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Second order conditions in optimal control problems with mixed equality-type constraints on a variable time interval*†

by

Nikolai P. Osmolovskii

Systems Research Institute, Polish Academy of Sciences,
ul. Newelska 6, 01-447, Warszawa, Poland
Politechnika Radomska
26-600 Radom, ul. Malczewskiego 20A, Poland
Akademia Podlaska
ul. 3 Maja 54, 08-110 Siedlce, Poland
email: osmolovski@ap.siedlce.pl

Abstract: This paper provides an analysis of second-order necessary or sufficient optimality conditions of Pontryagin or bounded strong minima, for optimal control problems of ordinary differential equations, considered on a nonfixed time interval, with constraints on initial-final time-state as well as mixed state-control constraints of equality type satisfying condition of linear independence of gradients w.r.t. control.

Keywords: piecewise continuous control, maximum principle, second order necessary or sufficient conditions, critical cone, quadratic form.

1. Introduction

In this paper, we discuss quadratic (second order) optimality conditions, both necessary and sufficient, in optimal control problems on a variable interval of time, with control appearing nonlinearly. There exists an extensive literature on this subject for optimal control problems considered on a fixed time interval; see Arutyunov and Karamzin (2005), Bonnans and Hermant (2007), Dunn (1995, 1996), Levitin, Milyutin and Osmolovskii (1978), Maurer (1981), Maurer and Pickenhain (1995), Maurer and Oberle (2002), Milyutin and Osmolovskii (1998), Osmolovskii (1988, 2004), Osmolovskii and Lempio (2000, 2002), Zeidan (1994) and further literature cited in these papers. Necessary (sufficient) second

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order conditions require that a certain quadratic form be positive semidefinite (positive definite) on the so called critical cone. In practice, the test for second order sufficient conditions can be performed by checking whether an associated matrix Riccati equation has a bounded solution under appropriate boundary conditions. The Riccati approach has been extended to discontinuous controls (broken extremals) by Osmolovskii and Lempio (2002). The second order sufficient conditions play important role in the sensitivity analysis of parametric optimal control problems; see Malanowski (1992, 1993, 1994, 2001), Malanowski and Maurer (1996, 1998, 2001), Dontchev et al. (1995), Augustin and Maurer (2001).

We study the question of optimality of extremal with discontinuous control in the problem which can be reduced to the following form, given in the book by Milyutin and Osmolovskii (1998), p. 1: let $\mathcal{T}$ denote a *trajectory* $(x(t), u(t) \mid t \in [t_0, t_1])$, where the state variable $x(\cdot)$ is a Lipschitz continuous function, and the control variable $u(\cdot)$ is a bounded measurable function on a time interval $\Delta = [t_0, t_1]$; the interval Δ is not fixed; for each trajectory $\mathcal{T}$ we denote by $p = (t_0, x(t_0), t_1, x(t_1))$ the vector of the endpoints of time-state variable (t, x). It is required to find $\mathcal{T}$ minimizing the functional

$$\mathcal{J}(\mathcal{T}) := J(p) \to \min \tag{1}$$

subject to the constraints

$$F(p) \le 0, \quad K(p) = 0, \tag{2}$$

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{3}$$

$$g(t, x(t), u(t)) = 0, \tag{4}$$

$$p \in \mathcal{P}, \quad (t, x(t), u(t)) \in \mathcal{Q}, \tag{5}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are open sets, x, u, F, K, f, and g are vector-functions.

We assume that the functions J, F, and K are defined and twice continuously differentiable on $\mathcal{P}$, and the functions f and g are defined and twice continuously differentiable on $\mathcal{Q}$. It is also assumed that the gradients with respect to the control $g_{iu}(t, x, u)$, $i = 1, \dots, d(g)$ are linearly independent at each point $(t, x, u) \in \mathcal{Q}$ such that $g(t, x, u) = 0$. Here $d(g)$ is a dimension of the vector g.

This statement corresponds to the general canonical optimal control problem in the Dubovitskii–Milyutin form, but, in contrast to the latter, it does not contain pointwise (or 'local', in the Dubovitskii–Milyutin terminology) inequality-type constraints $\varphi(t, x, u) \le 0$. Precisely these constraints bring about the biggest difficulties in the study of quadratic conditions (see Osmolovskii, 1988) and, because of the absence of local inequalities, we place this problem not in the context of optimal control, but rather of the calculus of variations and call it the *general problem of the calculus of variations* (see Milyutin and Osmolovskii, 1998). Its statement is close to the Mayer problem, but the existence of endpoint inequality-type constraints determines its specifics.

On the other hand, this problem, even being referred to the calculus of variations, is sufficiently general, and its statement is close to optimal control problems, especially owing to the local relation $g(t, x, u) = 0$. In Milyutin and Osmolovskii (1998), it was shown how, by using quadratic conditions for problem (1)-(5), one can obtain quadratic (necessary or sufficient) conditions in optimal control problems in which the controls enter linearly and the constraint on the control is given in the form of a convex polyhedron under the assumption that the optimal control is piecewise-constant and (outside the switching points) belongs to vertices of the polyhedron (the so-called bang-bang control). To show this, in Milyutin and Osmolovskii (1998), we first used the property that the set V of vertices of a polyhedron U can be given by a nondegenerate relation $g(u) = 0$ on an open set $\mathcal{Q}$ consisting of disjoint open neighborhoods of vertices. This allowed us to write quadratic necessary conditions for bang-bang controls. Further, in Milyutin and Osmolovskii (1998), it was shown that a sufficient minimality condition on V guarantees (in the problem linear in the control) the minimum on its convexification $U = \operatorname{co} V$. In this way, the quadratic sufficient conditions were obtained for bang-bang controls.

However, in Milyutin and Osmolovskii (1998), there is a substantial gap stemming from the fact that, in order to avoid making the book too large, the authors decided not to present the proofs of quadratic conditions for the general problem of calculus of variations and restricted themselves to their formulation and the presentation of proofs only in the case of the simplest problem.

The most important and considerable step to remove this gap was done in Osmolovskii (2004), where complete proofs of quadratic extremality conditions for discontinuous controls were presented in problem (1)-(5) on a *fixed* time interval $[t_0, t_1]$:

$$J(x(\cdot), u(\cdot)) = J(x(t_0), x(t_1)) \to \min,$$
$$F(x(t_0), x(t_1)) \le 0, \quad K(x(t_0), x(t_1)) = 0,$$
$$\dot{x} = f(t, x, u), \quad g(t, x, u) = 0, \quad (t, x, u) \in \mathcal{Q},$$

under the assumptions of C^2 smoothness of functions J, F, K, f, and g and the full rank condition for the matrix g_u on the surface $g = 0$. (The proofs were based on the general theory of conditions of higher order by Levitin, Milyutin, and Osmolovskii, 1978). The aim of the present paper is to extend the results obtained in Osmolovskii (2004) (for the problem on the fixed time interval) to the problem (1)-(5) on a variable interval of time $[t_0, t_1]$. Thus, the present paper together with Osmolovskii (2004) can be considered as a necessary supplement to the book of Milyutin and Osmolovskii (1998), completely removing the mentioned gap in the book.

We briefly recall different notions of minimum in the problem (1)-(5) on a variable interval $[t_0, t_1]$. First let us recall the definition of Pontryagin minimum, given in Milyutin and Osmolovskii (1998), pp. 2-3.

DEFINITION 1.1 The trajectory $\mathcal{T}$ affords a *Pontryagin minimum* if there is no sequence of admissible trajectories $\mathcal{T}^n = (x^n(t), u^n(t) \mid t \in [t_0^n, t_1^n])$, $n = 1, 2, \ldots$ such that
(a) $\mathcal{J}(\mathcal{T}^n) < \mathcal{J}(\mathcal{T}) \quad \forall n$;
(b) $t_0^n \to t_0, \quad t_1^n \to t_1 \quad (n \to \infty)$;
(c) $\max\limits_{\Delta^n \cap \Delta} |x^n(t) - x(t)| \to 0 \quad (n \to \infty), \quad$ where $\Delta^n = [t_0^n, t_1^n]$;
(e) $\int\limits_{\Delta^n \cap \Delta} |u^n(t) - u(t)| \, dt \to 0 \quad (n \to \infty)$;
(d) there exists a compact set $\mathcal{C} \subset \mathcal{Q}$ such that $(t, x^n(t), u^n(t)) \in \mathcal{C}$ a.e. on Δ^n for all n.

For convenience, let us give an equivalent definition of the Pontryagin minimum.

DEFINITION 1.2 The trajectory $\mathcal{T}$ affords a *Pontryagin minimum* if for each compact set $\mathcal{C} \subset \mathcal{Q}$ there exists $\varepsilon > 0$ such that $\mathcal{J}(\tilde{\mathcal{T}}) \geq \mathcal{J}(\mathcal{T})$ for all admissible trajectories $\tilde{\mathcal{T}} = (\tilde{x}(t), \tilde{u}(t) \mid t \in [\tilde{t}_0, \tilde{t}_1])$ satisfying the conditions
(a) $|\tilde{t}_0 - t_0| < \varepsilon, \quad |\tilde{t}_1 - t_1| < \varepsilon$,
(b) $\max\limits_{\tilde{\Delta} \cap \Delta} |\tilde{x}(t) - x(t)| < \varepsilon, \quad$ where $\tilde{\Delta} = [\tilde{t}_0, \tilde{t}_1]$,
(c) $\int\limits_{\tilde{\Delta} \cap \Delta} |\tilde{u}(t) - u(t)| \, dt < \varepsilon$,
(d) $(t, \tilde{x}(t), \tilde{u}(t)) \in \mathcal{C} \quad$ a.e. on $\tilde{\Delta}$.

