Control and Cybernetics

vol. **38** (2009) No. 4B

Optimal regularity and optimal control of a thermoelastic structural acoustic model with point control and clamped boundary conditions*

by

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Abstract: In this paper we consider point control of a structural acoustic model with thermoelastic effects. The key feature of this paper is that the two-dimensional plate modeling the active wall of the acoustic chamber has clamped boundary conditions. For this case a new optimal regularity result has recently become available (Triggiani, 2008). Using this new result for the plate alone, we derive a sharp (optimal) regularity result for the overall coupled system of wave and thermoelastic plate equations, after overcoming a series of additional technical difficulties. This allows for the study of an optimal control problem of the coupled system.

Keywords: point control, parabolic/hyperbolic thermoelastic system, hyperbolic/hyperbolic chamber/wall coupling.

1. Introduction

In this paper we consider point control of a structural acoustic model with thermoelastic effects. This type of model arises in engineering applications in which one is attempting to control noise in an acoustic chamber by means of a feedback mechanism on the active wall. Here we examine a canonical case where the 2-dimensional active wall is modeled by means of a *clamped* thermoelastic plate. The natural equation for the thermoelastic structural acoustic problem is a three-dimensional chamber with an active wall modeled by a two-dimensional plate with clamped boundary conditions. This is precisely the case for which a new optimal regularity result has recently become available (Triggiani, 2008). Indeed, optimal interior regularity for the thermoelastic equations of dimension

*Submitted: April 2009; Accepted: November 2009.

$n = 1$ and $n = 3$ was shown in Triggiani (2007b), but the $n = 2$ case suffered from a loss of ϵ due to the incompatibility of the boundary conditions of the spaces $H_0^{\frac{3}{2}}(\Omega)$ and $H_{00}^{\frac{3}{2}}(\Omega)$; see Eqns. (3.6) and (3.7) below. This loss was rectified in Triggiani (2008), where the optimal regularity was derived using technical analysis based on sharp trace regularity theory of the Kirchhoff and wave equations (Lasiecka, Lions and Triggiani, 1986; Lasiecka and Triggiani, 1991, 2000; Ourada and Triggiani, 1991; Lagnese, 1989). The approach used in Triggiani (2008) to analyze the $n = 2$ case is very different from that taken in Triggiani (2007b). For the cases of $n = 1$ and 3, the optimal result is produced—in the final analysis, after decoupling the wave and thermal dynamics—by a thermoelastic semigroup approach as applied to a 'right-hand side input.' However, for $n = 2$, this last step is responsible for a loss of ε in the final regularity, as noted before. Hence, one needs to work from boundary to interior, first changing variables to obtain a purely elastic problem and an associated z-thermoelastic problem. Then, optimal interior regularity for this new z-variable is obtained by using a very special boundary regularity for $\Delta z|_\Sigma$. This trace regularity requires a technical argument involving two pseudodifferential operators (Triggiani, 2008).

