *over* $[0,\infty)$*, for which the integral in problem (29) is finite, must automatically satisfy the condition in (29) that* $e^{-\rho T}x(T) \to 0$ *as* $T \to \infty$*.*

(b) If $(x(\cdot), r(\cdot))$ *is an* M^+*-trajectory starting from a point* (x_0, r_0) *near enough to* $(0,0)$*, then* $x(\cdot)$ *is the unique solution to problem (29) for* x_0*.*

(c) There is a neighborhood of $(0,0)$ *in which* M^+ *is the graph of a function* $F^+ : x_0 \mapsto r_0$ *that maps an* x_0*-neighborhood of 0 homeomorphically onto an* r_0*-neighborhood of 0.*

This comes from Rockafellar (1976b), where an analogous, but flawed claim was made about M^-. For our economic applications we are not really concerned with M^-, but being able to have both M^+ and M^- in the picture would better support the notion of "saddle point dynamics." Results that include M^-, but with zero discounting, will be viewed shortly.

The mapping in (c) of Theorem 3 has an important "feedback" interpretation. It informs us that as we follow an M^+-trajectory we always have $r(t) = F^+(x(t))$, inasmuch as any time can just as well be the starting time in an autonomous dynamical system.

Theorem 4 (feedback rule) *Under the assumptions of Theorem 3, there is a neighborhood of* 0 *in* $\mathbb{R}^n$ *such that, starting from any* x_0 *in that neighborhood, the unique solution to problem (29) for* x_0 *is the unique solution to the differential inclusion*

$$\dot{x}(t) \in \partial_r \overline{H}_0(x(t), F^+(x(t))), \qquad x(0) = x_0. \tag{32}$$

Of course, this translates back through the change of variables $k(t) = \bar{k} + x(t)$ to the context of problem (29). The strong concavity-convexity assumption on $\overline{H}_0$ in a neighborhood of $(0,0)$ corresponds to the same assumption for H_0 in a neighborhood of $(\bar{k}, \bar{q})$. The feedback rule then takes the form

$$\dot{k}(t) \in \partial_q H_0(k(t), F^+(k(t) - \bar{k})), \qquad k(0) = k_0. \tag{33}$$

Theorem 5 (zero discounting) *Let* $\rho = 0$ *and suppose there is a neighborhood of* $(0,0)$ *in which* $\overline{H}_0$ *is strictly concave in its first argument and strictly convex in its second argument.*

(a) Any arc $x(\cdot)$ *over* $[0, \infty)$*, for which the integral in problem (29) is finite, must automatically satisfy the condition in (29) that* $x(T) \to 0$ *as* $T \to \infty$*.*

(b) If $(x(\cdot), r(\cdot))$ *is an* M^+*-trajectory starting from a point* (x_0, r_0) *near enough to* $(0,0)$*, then* $x(\cdot)$ *is the unique solution to problem (29) for* x_0*.*

(c) There is a neighborhood of $(0,0)$ *in which* M^+ *is the graph of a function* $F^+ : x_0 \mapsto r_0$ *that maps an* x_0*-neighborhood of 0 homeomorphically onto an* r_0*-neighborhood of 0.*

(d) There is, likewise, a neighborhood of $(0,0)$ *in which* M^- *is the graph of a function* $F^- : x_0 \mapsto r_0$ *that maps an* x_0*-neighborhood of 0 homeomorphically onto an* r_0*-neighborhood of 0.*

(e) These n*-dimensional manifolds in* $\mathbb{R}^n \times \mathbb{R}^n$ *provided locally around* $(0,0)$ *by* M^+ *and* M^- *meet only at* $(0,0)$*.*

This result is from Rockafellar (1973), where it is also explained that M^--trajectories correspond to optimality in an analogous minimization problem over $(-\infty, 0]$. For that problem there is a feedback rule with F^-.

The statements so far are localized around the rest point, with neighborhoods "sufficiently small." It would be interesting to know, especially for the sake of macroeconomic applications, whether global versions might be possible, that is, with respect to the effective domain of the Hamiltonian (the set where it is finite). There is no good answer at present, when primal and/or dual state constraints are in the background, i.e., when the effective domain is not the whole space. But when it is the whole space, very satisfying and powerful global results are now available from Goebel (2005).