Next, we recall the definition of a bounded strong minimum, given in Milyutin and Osmolovskii (1998), pp. 290-291. To this end, let us define the notions of essential and unessential components of vector x.

DEFINITION 1.3 The ith component x_i of vector x is called *unessential* if the functions f and g do not depend on this component and the functions J, F, and K are affine in $x_{i0} = x_i(t_0)$, $x_{i1} = x(t_1)$; otherwise the component x_i is called *essential*.

We denote by $\underline{x}$ a vector composed of all essential components of vector x.

DEFINITION 1.4 We say that the trajectory $\mathcal{T}$ affords *a bounded strong minimum* if there is no sequence of admissible trajectories

$$\mathcal{T}^n = (x^n(t), u^n(t) \mid t \in [t_0^n, t_1^n]), \quad n = 1, 2, \ldots$$

such that
a) $\mathcal{J}(\mathcal{T}^n) < \mathcal{J}(\mathcal{T})$,
b) $t_0^n \to t_0, \quad t_1^n \to t_1, \quad x^n(t_0^n) \to x(t_0) \quad (n \to \infty)$,
c) $\max\limits_{\Delta^n \cap \Delta} |\underline{x}^n(t) - \underline{x}(t)| \to 0$ $(n \to \infty)$, where $\Delta^n = [t_0^n, t_1^n]$,
d) there exists a compact set $\mathcal{C} \subset \mathcal{Q}$ such that

$$(t, x^n(t), u^n(t)) \in \mathcal{C} \text{ a.e. on } \Delta^n \quad \forall n.$$

An equivalent definition has the following form:

Definition 1.5 The trajectory $\mathcal{T}$ affords *a bounded strong minimum* if for each compact set $\mathcal{C} \subset \mathcal{Q}$ there exists $\varepsilon > 0$ such that $\mathcal{J}(\tilde{\mathcal{T}}) \geq \mathcal{J}(\mathcal{T})$ for all admissible trajectories $\tilde{\mathcal{T}} = (\tilde{x}(t), \tilde{u}(t) \mid t \in [\tilde{t}_0, \tilde{t}_1])$ satisfying the conditions

(a) $|\tilde{t}_0 - t_0| < \varepsilon, \quad |\tilde{t}_1 - t_1| < \varepsilon, \quad |\tilde{x}(\tilde{t}_0) - x(t_0)| < \varepsilon,$

(b) $\max\limits_{\tilde{\Delta}\cap\Delta} |\underline{\tilde{x}}(t) - \underline{x}(t)| < \varepsilon, \quad$ where $\tilde{\Delta} = [\tilde{t}_0, \tilde{t}_1]$,

(c) $(t, \tilde{x}(t), \tilde{u}(t)) \in \mathcal{C}$ a.e. on $\tilde{\Delta}$.

The *strict* bounded strong minimum is defined in a similar way, with the nonstrict inequality $\mathcal{J}(\tilde{\mathcal{T}}) \geq \mathcal{J}(\mathcal{T})$ replaced by the strict one and the trajectory $\tilde{\mathcal{T}}$ required to be different from $\mathcal{T}$.

Finally, we define a *(strict) strong minimum* in the same way, but omitting condition (c) in the last definition. The following statement is quite obvious.

Proposition 1.1 *If there exists a compact set $\mathcal{C} \subset \mathcal{Q}$ such that*

$$\{(t, x, u) \in \mathcal{Q} \mid g(t, x, u) = 0\} \subset \mathcal{C},$$

then a (strict) strong minimum is equivalent to a (strict) bounded strong minimum.

Obviously, any of the following concepts is implied by the previous one: strong, bounded strong, Pontryagin, weak minimum. In the sequel, we will present necessary conditions of Pontryagin minimum and sufficient conditions of a bounded strong minimum in problem (1)-(5).

2. Necessary conditions of a Pontryagin minimum

Let $\mathcal{T}$ be a fixed admissible trajectory such that the control $u(\cdot)$ is a piecewise Lipschitz-continuous function on the interval Δ with the set of discontinuity points

$$\Theta = \{t^1, \dots, t^s\}, \quad t_0 < t^1 < \cdots < t^s < t_1.$$

In order to make the notations simpler we do not use such symbols and indices as zero, hat or asterisk to distinguish this trajectory from others.

Let us formulate a first-order necessary condition for optimality of the trajectory $\mathcal{T}$. To this end, let us introduce the *Pontryagin function*

$$H(t, x, u, \psi) = \psi f(t, x, u) \tag{6}$$

and the *augmented Pontryagin function*

$$\bar{H}(t, x, u, \psi, \nu) = H(t, x, u, \psi) - \nu g(t, x, u), \tag{7}$$

where ψ and ν are row-vectors of the dimensions $d(x)$ and $d(g)$, respectively.

Let us define the *end-point Lagrange function*

$$l(p, \alpha_0, \alpha, \beta) = \alpha_0 J(p) + \alpha F(p) + \beta K(p), \tag{8}$$

where $p = (t_0, x_0, t_1, x_1)$, $x_0 = x(t_0)$, $x_1 = x(t_1)$, $\alpha_0 \in \mathbb{R}$, $\alpha \in (\mathbb{R}^{d(F)})^*$, $\beta \in (\mathbb{R}^{d(K)})^*$. By $\mathbb{R}^{n*}$ we denote the space of horizontal vectors of the dimension n.

We introduce a tuple of Lagrange multipliers

$$\lambda = (\alpha_0, \alpha, \beta, \psi(\cdot), \psi_0(\cdot), \nu(\cdot)) \tag{9}$$

such that $\psi(\cdot) : \Delta \to (\mathbb{R}^{d(x)})^*$ and $\psi_0(\cdot) : \Delta \to \mathbb{R}^1$ are piecewise smooth functions, continuously differentiable on each interval of the set $\Delta \setminus \Theta$, and $\nu(\cdot) : \Delta \to (\mathbb{R}^{d(g)})^*$ is a piecewise continuous function, Lipschitz continuous on each interval of the set $\Delta \setminus \Theta$.

Denote by M_0 the set of the normed tuples λ satisfying the conditions of the maximum principle for the trajectory $\mathcal{T}$:

$$\begin{aligned}
&\alpha_0 \geq 0,\ \alpha \geq 0,\ \alpha F(p) = 0,\ \alpha_0 + \sum \alpha_i + \sum |\beta_j| = 1,\\
&\dot{\psi} = -\bar{H}_x,\ \dot{\psi}_0 = -\bar{H}_t,\ \bar{H}_u = 0,\ t \in \Delta \setminus \Theta,\\
&\psi(t_0) = l_{x_0},\ \psi(t_1) = -l_{x_1},\ \psi_0(t_0) = l_{t_0},\ \psi_0(t_1) = -l_{t_1},\\
&\max_{u \in U(t,x(t))} H(t, x(t), u, \psi(t)) = H(t, x(t), u(t), \psi(t)),\ t \in \Delta \setminus \Theta,\\
&H(t, x(t), u(t), \psi(t)) + \psi_0(t) = 0,\ t \in \Delta \setminus \Theta,
\end{aligned} \tag{10}$$

where $U(t, x) = \{u \in \mathbb{R}^{d(u)} \mid g(t, x, u) = 0,\ (t, x, u) \in \mathcal{Q}\}$. The derivatives l_{x_0} and l_{x_1} are at $(p, \alpha_0, \alpha, \beta)$, where $p = (t_0, x(t_0), t_1, x(t_1))$, and the derivatives $\bar{H}_x$, $\bar{H}_u$, and $\bar{H}_t$ are at $(t, x(t), u(t), \psi(t), \nu(t))$, where $t \in \Delta \setminus \Theta$. (Condition $\bar{H}_u = 0$ follows from the other conditions in this definition, and therefore could be excluded; yet we need to use it later.)