Our current work considers the case where such a clamped thermoelastic plate equation is coupled to a three-dimensional acoustic chamber. More precisely, part of the wall of the chamber is modelled by a thermoelastic plate, and the other part is considered a 'hard' wall. Control of the wave equation inside the chamber is accomplished by means of a point control on the thermoelastic wall. Various related models of plate and waves have been studied for some time (Avalos and Lasiecka, 1996, 1997, 2003; Bucci, 2007; Camurdan, 1999; Camurdan and Ji, 2000; Camurdan and Triggiani, 1999; Triggiani, 1997; Lasiecka, 2002; Lebiedzik, 2000, 2001; Lasiecka and Lebiedzik, 1999; Lasiecka and Triggiani, 2000c,d, Section 9.10, p. 844) with or without thermal effects. However, unlike other models of the literature, the present structural acoustic system constitutes a coupling of *hyperbolic*- thermally damped *hyperbolic* dynamics; *vis-á-vis* the *parabolic/hyperbolic* PDE models seen in the existing literature for the modelling of structural acoustic flows. Thermoelasticity has a natural damping effect which can be used to stabilize these systems (Avalos and Lasiecka, 1997; Lasiecka, 2002; Lasiecka and Lebiedzik, 1999; Lebiedzik, 2000, 2001). In analyzing these coupled systems of waves and plates, it can be seen that the coupling of the two equations can produce phenomena that do not appear with waves and plates alone. The precursor was the paper of Avalos and Lasiecka (1996), where the flexible wall is a damped Euler-Bernoulli plate equation, so that the overall system is *parabolic/hyperbolic*. Problems studied include: regularity, stabilization, singular estimates, optimal control and min-max game theory problems. More relevant to the present work is the paper by Camurdan and Triggiani (1999) (see also Triggiani, 1997), where optimal regularity of the overall structural acoustic model is studied, with a point control acting on the *elastic* (flexible) wall. Here we generalize this result to the case where the flexible wall is, in fact, *thermoelastic*. We note explicitly the important feature that, due to the presence of the constant $\gamma > 0$ in equation (2.1d) below, the thermoelastic problem (2.1d,e,f) on the active wall with $u \equiv 0$ and no coupling term $-z_t$ is *hyperbolic-dominated*: its free dynamics is described by a C_0- contractive uniformly stable *group* of operators (based on only the mechanical variables) perturbed by a *compact* term (Lasiecka and Triggiani, 2000b). The constant γ accounts for rotational forces in the model and is proportional to the square of the thickness of the plate in the two-dimensional case. In contrast, the case $\gamma = 0$ in the model corresponds to an *analytic* semigroup, that is, parabolic behavior (Lasiecka and Triggiani, 2000c, Appendices 3E, 3F, 3G, 3H, 3I to Chapter 3, pp. 324–401, 1998a,b,c, 2001) in fact under all canonical boundary conditions.

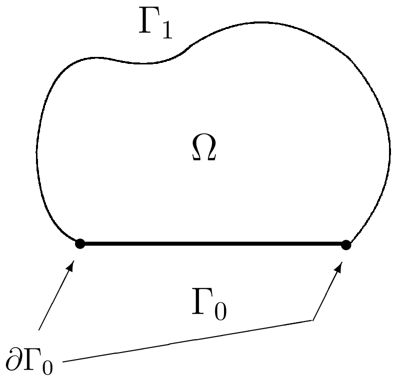

Figure 1. Cross-section of a sample domain Ω

Orientation. Following the approach in Camurdan and Triggiani (1999), Lasiecka and Triggiani (2000d, Section 9.10, p. 884), Triggiani (2007a,b, 2008), the first step of the present work consists of decoupling the wave and the thermoelastic equation to look at the optimal regularity of each part separately. The advantage of this procedures is that the point control is dealt with only within the thermoelastic model alone, rather than in the overall coupled structural acoustic-with-thermoelastic-wall problem. The overall solution of the problem of the main Theorem 2.1, or Theorem 4.3 encounters an unexpected number of technical issues and difficulties: the sharp regularity under point control of the (uncoupled) thermoelastic problem in dimension 2 (Triggiani, 2008); the sharp regularity of the mixed problem for the wave equation with Neumann boundary control (Lasiecka and Triggiani, 1990, 1991a, 1994; Tataru, 1998); and the role played by the factor space $\tilde{L}_2(\Gamma_0)$ (Laciecka and Triggiani, 2001) The definition of $\tilde{L}_2(\Gamma_0)$ is recalled in Step 4, equations (5.19), properties *(i), (ii)* below equation (5.20), in the proof of Theorem 4.3 in Section 5. The latter appears precisely due to the presence of clamped boundary conditions. Due to the pathology associated with clamped boundary conditions expressed by the space $\tilde{L}_2(\Gamma_0)$ (the "visible" portion of $L_2(\Gamma_0)$), we need to establish a new additional interpolation result (Proposition 3.1b below) to obtain the optimal regularity result of Theorem 4.3 (see also Proposition B.1, Appendix B). All this requires the preliminary critical "trick" in rewriting the semigroup solution as in (5.43) through (5.45).

2. Model equations. Main regularity result: a Sobolev space version. Significance

Mathematical model. Let Ω be a general three-dimensional bounded domain, whose smooth boundary Γ is divided into two parts, Γ_0 and Γ_1, $\Gamma = \Gamma_0 \cup \Gamma_1$. The portion of the boundary acting as the hard wall is Γ_1, and Γ_0 is the flat portion acting as the moving wall clamped at its edges.