To state these results—tailored to our context and elaborated in an easy way beyond their published statements—we need to rely on a growth assumption, which may at first seem obscure, but is actually very broad for a host of purposes. In particular, it exactly fits the demands of Hamilton-Jacobi theory under full convexity as developed in Rockafellar and Wolenski (2000). The condition stipulates the existence of a finite convex function ψ on $I\!R^n$ with $\min \psi = \psi(0) = 0$ along with a $\gamma > 0$ such that

$$-\psi(x) - \gamma|x||r| \leq \overline{H}_0(x,r) \leq \psi(r) + \gamma|x||r| \text{ for all } (x,r) \in I\!R^n \times I\!R^n. \quad (34)$$

What this means for the Lagrangian $\overline{L}_0$, and, ultimately, for the originating L_0, is something very natural which can be found in the paper just cited. We state the main saddle point result *qualitatively*—for the details see Goebel (2005).

Theorem 6 (global saddle point picture) *Let $\rho = 0$ and suppose there is a neighborhood of $(0,0)$ on which $\overline{H}_0$ is strictly concave-convex. Under the growth condition (34),*

(a) M^+-trajectories uniquely solve (29), whereas M^- trajectories uniquely solve an analogous backward-time problem.

(b) M^+ and M^- are the graphs of global homeomorphisms F^+ and F^- of $I\!R^n$ with itself. These n-dimensional manifolds in $I\!R^n \times I\!R^n$ meet only at $(0,0)$.

(c) F^+ and F^- arise from the gradient mappings of value functions that solve a stationary Hamilton-Jacobi equation for $\overline{H}_0$, or equivalently for H_0.

In Goebel (2005) there is actually no M^-, but the assertions about it here are immediately derivable through a simple time reversal. (Unfortunately, time reversal does not help with the case of positive discounting, because it would become negative discounting.)

7. Application to Ramsey's problem

Ramsey's problem (1) is one-dimensional, i.e., has merely $n = 1$. The functions f and u that it involves come from the class (2), for which the limits

$$a_f := \lim_{k\to\infty} f'(k) \geq 0, \qquad a_u := \lim_{c\to\infty} u'(c) \geq 0, \quad (35)$$

are sure to exist. These limits will have a role in the discussion. Note that $f'(k) > a_f$ and $u'(c) > a_u$ for all $k \in (0,\infty)$ and $c \in (0,\infty)$.

As recast in (3), Ramsey's problem has been seen to correspond to the Lagrangian L_0 in (6). What is the associated Hamitonian H_0? In terms of

$$c = f(k) - \sigma k - \dot{k}, \tag{36}$$

we can calculate H_0 by applying the Legendre-Fenchel transform in the $\dot{k}$ argument as in (12) (but with q in place of p for the reasons that have emerged in connection with rest points). Because of the second line in formula (6), we get $H_0(k,q) = -\infty$ when $k < 0$, whereas for $k \geq 0$ we get

$$\begin{aligned} H_0(k,q) &= \sup{}_{\dot{k}}\Big\{q\dot{k} - L_0(k,\dot{k})\Big\} \\ &= \sup{}_{\dot{k}}\Big\{q\dot{k} + u(f(k) - \sigma k - \dot{k})\Big\} = \sup{}_c\Big\{q[f(k) - \sigma k - c] + u(c)\Big\} \\ &= q[f(k) - \sigma k] - \inf{}_c\Big\{qc - u(c)\Big\} = q[f(k) - \sigma k] - u^*(q)\Big\}, \end{aligned} \tag{37}$$

where u^* is the concave function that is *conjugate* to u. Because u comes from the class (2), u^* has the properties that

$$\begin{aligned} &u^* \text{ is finite on } (a_u,\infty) \text{ with } u^{*\prime\prime}(q) < 0,\ u^{*\prime}(q) > 0, \\ &\text{and, moreover, } \ c = u^{*\prime}(q) \text{ if and only if } q = u'(c). \end{aligned} \tag{38}$$

The endpoint value $u^*(a_u)$ equals the limit of $u^*(q)$ as q decreases to a_u, which may be finite or $-\infty$. For $q < a_u$, necessarily $u^*(q) = -\infty$. The negativity and continuity of the second derivatives of f and u^* ensures that H_0 *is strongly concave-convex* in a neighborhood of every point of $(0,\infty)\times(a_u,\infty)$.