The condition $M_0 \neq \emptyset$ is equivalent to the Pontryagin's maximum principle. It is a first-order necessary condition of Pontryagin minimum for the trajectory $\mathcal{T}$, see Milyutin and Osmolovskii (1998), pp. 24-25 and 32-40. Thus, the following theorem holds:

THEOREM 2.1 *If the trajectory $\mathcal{T}$ affords a Pontryagin minimum, then the set M_0 is nonempty.*

Assume that M_0 is nonempty. Using the definition of the set M_0 and the full rank condition of the matrix g_u on the surface $g = 0$ one can easily prove the following statement:

PROPOSITION 2.1 *The set M_0 is a finite-dimensional compact set, and the mapping $\lambda \mapsto (\alpha_0, \alpha, \beta)$ is injective on M_0.*

Continuity of the functions ψ and ψ_0 at the points $t^k \in \Theta$ (for $\lambda \in M_0$) constitute the Weierstrass–Erdmann necessary conditions for broken extremal. Let us formulate one more condition of this type. To this end, for each $\lambda \in M_0, t^k \in \Theta$, we set

$$D^k(\bar{H}) = \bar{H}_x^{k+}\bar{H}_\psi^{k-} - \bar{H}_x^{k-}\bar{H}_\psi^{k+} + [\bar{H}_t]^k, \tag{11}$$

where $\bar{H}_x^{k-} = \bar{H}_x(t^k, x(t^k), u(t^k-), \psi(t^k), \nu(t^k-))$,
$\bar{H}_x^{k+} = \bar{H}_x(t^k, x(t^k), u(t^k+), \psi(t^k), \nu(t^k+))$, $[\bar{H}_t]^k = \bar{H}_t^{k+} - \bar{H}_t^{k-}$, etc.

Theorem 2.2 *For each $\lambda \in M_0$ the following conditions hold:*

$$D^k(\bar{H}) \geq 0, \quad k = 1, \ldots, s. \tag{12}$$

Thus, conditions (12) follow from the maximum principle conditions (10). An alternative method to calculate $D^k(\bar{H})$ is the following. For $\lambda \in M_0$, $t^k \in \Theta$, consider the function

$$\begin{aligned}(\Delta_k\bar{H})(t) = \bar{H}(t^k, x(t), u(t^k+), \psi(t), \nu(t^k+))\\ - \bar{H}(t^k, x(t), u(t^k-), \psi(t), \nu(t^k-)).\end{aligned}$$

Proposition 2.2 *For each $\lambda \in M_0$ the following equalities hold*

$$\frac{d}{dt}(\Delta_k\bar{H})\big|_{t=t^k-} = \frac{d}{dt}(\Delta_k\bar{H})\big|_{t=t^k+} = D^k(\bar{H}), \quad k = 1, \ldots, s. \tag{13}$$

Hence, for $\lambda \in M_0$ the function $(\Delta_k\bar{H})(t)$ has a derivative at the point $t^k \in \Theta$ equal to $D^k(\bar{H})$, $k = 1, \ldots, s$.

Let us formulate a quadratic necessary condition of a Pontryagin minimum for the trajectory $\mathcal{T}$. First, for this trajectory, we introduce a Hilbert space $\mathcal{Z}_2(\Theta)$ and the critical cone $\mathcal{K} \subset \mathcal{Z}_2(\Theta)$.

Denote by $P_\Theta W^{1,2}(\Delta, \mathbb{R}^{d(x)})$ the Hilbert space of piecewise continuous functions $\bar{x}(\cdot) : \Delta \to \mathbb{R}^{d(x)}$, absolutely continuous on each interval of the set $\Delta \setminus \Theta$ and such that their first derivative is square integrable. For each $\bar{x} \in P_\Theta W^{1,2}(\Delta, \mathbb{R}^{d(x)})$, $t^k \in \Theta$ we set $\bar{x}^{k-} = \bar{x}(t^k-)$, $\bar{x}^{k+} = \bar{x}(t^k+)$, $[\bar{x}]^k = \bar{x}^{k+} - \bar{x}^{k-}$. Further, we denote $\bar{z} = (\bar{t}_0, \bar{t}_1, \bar{\xi}, \bar{x}, \bar{u})$, where

$$\bar{t}_0 \in \mathbb{R}^1, \quad \bar{t}_1 \in \mathbb{R}^1, \quad \bar{\xi} \in \mathbb{R}^s, \quad \bar{x} \in P_\Theta W^{1,2}(\Delta, \mathbb{R}^{d(x)}), \quad \bar{u} \in L^2(\Delta, \mathbb{R}^{d(u)}).$$

Thus,

$$\bar{z} \in \mathcal{Z}_2(\Theta) := \mathbb{R}^2 \times \mathbb{R}^s \times P_\Theta W^{1,2}(\Delta, \mathbb{R}^{d(x)}) \times L^2(\Delta, \mathbb{R}^{d(u)}).$$

Moreover, for given $\bar{z}$ we set

$$\bar{w} = (\bar{x}, \bar{u}), \quad \bar{x}_0 = \bar{x}(t_0), \quad \bar{x}_1 = \bar{x}(t_1), \tag{14}$$

$$\bar{\bar{x}}_0 = \bar{x}(t_0) + \bar{t}_0\dot{x}(t_0), \quad \bar{\bar{x}}_1 = \bar{x}(t_1) + \bar{t}_1\dot{x}(t_1), \quad \bar{\bar{p}} = (\bar{\bar{x}}_0, \bar{t}_0, \bar{\bar{x}}_1, \bar{t}_1). \tag{15}$$

By $I_F(p) = \{i \in \{1,\ldots,d(F)\} \mid F_i(p) = 0\}$ we denote the set of active indices of the constraints $F_i(p) \le 0$.

Let $\mathcal{K}$ be the set of all $\bar z \in \mathcal{Z}_2(\Theta)$ satisfying the following conditions:

$$\begin{aligned}
&J'(p)\bar{\bar p} \le 0, \quad F_i'(p)\bar{\bar p} \le 0 \;\forall i \in I_F(p), \quad K'(p)\bar{\bar p} = 0,\\
&\dot{\bar x}(t) = f_w(t,w(t))\bar w(t), \text{ for a.a. } t \in [t_0,t_1],\\
&[\bar x]^k = [\dot x]^k \bar\xi_k, \quad k = 1,\ldots,s\\
&g_w(t,w(t))\bar w(t) = 0, \text{ for a.a. } t \in [t_0,t_1],
\end{aligned} \tag{16}$$

where $p = (t_0, x(t_0), t_1, x(t_1))$, $w = (x,u)$. It is obvious that $\mathcal{K}$ is a convex cone in the Hilbert space $Z_2(\Theta)$, and we call it *the critical cone.* If the interval Δ is fixed, then we set $p := (x_0,x_1) = (x(t_0),x(t_1))$, and in the definition of $\mathcal{K}$ we have $\bar t_0 = \bar t_1 = 0$, $\bar{\bar x}_0 = \bar x_0$, $\bar{\bar x}_1 = \bar x_1$, and $\bar{\bar p} = \bar p := (\bar x_0, \bar x_1)$.

Let us introduce a quadratic form on $\mathcal{Z}_2(\Theta)$. For $\lambda \in M_0$ and $\bar z \in \mathcal{K}$, we set

$$\begin{aligned}
\omega_e(\lambda,\bar z) \;\;=\;\; &\langle l_{pp}\bar{\bar p}, \bar{\bar p}\rangle + 2\dot\psi(t_1)\bar x(t_1)\bar t_1 + \Big(\dot\psi(t_1)\dot x(t_1) + \dot\psi_0(t_1)\Big)\bar t_1^2\\
&-2\dot\psi(t_0)\bar x(t_0)\bar t_0 - \Big(\dot\psi(t_0)\dot x(t_0) + \dot\psi_0(t_0)\Big)\bar t_0^2,
\end{aligned} \tag{17}$$

where $l_{pp} = l_{pp}(p,\alpha_0,\alpha,\beta)$, $p = (t_0,x(t_0),t_1,x(t_1))$. We also set

$$\omega(\lambda,\bar z) = \omega_e(\lambda,\bar z) - \int_{t_0}^{t_1} \langle \bar H_{ww}\bar w(t), \bar w(t)\rangle\, dt, \tag{18}$$

where $\bar H_{ww} = \bar H_{ww}(t,x(t),u(t),\psi(t),\nu(t))$. Finally, we set

$$\Omega(\lambda,\bar z) = \omega(\lambda,\bar z) + \sum_{k=1}^{s}\Big(D^k(\bar H)\bar\xi_k^2 + 2[\dot\psi]^k \bar x_{\mathrm{av}}^k \bar\xi_k\Big), \tag{19}$$

where

$$\bar x_{\mathrm{av}}^k = \frac{1}{2}(\bar x^{k-} + \bar x^{k+}), \quad [\dot\psi]^k = \dot\psi^{k+} - \dot\psi^{k-}.$$

Now, we formulate the main necessary quadratic condition of Pontryagin minimum in the problem on a variable time interval (see Theorem 10.1 in Milyutin and Osmolovskii, 1998, Part 2, p. 289, given there without proof).

Theorem 2.3 *If the trajectory $\mathcal{T}$ yields a Pontryagin minimum, then the following Condition $\mathcal{A}$ holds: the set M_0 is nonempty and*

$$\max_{\lambda \in M_0} \Omega(\lambda,\bar z) \ge 0 \textit{ for all } \bar z \in \mathcal{K}.$$

3. Sufficient conditions of a bounded strong minimum

Let us formulate a sufficient optimality condition $\mathcal{B}$, which is a natural strengthening of the necessary condition $\mathcal{A}$. The condition $\mathcal{B}$ is sufficient not only for a Pontryagin minimum, but also for a strict bounded strong minimum.