The model considered consists of the acoustic wave equation on Ω in the variable z coupled to the thermoelastic equation on Γ_0. Here, w is the vertical displacement of the wall, θ is the thermal stress resultant, $\gamma > 0$ is a constant (see Introduction), $u(t)$ is the scalar control function, and $\delta(x_0)$ is the Dirac δ-function at x_0.

$$\text{chamber } \Omega : \begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T] \times \Omega \equiv Q, & (2.1\text{a}) \\ \dfrac{\partial}{\partial \nu} z = 0 \quad \text{on } (0,T] \times \Gamma_1 \equiv \Sigma_1, & (2.1\text{b}) \\ \dfrac{\partial}{\partial \nu} z = -w_t \quad \text{on } (0,T] \times \Gamma_0 \equiv \Sigma_0; & (2.1\text{c}) \end{cases}$$

$$\text{wall } \Gamma_0 : \begin{cases} w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w + \Delta\theta - z_t = \delta(x_0) u(t) \quad \text{on } \Sigma_0, & (2.1\text{d}) \\ \theta_t - \Delta\theta - \Delta w_t = 0 \quad \text{on } \Sigma_0, & (2.1\text{e}) \\ w = 0; \quad \dfrac{\partial}{\partial \nu} w = 0; \quad \theta = 0 \quad \text{on } (0,T] \times \partial\Gamma_0; & (2.1\text{f}) \end{cases}$$

$$\text{I.C.} \quad z(0,\cdot) = z_0;\ z_t(0,\cdot) = z_1 \text{ in } \Omega;\ w(0,\cdot) = w_0;\ w_t(0,\cdot) = w_1; \quad \theta(0,\cdot) = \theta_0 \text{ in } \Gamma_0. \qquad (2.1\text{g})$$

REMARK 2.1 *Integrating equation (2.1a) in* $(0,t] \times \Omega$ *yields the following invariance property of the dynamics*

$$\int_\Omega z_t(t)\, d\Omega + \int_{\Gamma_0} w(t)\, d\Gamma_0 \equiv \int_\Omega z_1\, d\Omega + \int_{\Gamma_0} w_0\, d\Gamma_0, \qquad \forall t. \qquad (2.2)$$

Main regularity result. The main regularity result of the present paper, in a preliminary version, is given next. Its achievement will then allow us to introduce, formulate, and solve an optimal control problem in Section 4, in line with the main aim of the September 2008 Conference "50 Years of Optimal Control Theory," held at Będlewo, Poland.

THEOREM 2.1 *With reference to problem (2.1a-g), let*

$$[z_0,\ z_1,\ w_0,\ w_1,\ \theta_0] = 0, \qquad u \in L_2(0,T). \tag{2.3}$$

Then, the corresponding PDE solution satisfies

$$\begin{aligned}&[z(t), z_t(t), w(t), w_t(t), \theta(t)]\\ &\in C([0,T]; H^{3/2}(\Omega)/\mathbb{R} \times H^{1/2}(\Omega) \times \left[H^{5/2}(\Gamma_0) \cap H_0^2(\Gamma_0)\right]\\ &\qquad\qquad\qquad \times H_{00}^{3/2}(\Gamma_0) \times H_0^1(\Gamma_0)).\end{aligned} \tag{2.4}$$

Actually, the regularity of the thermal variable $\theta(t)$ can then be further boosted to read:

$$\theta(t) \in C([0,T]; H^{3/2-\varepsilon}(\Gamma_0)) \cap L_p(0,T; H_0^{3/2}(\Gamma_0)) \tag{2.5}$$

for any $\varepsilon > 0$ and $1 < p < \infty$.

A more detailed version of the regularity results for problem (2.1a-g), in terms of domains of fractional powers of relevant operators is postponed to Section 4, Theorem 4.3, after these concepts and notations have been introduced in Section 3 and related to Sobolev spaces. Moreover, Proposition 4.1 states the basic semigroup well posedness result, in the natural energy space Y_γ in equation (3.20) below. A corresponding optimal control problem—to hinge on these regularity results—is given in Theorem 4.4 below.