In this setting the Hamiltonian subgradients reduce to derivatives:

$$(\partial_k H_0(k,q), \partial_q H_0(k,q)) = \\ \begin{cases} \Big\{(q[f'(k) - \sigma], f(k) - \sigma k - u^{*\prime}(q))\Big\} & \text{for } (k,q) \in (0,\infty)\times(a_u,\infty), \\ \emptyset & \text{otherwise.} \end{cases}$$

The discounted Hamiltonian dynamical system (14) thus operates only in the open set $(0,\infty)\times(a_u,\infty)$ and takes the form that

$$\dot{k}(t) = f(k(t)) = \sigma k(t) - u^{*\prime}(q(t)), \qquad \dot{q}(t) = q(t)[\sigma + \rho - f(k(t))]. \tag{39}$$

Here $c(t) = u^{*\prime}(q(t))$ gives the ongoing rate of consumption, with $q(t) = u'(c(t))$ by (38). The rest point conditions $0 \in \partial_q H_0(\bar{k},\bar{q})$, $\rho\bar{q} \in \partial_k H_0(\bar{k},\bar{q})$, in (18) require $\bar{k} \in (0,\infty)$, $\bar{q} \in (a_u,\infty)$, and mean that

$$0 = f(\bar{k}) - \sigma\bar{k} - u^{*\prime}(\bar{q}), \qquad 0 = \bar{q}[\sigma + \rho - f'(\bar{k})].$$

These relations are easy to analyze and yield an explicit result of existence and uniqueness.

THEOREM 7 (REST POINTS IN RAMSEY'S MODEL) *Under the assumption that* $\sigma + \rho > a_f$*, there is a unique rest point* $(\bar{k}, \bar{q})$ *for the discounted Hamiltonian dynamics which lies in* $(0, \infty) \times (a_u, \infty)$ *and is obtained as follows:*

(a) Take $\bar{k}$ *to be the unique solution to* $f'(k) = \sigma + \rho$*.*

(b) Take $\bar{c} = f(\bar{k}) - \sigma\bar{k}$ *and then* $\bar{q} = u(\bar{c})$*.*

The interpretation is that $c(t) \equiv \bar{c}$ gives a steady rate of consumption under which the accumulated capital is self-sustaining: $k(t) \equiv \bar{k}$.

The strong concavity-convexity of H_0 around the rest point $(\bar{k}, \bar{q})$ opens the way to applying Theorem 3 by taking ρ sufficiently small. In fact, moduli α and β of strong concavity in k and strong convexity in q in a neighborhood of $(\bar{k}, \bar{q})$ can be obtained simply by taking $\alpha < -\bar{q} f''(\bar{k})$ and $\beta < -u^{*\prime\prime}(\bar{q})$, which is equivalent to $\beta < -1/u''(\bar{c})$, because the conjugate functions have reciprocal derivatives at conjugate points.

THEOREM 8 (CONVERGENCE TO A STEADY-STATE) *Suppose that* $\sigma + \rho > a_f$ *and, with respect to the rest point described in Theorem 7, that*

$$0 < \rho < 2\sqrt{\bar{q} f''(\bar{k})/u''(\bar{c})}. \tag{40}$$

Let M^+ *denote the set of all* (k_0, q_0)*, from which the system (39) has a trajectory* $(k(\cdot), q(\cdot))$ *that tends over infinite time to* $(\bar{k}, \bar{q})$*.*

(a) If $(k(\cdot), q(\cdot))$ *is an* M^+*-trajectory starting from a pair* (k_0, q_0) *near enough to* $(0, 0)$*, then* $k(\cdot)$ *is the unique solution to Ramsey's problem (1) from* k_0 *under the terminal constraint that* $e^{\rho T} k(T) \to 0$ *as* $T \to \infty$*.*

(b) There is an interval around $\bar{k}$*, on which* M^+ *is the graph of a homeomorphism* F^+ *onto an interval around* $\bar{q}$*, with* $F^+(\bar{k}) = \bar{q}$*.*

The inequality (40) ensures that the upper bound (31) is respected when α and β are chosen as described ahead of the theorem. Here we have translated from the (x, r) context of Theorem 3 back to the (k, q) original context, taking advantage of Theorem 2.

The result says that optimal behavior of an "economy" in the sense of the utility maximization in (1), if initiated in a capital state near enough to the ideal state $\bar{k}$, will move the economy toward that state, and moreover this optimal behavior corresponds to applying a uniquely determined feedback rule (coming out of (b)). It would be nice to know whether a more than local statement is true, but nothing in the general results presented in earlier sections is able to cover that.

In conclusion, it is worth mentioning that the issues in growth models do not stop with convexity. A very interesting example of saddle point analysis requiring nonsmooth analysis beyond convex analysis is in the paper of Clark, Clarke and Munro (1979). For general macroeconomic background on growth, see the books of Barro and Howitt (1998) and Romer (1996). Of course, there is much in the economics literature which relates to the topic addressed here, and apologies are due for not surveying it.

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