To formulate the condition $\mathcal{B}$, we introduce, for $\lambda \in M_0$, the following conditions of the *strict maximum principle*:

$$(MP^+_{\Delta\setminus\Theta}): \qquad H(t,x(t),u,\psi(t)) < H(t,x(t),u(t),\psi(t))$$

if $t \in \Delta \setminus \Theta$, $u \neq u(t)$, $u \in U(t,x(t))$,

$$(MP^+_{\Theta}): \qquad H(t^k,x(t^k),u,\psi(t^k)) < H^k$$

if $t^k \in \Theta$, $u \in U(t^k,x(t^k))$, $u \neq u(t^k-)$, $u \neq u(t^k+)$, where

$$H^k := H^{k-} = H^{k+},$$

$$H^{k-} = H(t^k,x(t^k),u(t^k-),\psi(t^k)), \quad H^{k+} = H(t^k,x(t^k),u(t^k+),\psi(t^k)).$$

We denote by M_0^+ the set of all $\lambda \in M_0$ satisfying conditions $(MP^+_{\Delta\setminus\Theta})$ and (MP^+_{Θ}).

For $\lambda \in M_0$ we also introduce the strengthened Legendre-Clebsch conditions:

Condition $(SLC_{\Delta\setminus\Theta})$: for each $t \in \Delta \setminus \Theta$ the quadratic form

$$-\langle \bar{H}_{uu}(t,x(t),u(t),\psi(t),\nu(t))\bar{u},\bar{u}\rangle$$

is positive definite on the subspace of vectors $\bar{u} \in \mathbb{R}^{d(u)}$ such that

$$g_u(t,x(t),u(t))\bar{u} = 0.$$

Condition (SLC^{k-}): for $t^k \in \Theta$, the quadratic form

$$-\langle \bar{H}_{uu}(t^k,x(t^k),u(t^k-),\psi(t^k),\nu(t^k-))\bar{u},\bar{u}\rangle$$

is positive definite on the subspace of vectors $\bar{u} \in \mathbb{R}^{d(u)}$ such that

$$g_u(t^k,x(t^k),u(t^k-))\bar{u} = 0.$$

Condition (SLC^{k+}) — this condition is symmetric to condition (SLC^{k-}): (t^k-) must be replaced everywhere by (t^k+).

Note that for each $\lambda \in M_0$ the non strengthened Legendre–Clebsch conditions hold, i.e., the same quadratic forms are *nonnegative* on the corresponding subspaces.

We denote by $\mathrm{Leg}_+(M_0^+)$ the set of all $\lambda \in M_0^+$ satisfying the strengthened Legendre–Clebsch conditions $(SLC_{\Delta\setminus\Theta})$, (SLC^{k-}), (SLC^{k+}), $k = 1,\dots,s$, and also the conditions

$$D^k(\bar{H}) > 0 \text{ for all } k = 1,\dots,s. \tag{20}$$

Let us introduce the functional

$$\bar{\gamma}(\bar{z}) = \bar{t}_0^2 + \bar{t}_1^2 + \langle\bar{\xi},\bar{\xi}\rangle + \langle\bar{x}(t_0),\bar{x}(t_0)\rangle + \int_{t_0}^{t_1}\langle\bar{u}(t),\bar{u}(t)\rangle\,dt, \tag{21}$$

which is equivalent to the norm squared on the subspace

$$\dot{\bar{x}} = f_w(t, x(t), u(t))\bar{w}; \quad [\bar{x}]^k = [\dot{x}]^k \bar{\xi}_k, \quad k = 1, \dots, s \tag{22}$$

of Hilbert space $\mathcal{Z}_2(\Theta)$. Recall that the critical cone $\mathcal{K}$ is contained in the subspace (22). The following theorem is given without proof in Milyutin and Osmolovskii (1998), Part 2, Theorem 10.2, p. 293.

THEOREM 3.1 *For the trajectory $\mathcal{T}$, assume that the following Condition $\mathcal{B}$ holds: the set $\mathrm{Leg}_+(M_0^+)$ is nonempty and there exist a nonempty compact set $M \subset \mathrm{Leg}_+(M_0^+)$ and a number $C > 0$ such that*

$$\max_{\lambda \in M} \Omega(\lambda, \bar{z}) \geq C\bar{\gamma}(\bar{z}) \tag{23}$$

for all $\bar{z} \in \mathcal{K}$. Then the trajectory $\mathcal{T}$ affords a strict bounded strong minimum.

4. Proofs

We will give the proofs omitting some details. As it was already mentioned in Introduction, the proofs are based on the quadratic optimality conditions, obtained in Osmolovskii (2004) for problem (1)-(5) considered on a fixed interval of time.

In order to extend the results given in Osmolovskii (2004) to the case of a variable interval $[t_0, t_1]$ we use a simple change of the time variable. Namely, with the fixed admissible trajectory

$$\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_1])$$

in problem on a variable time interval (1)-(5) we associate a trajectory

$$\mathcal{T}^\tau = (v(\tau), t(\tau), x(\tau), u(\tau) \mid \tau \in [\tau_0, \tau_1]),$$

considered on a fixed interval $[\tau_0, \tau_1]$, where

$$\tau_0 = t_0, \quad \tau_1 = t_1, \quad t(\tau) \equiv \tau, \quad v(\tau) \equiv 1.$$

This is an admissible trajectory in the following problem on a fixed interval $[\tau_0, \tau_1]$: to minimize the cost function

$$\mathcal{J}(\mathcal{T}^\tau) := J(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) \to \min \tag{24}$$

subject to the constraints

$$F(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) \leq 0, \quad K(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) = 0, \tag{25}$$

$$\frac{dx(\tau)}{d\tau} = v(\tau) f(t(\tau), x(\tau), u(\tau)), \quad \frac{dt(\tau)}{d\tau} = v(\tau), \quad \frac{dv(\tau)}{d\tau} = 0, \tag{26}$$

$$g(t(\tau), x(\tau), u(\tau)) = 0, \tag{27}$$

$$(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) \in \mathcal{P}, \quad (t(\tau), x(\tau), u(\tau)) \in \mathcal{Q}. \tag{28}$$

In this problem, $x(\tau)$, $t(\tau)$, and $v(\tau)$ are state variables, and $u(\tau)$ is a control variable. For brevity, we will refer to the problem (1)-(5) as the problem P (on a variable interval $\Delta = [t_0, t_1]$), and to the problem (24)-(28) as the problem P^τ (on a fixed interval $[\tau_0, \tau_1]$). We denote by $\mathcal{A}^\tau$ the necessary quadratic condition $\mathcal{A}$ for problem P^τ on a fixed interval $[\tau_0, \tau_1]$, see Osmolovskii (2004), Theorem 4.1. Similarly, we denote by $\mathcal{B}^\tau$ the sufficient quadratic condition $\mathcal{B}$ for problem P^τ on a fixed interval $[\tau_0, \tau_1]$, see Osmolovskii (2004), Theorem 4.2.

Recall that the control $u(\cdot)$ is a piecewise Lipschitz-continuous function on the interval $\Delta = [t_0, t_1]$ with the set of discontinuity points $\Theta = \{t^1, \ldots, t^s\}$, where $t_0 < t^1 < \cdots < t^s < t_1$. Hence, for each $\lambda \in M_0$, the function $\nu(t)$ is also piecewise Lipschitz-continuous on the interval Δ, and, moreover, all discontinuity points of ν belong to Θ. This easily follows from the equation $\bar{H}_u = 0$ and the full rank condition for matrix g_u. Consequently, $\dot{u}$ and $\dot{\nu}$ are bounded measurable functions on Δ.

The proof of Theorem 2.3 is composed of the following chain of implications:

(i) A Pontryagin minimum is attained on the trajectory $\mathcal{T}$ in the problem $P \Rightarrow$

(ii) A Pontryagin minimum is attained on the trajectory $\mathcal{T}^\tau$ in the problem $P^\tau \Longrightarrow$

(iii) Condition $\mathcal{A}^\tau$ for the trajectory $\mathcal{T}^\tau$ in the problem $P^\tau \Longrightarrow$

(iv) Condition $\mathcal{A}$ for the trajectory $\mathcal{T}$ in the problem P.

The first implication is readily verified, the second follows from Theorem 4.1 in Osmolovskii (2004). The verification of the implication (iii) $\Rightarrow$ (iv) is not short and rather technical: we have to compare the sets of Lagrange multipliers, the critical cones and the quadratic forms in both problems. This will be done below.

In order to prove the sufficient conditions in the problem P, given by Theorem 3.1, we have to check the following chain of implications:

(v) Condition $\mathcal{B}$ for the trajectory $\mathcal{T}$ in problem $P \Longrightarrow$

(vi) Condition $\mathcal{B}^\tau$ for the trajectory $\mathcal{T}^\tau$ in problem $P^\tau \Longrightarrow$

(vii) A bounded strong minimum is attained on the trajectory $\mathcal{T}^\tau$ in problem $P^\tau \Longrightarrow$

($viii$) A boundeded strong minimum is attained on the trajectory $\mathcal{T}$ in problem P.