REMARK 2.2 *When dealing with a non-parabolic PDE problem subject to point control, in order to appreciate the regularity of the solution claimed, it is both instructive and enlightening to compare it with the result that can be obtained in a straightforward manner through the variation of parameters formula of the problem at hand. In the present case, such a formula is given for $[z_0, z_1, w_0, w_1, \theta_0] = 0$ by*

$$\begin{bmatrix} z(t) \\ z_t(t) \\ w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = \int_0^t e^{\mathbb{A}(t-\tau)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\delta u(\tau) \\ 0 \end{bmatrix} d\tau, \tag{2.6}$$

where $\mathbb{A}$ (see (3.18), (3.19) below) is the generator of the C_0-semigroup $e^{\mathbb{A}t}$ asserted by Proposition 4.1 on the natural finite energy space Y_γ, defined in (3.20) below.

The following considerations apply:

(a) The *weaker* (than Theorem 2.1) claim that

$$[z(t), z_t(t), w(t), w_t(t), \theta(t)] \in C([0,T]; Y_\gamma), \tag{2.7}$$

which is obtained in the present paper without taking advantage of the technical Proposition 3.1b below (in Step 8 of the proof of the main theorem), is only ε-better in space regularity over the result that can be obtained directly from (2.6) by simply using two ingredients: that $e^{\mathbb{A}t}$ is a C_0-semigroup on Y_γ and that $A^{-\frac{1}{2}}\delta \in L_2(\Gamma_0)$ in the present case of $\dim \Gamma_0 = 2$, see (A.4) of Appendix A. The fact that the operator $A^{\frac{1}{2}}B_\gamma^{-1}$ is not well-defined in $L_2(\Gamma_0)$ is an obstacle that prevents one from obtaining (2.7) directly from (2.6). Since Y_γ is the natural finite energy space of the free dynamics of the present problem ((2.1a-g) with $u \equiv 0$, the regularity result (2.7) is physically satisfactory and it balances off with that of the semigroup $e^{\mathbb{A}t}$, for initial conditions $[z_0, z_1, w_0, w_1, \theta] \in Y_\gamma$. But it is not optimal.

(b) In contrast, the optimal regularity obtained in (2.1), (2.5) of Theorem 2.1 yields for the mechanical variables $\{w, w_t\}$ precisely the *same* regularity result that holds true for the elastic Kirchhoff equation with clamped boundary conditions *alone* (Triggiani, 1993; Lasiecka and Triggiani, 2001), as well as for the corresponding variables $\{w, w_t, \theta\}$ of the thermoelastic problem (Triggiani, 2008), under point control, all in dimension equal to two. Accordingly, for these mechanical variables $\{w, w_t\}$, their optimal regularity given in Theorem 2.1 is, in fact, $\left(\frac{1}{2} + \varepsilon\right)$ better than that claimed through the variation of parameter formula (2.6), by the direct procedure explained in point (a).

3. Abstract model. Preliminary technical results

Abstract settings. (Lasiecka and Triggiani, 2000d, Section 9.10, p. 884). We now introduce an abstract setting for problem (2.1) following the notation of Triggiani (2008). We define:

(i) the positive, self-adjoint operators B, B_γ, on $L_2(\Gamma_0)$:

$$Bf \;=\; -\Delta f; \quad \mathcal{D}(B) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0), \tag{3.1a}$$

$$B_\gamma \;=\; (I + \gamma B); \qquad \mathcal{D}(B_\gamma^{\frac{1}{2}}) = \mathcal{D}(B^{\frac{1}{2}}) = H_0^1(\Gamma_0); \tag{3.1b}$$

(ii) the positive, self-adjoint elastic operator A on $L_2(\Gamma_0)$,

$$Af = \Delta^2 f; \qquad \mathcal{D}(A) = \left\{ f \in H^4(\Gamma_0);\; f|_{\partial\Gamma_0} = \frac{\partial}{\partial\nu} f\bigg|_{\partial\Gamma_0} = 0 \right\}. \tag{3.2}$$

For these, we recall that (Triggiani, 1991, 2008)

$$\mathcal{D}(B_\gamma^{\frac{1}{2}}) \;=\; \mathcal{D}(B^{\frac{1}{2}}) = H_0^1(\Gamma_0); \tag{3.3}$$