The verification of the first implication here (v) $\Rightarrow$ (vi) is similar to the verification of the third implication (iii) $\Rightarrow$ (iv) in the proof of the necessary conditions, the second implication (vi) $\Rightarrow$ (vii) follows from Theorem 4.2 in Osmolovskii (2004), the third one (vii) $\Rightarrow$ ($viii$) is readily verified.

Thus, it remains to compare the sets of Lagrange multipliers, the critical cones and the quadratic forms in problems P and P^τ for the trajectories $\mathcal{T}$ and $\mathcal{T}^\tau$, respectively.

Comparison of the sets of Lagrange multiplies. Let us formulate the Pontryagin maximum principle in the problem P^τ for the trajectory $\mathcal{T}^\tau$. The endpoint Lagrange function l, the Pontryagin function H and the augmented Pontryagin function $\bar{H}$ (all of them are equipped with the superscript τ) have the form:

$$l^\tau = \alpha_0 J + \alpha F + \beta K = l,$$
$$H^\tau = \psi v f + \psi_0 v + \psi_v \cdot 0 = v(\psi f + \psi_0), \quad \bar{H}^\tau = H^\tau - \nu g.$$

According to Osmolovskii (2004), the set M_0^τ in the problem P^τ for the trajectory $\mathcal{T}^\tau$ consists of all tuples of Lagrange multipliers $\lambda^\tau = (\alpha_0, \alpha, \beta, \psi, \psi_0, \psi_v, \nu)$ such that the following conditions hold:

$$\begin{aligned}
&\alpha_0 + |\alpha| + |\beta| = 1, \\
&-\tfrac{d\psi}{d\tau} = v\psi f_x - \nu g_x, \quad -\tfrac{d\psi_0}{d\tau} = v\psi f_t - \nu g_t, \quad -\tfrac{d\psi_v}{d\tau} = \psi f + \psi_0 \\
&\psi(\tau_0) = l_{x_0}, \quad -\psi(\tau_1) = l_{x_1}, \quad \psi_0(\tau_0) = l_{\tau_0}, \quad -\psi_0(\tau_1) = l_{t_1}, \\
&\psi_v(\tau_0) = \psi_v(\tau_1) = 0, \quad v\psi f_u - \nu g_u = 0, \\
&v(\tau)\big(\psi(\tau) f(t(\tau), x(\tau), u) + \psi_0(\tau)\big) \\
&\le v(\tau)\big(\psi(\tau) f(t(\tau), x(\tau), u(\tau)) + \psi_0(\tau)\big).
\end{aligned} \tag{29}$$

The last inequality holds for all $u \in \mathbb{R}^{d(u)}$ such that $g(t(\tau), x(\tau), u) = 0$, $(t(\tau), x(\tau), u) \in \mathcal{Q}$. Recall that here

$$v(\tau) \equiv 1, \quad t(\tau) \equiv \tau, \quad \tau_0 = t_0, \quad \tau_1 = t_1.$$

In (29), the function f and its derivatives f_x, f_u, f_t, as well as g_x g_u, g_t are taken at $(t(\tau), x(\tau), u(\tau))$, $\tau \in [\tau_0, \tau_1] \setminus \Theta$, while the derivatives l_{t_0}, l_{x_0}, l_{t_1} l_{x_1} are calculated at $(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) = (t_0, x(t_0), t_1, x(t_1))$.

Conditions $-d\psi_v/d\tau = \psi f + \psi_0$ and $\psi_v(\tau_0) = \psi_v(\tau_1) = 0$ imply that $\int_{\tau_0}^{\tau_1} (\psi f + \psi_0)\, d\tau = 0$. As is well-known, conditions (29) of the maximum principle also imply that $\psi f + \psi_0 = \text{const}$, whence $\psi f + \psi_0 = 0$ and $\psi_v = 0$. Taking this fact into account and comparing the definitions of the sets M_0^τ (29) and M_0 (10) we see that the projector

$$\big(\alpha_0, \alpha, \beta, \psi, \psi_0, \psi_v, \nu\big) \to \big(\alpha_0, \alpha, \beta, \psi, \psi_0, \nu\big) \tag{30}$$

realizes a one-to-one correspondence between these two sets. (Moreover, in the definition of the set M_0^τ we could replace the relations $-d\psi_v/d\tau = \psi f + \psi_0$ and $\psi_v(\tau_0) = \psi_v(\tau_1) = 0$ by $\psi f + \psi_0 = 0$ and thus identify M_0^τ with M_0).

We shall say that an element $\lambda^\tau \in M_0^\tau$ *corresponds* to an element $\lambda \in M_0$ if λ is the projection of λ^τ under the mapping (30).

Comparison of the critical cones. For brevity, we set

$$\varrho = (v, t, x, u) = (v, t, w).$$

According to Osmolovskii (2004), the critical cone $\mathcal{K}^\tau$ in the problem P^τ for the trajectory $\mathcal{T}^\tau$ consists of all tuples $(\bar\xi, \bar v, \bar t, \bar x, \bar u) = (\bar\xi, \bar\varrho)$ satisfying the relations:

$$J_{t_0}\bar t(\tau_0) + J_{x_0}\bar x(\tau_0) + J_{t_1}\bar t(\tau_1) + J_{x_1}\bar x(\tau_1) \le 0, \tag{31}$$
$$F_{it_0}\bar t(\tau_0) + F_{ix_0}\bar x(\tau_0) + F_{it_1}\bar t(\tau_1) + F_{ix_1}\bar x(\tau_1) \le 0, \quad i \in I_F(p), \tag{32}$$
$$K_{t_0}\bar t(\tau_0) + K_{x_0}\bar x(\tau_0) + K_{t_1}\bar t(\tau_1) + K_{x_1}\bar x(\tau_1) = 0, \tag{33}$$
$$\frac{d\bar x}{d\tau} = \bar v f + v\big(f_t\bar t + f_x\bar x + f_u\bar u\big),\ [\bar x]^k = [\dot x]^k\bar\xi_k,\ k = 1,\dots s, \tag{34}$$
$$\frac{d\bar t}{d\tau} = \bar v,\ [\bar t]^k = 0,\ k = 1,\dots s, \quad \frac{d\bar v}{d\tau} = 0,\ [\bar v]^k = 0,\ k = 1,\dots s, \tag{35}$$
$$g_t\bar t + g_x\bar x + g_u\bar u = 0, \tag{36}$$

where the derivatives J_{t_0}, J_{x_0}, J_{t_1} J_{x_1}, etc. are calculated at

$$(t(\tau_0), x(\tau_0), t(\tau_1), x(\tau_1)) = (t_0, x(t_0), t_1, x(t_1)),$$

while f, f_t, f_x, f_u g_t, g_x, and g_u are taken at $(t(\tau), x(\tau), u(\tau))$, $\tau \in [\tau_0, \tau_1] \setminus \Theta$.

Let $(\bar\xi, \bar v, \bar t, \bar x, \bar u)$ be an element of the critical cone $\mathcal{K}^\tau$. We will make use of the following change of variables:

$$\tilde x = \bar x - \bar t\dot x, \quad \tilde u = \bar u - \bar t\dot u, \tag{37}$$

or briefly

$$\tilde w = \bar w - \bar t\dot w. \tag{38}$$

Since $v = 1$, $\dot x = f$, and $t = \tau$, equation (34) is equivalent to the equation

$$\frac{d\bar x}{dt} = \bar v\dot x + f_t\bar t + f_w\bar w. \tag{39}$$

Using the relation $\bar x = \tilde x + \bar t\dot x$ in this equation along with $\dot{\bar t} = \bar v$, we get

$$\dot{\tilde x} + \bar t\ddot x = \bar t f_t + f_w\bar w. \tag{40}$$

By differentiating the equation $\dot x(t) = f(t, w(t))$, we obtain

$$\ddot x = f_t + f_w\dot w. \tag{41}$$

Using this relation in (40), we get

$$\dot{\tilde x} = f_w\tilde w. \tag{42}$$

The relations

$$[\bar x]^k = [\dot x]^k\bar\xi_k, \quad \bar x = \tilde x + \bar t\dot x$$

imply

$$[\tilde{x}]^k = [\dot{x}]^k \tilde{\xi}_k, \tag{43}$$

where

$$\tilde{\xi}_k = \bar{\xi}_k - \bar{t}^k, \quad \bar{t}^k = \bar{t}(t^k), \quad k = 1, \dots, s. \tag{44}$$

Further, relation (36) may be written as

$$g_t \bar{t} + g_w \bar{w} = 0.$$

Differentiating the relation $g(t, w(t)) = 0$ we obtain

$$g_t + g_w \dot{w} = 0. \tag{45}$$

These relations along with (38) imply that

$$g_w \tilde{w} = 0. \tag{46}$$

Finally, note that since $\bar{x} = \tilde{x} + \bar{t}\dot{x}$, and $\tau_0 = t_0$, $\tau_1 = t_1$, we have

$$\bar{p} = \big(\bar{t}_0, \bar{x}(t_0), \bar{t}_1, \bar{x}(t_1)\big) = \big(\bar{t}_0, \tilde{x}(t_0) + \bar{t}_0\dot{x}(t_0), \bar{t}_1, \tilde{x}(t_1) + \bar{t}_1\dot{x}(t_1)\big), \tag{47}$$

where $\bar{t}_0 = \bar{t}(t_0)$ and $\bar{t}_1 = \bar{t}(t_1)$. The vector in the r.h.s. of the last equality has the same form as the vector $\bar{\bar{p}}$ in definition (15). Consequently, all relations in definition (16) of the critical cone $\mathcal{K}$ in problem P are satisfied for the element $\tilde{z} = (\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$. We have proved that the thus obtained element $\tilde{z}$ belongs to the critical cone $\mathcal{K}$ in problem P.

Vice versa, if $(\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ is an element of the critical cone in problem P, then by setting

$$\bar{v} = \frac{\bar{t}_1 - \bar{t}_0}{t_1 - t_0}, \quad \bar{t} = \bar{v}(\tau - \tau_0) + \bar{t}_0, \quad \bar{w} = \tilde{w} + \bar{t}\dot{w}, \quad \bar{\xi}_k = \tilde{\xi}_k + \bar{t}(\tau_k),\ k = 1, \dots, s,$$

we obtain an element $(\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ of the critical cone (31)-(36) in the problem P^τ. Thus, we have proved the following lemma:

Lemma 4.1 *If $(\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ is an element of the critical cone (31)-(36) in problem P^τ for the trajectory $\mathcal{T}^\tau$ and*

$$\begin{aligned} &\bar{t}_0 = \bar{t}(t_0), \quad \bar{t}_1 = \bar{t}(t_1), \quad \tilde{w} = \bar{w} - \bar{t}\dot{w}, \\ &\tilde{\xi}_k = \bar{\xi}_k - \bar{t}(t^k), \quad k = 1, \dots, s, \end{aligned} \tag{48}$$

then $(\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ is an element of the critical cone (16) in the problem P for the trajectory $\mathcal{T}$. Moreover, relations (48) define a one-to-one correspondence between elements of the critical cones in problems P^τ and P.

We shall say that an element $(\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ of the critical cone in problem P^τ *corresponds* to an element $(\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ of the critical cone in problem P if relations (48) hold.

Comparison of the quadratic forms. Assume that the element $\lambda^\tau \in M_0^\tau$ corresponds to the element $\lambda \in M_0$. Let us show that the quadratic form $\Omega^\tau(\lambda^\tau, \cdot)$, calculated on the element $(\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ of the critical cone in the problem P^τ for the trajectory $\mathcal{T}^\tau$ (see Osmolovskii, 2004), can be transformed to the quadratic form $\Omega(\lambda, \cdot)$ calculated on the corresponding element $(\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ of the critical cone in the problem P for the trajectory $\mathcal{T}$.

(i) The relations

$$\bar{H}^\tau = v(H + \psi_0) - \nu g, \quad \bar{H} = H - \nu g, \quad v = 1$$

imply

$$\langle \bar{H}^\tau_{\varrho\varrho}\bar{\varrho}, \bar{\varrho}\rangle = \langle \bar{H}_{ww}\bar{w}, \bar{w}\rangle + 2\bar{H}_{tw}\bar{w}\bar{t} + \bar{H}_{tt}\bar{t}^2 + 2\bar{v}(H_w\bar{w} + H_t\bar{t}), \tag{49}$$

where $\varrho = (v, t, w)$, $\bar{\varrho} = (\bar{v}, \bar{t}, \bar{w})$. Since $\tilde{w} = \bar{w} - \bar{t}\dot{w}$, we have

$$\langle \bar{H}_{ww}\bar{w}, \bar{w}\rangle = \langle \bar{H}_{ww}\tilde{w}, \tilde{w}\rangle + 2\langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle\bar{t} - \langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle\bar{t}^2. \tag{50}$$

Moreover, using the relations

$$\begin{aligned}
&H_w = \bar{H}_w + \nu g_w, \quad H_t = \bar{H}_t + \nu g_t, \quad g_w\bar{w} + g_t\bar{t} = 0,\\
&-\dot{\psi} = \bar{H}_x, \quad -\dot{\psi}_0 = \bar{H}_t, \quad \bar{H}_u = 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
&H_w\bar{w} + H_t\bar{t} = \bar{H}_w\bar{w} + \bar{H}_t\bar{t} + \nu(g_w\bar{w} + g_t\bar{t})\\
&= \bar{H}_w\bar{w} + \bar{H}_t\bar{t} = \bar{H}_x\bar{x} + \bar{H}_t\bar{t} = -\dot{\psi}\bar{x} - \dot{\psi}_0\bar{t}.
\end{aligned} \tag{51}$$

Relations (49)-(51) imply

$$\begin{aligned}
&\langle \bar{H}^\tau_{\varrho\varrho}\bar{\varrho}, \bar{\varrho}\rangle = \langle \bar{H}_{ww}\tilde{w}, \tilde{w}\rangle + 2\langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle\bar{t} + 2\bar{H}_{tw}\bar{w}\bar{t}\\
&-\langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle\bar{t}^2 + \bar{H}_{tt}\bar{t}^2 - 2\bar{v}\big(\dot{\psi}\bar{x} + \dot{\psi}_0\bar{t}\big).
\end{aligned} \tag{52}$$

(ii) Let us transform the terms $2\langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle\bar{t} + 2\bar{H}_{tw}\bar{w}\bar{t}$ in (52). By differentiating the equation $-\dot{\psi} = \bar{H}_x$ with respect to t, we obtain

$$-\ddot{\psi} = \bar{H}_{tx} + (\dot{w})^*\bar{H}_{wx} + \dot{\psi}\bar{H}_{\psi x} + \dot{\nu}\bar{H}_{\nu x}.$$

Here we have $\bar{H}_{\psi x} = f_x$ and $\bar{H}_{\nu x} = -g_x$. Therefore

$$-\ddot{\psi} = \bar{H}_{tx} + (\dot{w})^*\bar{H}_{wx} + \dot{\psi}f_x - \dot{\nu}g_x. \tag{53}$$

Similarly, by differentiating the equation $\bar{H}_u = 0$ with respect to t, we obtain

$$0 = \bar{H}_{tu} + (\dot{w})^*\bar{H}_{wu} + \dot{\psi}f_u - \dot{\nu}g_u. \tag{54}$$

Multiplying equation (53) by $\bar{x}$ and equation (54) by $\bar{u}$ and summing the results we get

$$-\ddot{\psi}\bar{x} = \bar{H}_{tw}\bar{w} + \langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle + \dot{\psi} f_w \bar{w} - \dot{\nu} g_w \bar{w}. \tag{55}$$

Since $(\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ is an element of the critical cone in the problem P^τ, from (34) and (36) we get

$$f_w\bar{w} = \dot{\bar{x}} - \bar{v}\dot{x} - f_t\bar{t}, \quad g_w\bar{w} = -g_t\bar{t}.$$

Therefore, equation (55) can be represented in the form

$$\bar{H}_{tw}\bar{w} + \langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle = \bar{v}(\dot{\psi}\dot{x}) - \frac{d}{dt}(\dot{\psi}\bar{x}) + \big(\dot{\psi} f_t - \dot{\nu} g_t\big)\bar{t}, \tag{56}$$

which implies

$$2\langle \bar{H}_{ww}\dot{w}, \bar{w}\rangle\bar{t} + 2\bar{H}_{tw}\bar{w}\bar{t} = 2\bar{t}\bar{v}(\dot{\psi}\dot{x}) - 2\bar{t}\frac{d}{dt}(\dot{\psi}\bar{x}) + 2\big(\dot{\psi} f_t - \dot{\nu} g_t\big)\bar{t}^2. \tag{57}$$

(iii) Let us transform the term $-\langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle\bar{t}^2$ in (52). Multiplying equation (53) by $\dot{x}$ and equation (54) by $\dot{u}$ and summing the results we obtain

$$-\ddot{\psi}\dot{x} = \bar{H}_{tw}\dot{w} + \langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle + \dot{\psi} f_w \dot{w} - \dot{\nu} g_w \dot{w}. \tag{58}$$

From (41) and (45) we get

$$f_w\dot{w} = \ddot{x} - f_t, \quad g_w\dot{w} = -g_t,$$

respectively. Then (58) implies

$$\bar{H}_{tw}\dot{w} + \langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle = -\frac{d}{dt}(\dot{\psi}\dot{x}) + \big(\dot{\psi} f_t - \dot{\nu} g_t\big). \tag{59}$$

Multiplying this relation by $-\bar{t}^2$ we get

$$-\langle \bar{H}_{ww}\dot{w}, \dot{w}\rangle\bar{t}^2 = \bar{H}_{tw}\dot{w}\bar{t}^2 + \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) - \big(\dot{\psi} f_t - \dot{\nu} g_t\big)\bar{t}^2. \tag{60}$$

(iv) Finally, let us transform the term $\bar{H}_{tt}\bar{t}^2$ in (52). Differentiating the equation $-\dot{\psi}_0 = \bar{H}_t$ with respect to t and using the relations $\bar{H}_{\psi t} = f_t$ and $\bar{H}_{\nu t} = -g_t$, we get

$$-\ddot{\psi}_0 = \bar{H}_{tt} + \bar{H}_{tw}\dot{w} + \big(\dot{\psi} f_t - \dot{\nu} g_t\big). \tag{61}$$

Consequently,

$$\bar{H}_{tt}\bar{t}^2 = -\ddot{\psi}_0\bar{t}^2 - \bar{H}_{tw}\dot{w}\bar{t}^2 - \big(\dot{\psi} f_t - \dot{\nu} g_t\big)\bar{t}^2. \tag{62}$$

(v) Summing equations (60) and (62) we obtain

$$-\langle \bar{H}_{ww}\dot{w},\dot{w}\rangle \bar{t}^2 + \bar{H}_{tt}\bar{t}^2 = -\ddot{\psi}_0\bar{t}^2 - 2\big(\dot{\psi}f_t - \dot{\nu}g_t\big)\bar{t}^2 + \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}). \tag{63}$$

Using relations (57) and (63) in (52) we get

$$\begin{aligned}&\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle = \langle \bar{H}_{ww}\tilde{w},\tilde{w}\rangle + 2\bar{t}\bar{v}(\dot{\psi}\dot{x})\\&-2\bar{t}\frac{d}{dt}(\dot{\psi}\bar{x}) - \ddot{\psi}_0\bar{t}^2 + \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) - 2\bar{v}\big(\dot{\psi}\bar{x}+\dot{\psi}_0\bar{t}\big).\end{aligned} \tag{64}$$

But

$$\begin{aligned}&\ddot{\psi}_0\bar{t}^2 + 2\bar{v}\bar{t}\dot{\psi}_0 = \frac{d}{dt}\big(\dot{\psi}_0\bar{t}^2\big), \quad \bar{t}\frac{d}{dt}(\dot{\psi}\bar{x}) + \bar{v}(\dot{\psi}\bar{x}) = \frac{d}{dt}\Big(\bar{t}\dot{\psi}\bar{x}\Big),\\&2\bar{t}\bar{v}(\dot{\psi}\dot{x}) + \bar{t}^2\frac{d}{dt}(\dot{\psi}\dot{x}) = \frac{d}{dt}(\dot{\psi}\dot{x}\bar{t}^2).\end{aligned}$$

Therefore,

$$\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle = \langle \bar{H}_{ww}\tilde{w},\tilde{w}\rangle + \frac{d}{dt}\Big((\dot{\psi}\dot{x})\bar{t}^2 - \dot{\psi}_0\bar{t}^2 - 2\dot{\psi}\bar{x}\bar{t}\Big). \tag{65}$$

Finally, using the change of the variable $\bar{x} = \tilde{x} + \bar{t}\dot{x}$ in the r.h.s. of this relation, we obtain

$$\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle = \langle \bar{H}_{ww}\tilde{w},\tilde{w}\rangle - \frac{d}{dt}\Big((\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}^2 + 2\dot{\psi}\tilde{x}\bar{t}\Big). \tag{66}$$

We have proved the following lemma.

LEMMA 4.2 *Let $(\bar{\xi},\bar{v},\bar{t},\bar{w}) = (\bar{\xi},\bar{\varrho})$ be an element of the critical cone $\mathcal{K}^{\tau}$ in the problem P^{τ} for the trajectory $\mathcal{T}^{\tau}$. Set $\tilde{w} = \bar{w} - \bar{t}\dot{w}$. Then, formula (66) holds.*

(vi) Recall that λ^{τ} is an arbitrary element of the set M_0^{τ} (consequently $\psi_v = 0$) and λ is the corresponding element of the set M_0, i.e., λ is the projection of λ^{τ} under the mapping (30). The quadratic form $\Omega^{\tau}(\lambda^{\tau},\cdot)$ in the problem P^{τ} for the trajectory $\mathcal{T}^{\tau}$ has the following representation (see Osmolovskii, 2004):

$$\begin{aligned}\Omega^{\tau}(\lambda^{\tau};\bar{\xi},\bar{\varrho}) \quad = \quad & \sum_{k=1}^{s}\Big(D^k(\bar{H}^{\tau})\bar{\xi}_k^2 + 2[\dot{\psi}]^k\bar{x}^k_{\mathrm{av}}\bar{\xi}_k + 2[\dot{\psi}_0]^k\bar{t}^k_{\mathrm{av}}\bar{\xi}_k\Big)\\&+\langle l_{pp}\bar{p},\bar{p}\rangle - \int_{\tau_0}^{\tau_1}\langle \bar{H}^{\tau}_{\varrho\varrho}\bar{\varrho},\bar{\varrho}\rangle\, d\tau.\end{aligned} \tag{67}$$

Comparing the definitions of $D^k(\bar{H}^{\tau})$ and $D^k(\bar{H})$ (see (13)) and taking into account that $\bar{H}^{\tau} = v(\psi f + \psi_0) - \nu g$ and $v = 1$, we get

$$D^k(\bar{H}^{\tau}) = D^k(\bar{H}). \tag{68}$$

Let $\bar{z}^\tau = (\bar{\xi}, \bar{\varrho}) = (\bar{\xi}, \bar{v}, \bar{t}, \bar{x}, \bar{u})$ be an element of the critical cone $\mathcal{K}^\tau$ in the problem P^τ for the trajectory $\mathcal{T}^\tau$ and let $\tilde{z} = (\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{x}, \tilde{u})$ be the corresponding element of the critical cone $\mathcal{K}$ in the problem P for the trajectory $\mathcal{T}$, i.e., relations (48) hold. Since $[\bar{t}]^k = 0$, $k = 1, \dots, s$, we have

$$\bar{t}^k_{\rm av} = \bar{t}^k, \quad k = 1, \dots, s \tag{69}$$

where $\bar{t}^k = \bar{t}(t^k)$, $k = 1, \dots, s$. Also recall that $\tau_0 = t_0$, $\tau_1 = t_1$, $t(\tau) = \tau$, $\quad dt = d\tau$. Since the functions $\dot{\psi}_0$, $\dot{\psi}$, $\dot{x}$, and $\bar{x}$ may have discontinuities only at the points of the set Θ, the following formula holds:

$$\begin{aligned} &\int_{t_0}^{t_1} \tfrac{d}{dt}\Big((\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}^2 + 2\dot{\psi}\bar{x}\bar{t}\Big)\,dt \\ &= \Big((\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}^2 + 2\dot{\psi}\tilde{x}\bar{t}\Big)\,\Big|_{t_0}^{t_1} - \sum_{k=1}^{s}\Big([\dot{\psi}_0 + \dot{\psi}\dot{x}]^k\bar{t}(t^k)^2 + 2[\dot{\psi}\tilde{x}]^k\bar{t}(t^k)\Big). \end{aligned} \tag{70}$$

Relations (66)-(70) imply the following representation of the quadratic form Ω^τ on the element $(\bar{\xi}, \bar{\varrho})$ of the critical cone $\mathcal{K}^\tau$:

$$\begin{aligned} \Omega^\tau(\lambda^\tau; \bar{\xi}, \bar{\varrho}) = &\sum_{k=1}^{s}\Big(D^k(\bar{H})\bar{\xi}_k^2 + 2[\dot{\psi}]^k\bar{x}^k_{av}\bar{\xi}_k + 2[\dot{\psi}_0]^k\bar{t}(t^k)\bar{\xi}_k \\ &- [\dot{\psi}_0 + \dot{\psi}\dot{x}]^k\bar{t}(t^k)^2 - 2[\dot{\psi}\tilde{x}]^k\bar{t}(t^k)\Big) + \langle l_{pp}\bar{p}, \bar{p}\rangle \\ &+ \Big((\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}^2 + 2\dot{\psi}\tilde{x}\bar{t}\Big)\,\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\langle \bar{H}_{ww}\tilde{w}, \tilde{w}\rangle\,d\tau. \end{aligned} \tag{71}$$

Let us transform the terms related to the discontinuity points t^k of the control $u(\cdot)$, $k = 1, \dots, s$. For any $\lambda \in M_0$, the following lemma holds.

Lemma 4.3 *Let $\bar{z} = (\bar{\xi}, \bar{\varrho}) = (\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ be an element of the critical cone $\mathcal{K}^\tau$ in the problem P^τ for the trajectory $\mathcal{T}^\tau$. Let the pair $(\tilde{\xi}, \tilde{x})$ be defined by the relations*

$$\tilde{\xi}_k = \bar{\xi}_k - \bar{t}(t^k), \quad k = 1, \dots, s, \quad \tilde{x} = \bar{x} - \bar{t}\dot{x}. \tag{72}$$

Then for any $k = 1, \dots, s$ the following formula holds

$$\begin{aligned} &D^k(\bar{H})\bar{\xi}_k^2 + 2[\dot{\psi}]^k\bar{x}^k_{av}\bar{\xi}_k + 2[\dot{\psi}_0]^k\bar{t}(t^k)\bar{\xi}_k - [\dot{\psi}_0 + \dot{\psi}\dot{x}]^k\bar{t}(t^k)^2 - 2[\dot{\psi}\tilde{x}]^k\bar{t}(t^k) \\ &= D^k(\bar{H})\tilde{\xi}_k^2 + 2[\dot{\psi}]^k\tilde{x}^k_{\rm av}\tilde{\xi}_k. \end{aligned} \tag{73}$$

Proof. Everywhere in this proof we will omit the subscript and superscript k. We will also write $\bar{t}$ instead of $\bar{t}(t^k)$. Set $a = D(\bar{H})$. Using the relations

$$\bar{\xi} = \tilde{\xi} + \bar{t}, \quad \bar{x}_{\rm av} = \tilde{x}_{\rm av} + \bar{t}\dot{x}_{\rm av}, \tag{74}$$

we obtain

$$\begin{aligned} & a\bar{\xi}^2 + 2[\dot{\psi}]\bar{x}_{av}\bar{\xi} + 2[\dot{\psi}_0]\bar{t}\bar{\xi} - [\dot{\psi}_0 + \dot{\psi}\dot{x}]\bar{t}^2 - 2[\dot{\psi}\tilde{x}]\bar{t} \\ = \; & a\tilde{\xi}^2 + 2a\tilde{\xi}\bar{t} + a\bar{t}^2 + 2[\dot{\psi}]\tilde{x}_{av}\bar{\xi} + 2[\dot{\psi}]\dot{x}_{av}\bar{t}\bar{\xi} \\ & +2[\dot{\psi}_0]\bar{t}\bar{\xi} - [\dot{\psi}_0 + \dot{\psi}\dot{x}]\bar{t}^2 - 2[\dot{\psi}\tilde{x}]\bar{t} \\ = \; & a\tilde{\xi}^2 + 2[\dot{\psi}]\tilde{x}_{av}\tilde{\xi} + r, \end{aligned} \tag{75}$$

where

$$r = 2a\tilde{\xi}\bar{t} + a\bar{t}^2 + 2[\dot{\psi}]\tilde{x}_{av}\bar{t} + 2[\dot{\psi}]\dot{x}_{av}\bar{t}\bar{\xi} + 2[\dot{\psi}_0]\bar{t}\bar{\xi} - [\dot{\psi}_0 + \dot{\psi}\dot{x}]\bar{t}^2 - 2[\dot{\psi}\tilde{x}]\bar{t}. \tag{76}$$

It suffices to show that $r = 0$. Using the relations (74) in formula (76), we get

$$\begin{aligned} r \; & = 2a(\bar{\xi} - \bar{t})\bar{t} + a\bar{t}^2 + 2[\dot{\psi}](\bar{x}_{av} - \bar{t}\dot{x}_{av})\bar{t} + 2[\dot{\psi}]\dot{x}_{av}\bar{t}\bar{\xi} + 2[\dot{\psi}_0]\bar{t}\bar{\xi} \\ & -[\dot{\psi}_0 + \dot{\psi}\dot{x}]\bar{t}^2 - 2[\dot{\psi}(\bar{x} - \bar{t}\dot{x})]\bar{t} \\ & = \bar{t}^2\big(- a - 2[\dot{\psi}]\dot{x}_{\mathrm{av}} - [\dot{\psi}_0] + [\dot{\psi}\dot{x}]\big) + 2\bar{t}\bar{\xi}\big(a + [\dot{\psi}]\dot{x}_{\mathrm{av}} + [\dot{\psi}_0]\big) \\ & +2\bar{t}\big([\dot{\psi}]\bar{x}_{\mathrm{av}} - [\dot{\psi}\bar{x}]\big). \end{aligned}$$

The coefficient of $\bar{t}^2$ in the r.h.s. of the last equality vanishes:

$$\begin{aligned} & [\dot{\psi}\dot{x}] - 2[\dot{\psi}]\dot{x}_{av} - a - [\dot{\psi}_0] = \dot{\psi}^+\dot{x}^+ - \dot{\psi}^-\dot{x}^- - (\dot{\psi}^+ - \dot{\psi}^-)(\dot{x}^+ + \dot{x}^-) \\ & -\big(\dot{\psi}^-\dot{x}^+ - \dot{\psi}^+\dot{x}^- - [\dot{\psi}_0]\big) - [\dot{\psi}_0] = 0. \end{aligned}$$

The coefficient of $2\bar{t}\bar{\xi}$ is equal to

$$\begin{aligned} & a + [\dot{\psi}]\dot{x}_{\mathrm{av}} + [\dot{\psi}_0] \\ & = \dot{\psi}^-\dot{x}^+ - \dot{\psi}^+\dot{x}^- - [\dot{\psi}_0] + \tfrac{1}{2}(\dot{\psi}^+ - \dot{\psi}^-)(\dot{x}^- + \dot{x}^+) + [\dot{\psi}_0] \\ & = -\tfrac{1}{2}\big(\dot{\psi}^+\dot{x}^- - \dot{\psi}^-\dot{x}^+\big) + \tfrac{1}{2}[\dot{\psi}\dot{x}]. \end{aligned}$$

The coefficient of $2\bar{t}$ is equal to

$$\begin{aligned} & [\dot{\psi}]\bar{x}_{\mathrm{av}} - [\dot{\psi}\bar{x}] \\ & = \tfrac{1}{2}(\dot{\psi}^+ - \dot{\psi}^-)(\bar{x}^- + \bar{x}^+) - (\dot{\psi}^+\bar{x}^+ - \dot{\psi}^-\bar{x}^-) \\ & = -\tfrac{1}{2}\dot{\psi}^+[\bar{x}] - \tfrac{1}{2}\dot{\psi}^-[\bar{x}] = -\dot{\psi}_{av}[\dot{x}]\bar{\xi}, \end{aligned}$$

since $[\bar{x}] = [\dot{x}]\bar{\xi}$. Consequently,

$$\begin{aligned} r \; = \; & 2\bar{t}\bar{\xi}\big(a + [\dot{\psi}]\dot{x}_{\mathrm{av}} + [\dot{\psi}_0]\big) + 2\bar{t}\big([\dot{\psi}]\bar{x}_{\mathrm{av}} - [\dot{\psi}\bar{x}]\big) \\ = \; & 2\bar{t}\bar{\xi}\big(- \tfrac{1}{2}\big(\dot{\psi}^+\dot{x}^- - \dot{\psi}^-\dot{x}^+\big) + \tfrac{1}{2}[\dot{\psi}\dot{x}] - \dot{\psi}_{av}[\dot{x}]\big) \\ = \; & \bar{t}\bar{\xi}\big(- (\dot{\psi}^+\dot{x}^- - \dot{\psi}^-\dot{x}^+) + (\dot{\psi}^+\dot{x}^+ - \dot{\psi}^-\dot{x}^-) \\ & -(\dot{\psi}^- + \dot{\psi}^+)(\dot{x}^+ - \dot{x}^-)\big) = 0. \end{aligned}$$

In view of (75) the equality $r = 0$ proves the lemma. ∎

Relation (71) along with equality (73) gives the following transformation of quadratic form Ω^τ (67) on the element $\bar{z}^\tau = (\bar{\xi}, \bar{\varrho})$ of the critical cone $\mathcal{K}^\tau$

$$\begin{aligned}\Omega^\tau(\lambda^\tau; \bar{\xi}, \bar{\varrho}) = \sum_{k=1}^{s} \left(D^k(\bar{H})\tilde{\xi}_k^2 + 2[\dot{\psi}]^k \tilde{x}_{\mathrm{av}}^k \tilde{\xi}_k\right) \\ +\langle l_{pp}\bar{p}, \bar{p}\rangle + \Big((\dot{\psi}_0 + \dot{\psi}\dot{x})\bar{t}^2 + 2\dot{\psi}\tilde{x}\bar{t}\Big)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \langle \bar{H}_{ww}\tilde{w}, \tilde{w}\rangle \, d\tau.\end{aligned} \tag{77}$$

Taking into account (47) and definitions (17)-(19) of quadratic forms ω_e, ω, and Ω, we see that the r.h.s. of (77) is the quadratic form $\Omega(\lambda, \tilde{z})$ (19) in problem P for the trajectory $\mathcal{T}$, where $\tilde{z} = (\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ is the corresponding element of the critical cone $\mathcal{K}$. Thus we have proved the following theorem.

THEOREM 4.1 *Let $\bar{z}^\tau = (\bar{\xi}, \bar{v}, \bar{t}, \bar{w})$ be an element of the critical cone $\mathcal{K}^\tau$ in the problem P^τ for the trajectory $\mathcal{T}^\tau$. Let $\tilde{z} = (\bar{t}_0, \bar{t}_1, \tilde{\xi}, \tilde{w})$ be the corresponding element of the critical cone $\mathcal{K}$ in the problem P for the trajectory $\mathcal{T}$, i.e., relations (48) holds. Then for any $\lambda^\tau \in M_0^\tau$ and the corresponding projection $\lambda \in M_0$ (under the mapping (30)) the following equality holds*

$$\Omega^\tau(\lambda^\tau, \bar{z}^\tau) = \Omega(\lambda, \tilde{z}).$$

This theorem proves the implications $(iii) \Rightarrow (iv)$ and $(v) \Rightarrow (vi)$ (see the beginning of this section), and thus completes the proofs of Theorems 2.3 and 3.1.

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