$$\begin{aligned}
\mathcal{D}(A^{\frac{3}{4}}) &\equiv H^3(\Gamma_0) \cap H_0^2(\Gamma_0) \\
&\equiv \left\{ f \in H^3(\Gamma_0) : f\Big|_{\partial\Gamma_0} = \frac{\partial f}{\partial \nu}\Big|_{\partial\Gamma_0} = 0 \right\}; \qquad (3.4) \\
\mathcal{D}(A^{\frac{1}{2}}) &\equiv H_0^2(\Gamma_0); \quad \mathcal{D}(A^{\frac{1}{4}}) \equiv H_0^1(\Gamma_0) = \mathcal{D}(B^{\frac{1}{2}}); \qquad (3.5) \\
\mathcal{D}(A^{\frac{3}{8}}) &= [\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A^{\frac{1}{4}})]_{\frac{1}{2}} = [H_0^2(\Gamma_0), H_0^1(\Gamma_0)]_{\frac{1}{2}} \equiv H_{00}^{\frac{3}{2}}(\Gamma_0) \qquad (3.6) \\
&\subset [\mathcal{D}(B), \mathcal{D}(B^{\frac{1}{2}})]_{\frac{1}{2}} \equiv \mathcal{D}(B^{\frac{3}{4}}) \equiv \mathcal{D}(B_\gamma^{\frac{3}{4}}) \equiv H_0^{\frac{3}{2}}(\Gamma_0); \qquad (3.7) \\
\mathcal{D}(A^{\frac{1}{8}}) &= [\mathcal{D}(A^{\frac{1}{4}}), L_2(\Gamma_0)]_{\frac{1}{2}} = [H_0^1(\Gamma_0), L_2(\Gamma_0)]_{\frac{1}{2}} = H_{00}^{\frac{1}{2}}(\Gamma_0) \qquad (3.8) \\
&= \mathcal{D}(B^{\frac{1}{4}}) \equiv \mathcal{D}(B_\gamma^{\frac{1}{4}}), \qquad (3.9) \\
\mathcal{D}(A^{\frac{5}{8}}) &= H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0). \qquad (3.10)
\end{aligned}$$

see Lions and Magenes (1972, p. 66) for these Sobolev spaces. The lack of compatibility of the boundary conditions between $H_{00}^{\frac{3}{2}}(\Gamma_0)$ and $H_0^{\frac{3}{2}}(\Gamma_0)$—whereby $H_{00}^{\frac{3}{2}}(\Gamma_0) \underset{\neq}{\subset} H_0^{\frac{3}{2}}(\Gamma_0)$ with a finer topology (Lions and Magenes, 1972, p. 66)—is a source of serious technical difficulties in the study of well-posedness of the thermoelastic problem alone in dim $\Gamma_0 = 2$, with *clamped boundary* conditions. This difficulty is not present in the case dim $\Gamma_0 = 3$ or dim $\Gamma_0 = 1$ (here, a different technical difficulty arises, Triggiani, 2007b), with clamped boundary conditions. Moreover, all these difficulties are *not* present in the case of *hinged* boundary conditions. See Triggiani (2007a).

(iii) In addition, we define the Neumann map N (harmonic extension in interior of a Neumann boundary datum) as $N: \ H^s(\Gamma_0) \to H^{s+\frac{3}{2}}(\Omega), \ s \in \mathbb{R}$

$$h = Ng \Longleftrightarrow \begin{cases} (\Delta - I)h = 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu} h = 0 & \text{on } \Gamma_1; \quad \dfrac{\partial}{\partial \nu} h = g \ \text{ on } \Gamma_0; \end{cases} \qquad (3.11)$$

the strictly positive, self-adjoint operator $\mathcal{A}_N$ defined by $(-\Delta)$ with Neumann homogeneous boundary conditions, that is, $\mathcal{A}_N : L_2^0(\Omega) \equiv L_2(\Omega)/\mathcal{N}(\mathcal{A}_N) \to L_2(\Omega)$,

$$\mathcal{A}_N f = -\Delta f; \quad \mathcal{D}(\mathcal{A}_N) = \left\{ f \in H^2(\Omega) : \ \frac{\partial}{\partial \nu} f\Big|_{\Gamma} = 0 \right\} \qquad (3.12a)$$

$$\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega)/\mathbb{R}. \qquad (3.12b)$$

In (3.11) and (3.12), $L_2^0(\Omega)$ is the factor space $L_2(\Omega)/\mathcal{N}(\mathcal{A}_N)$: $L_2(\Omega)$ factored by the one-dimensional null space of $\mathcal{N}(\mathcal{A}_N)$ defined by constant functions. Define also

$$\tilde{\mathcal{A}_N} = \mathcal{A}_N + I. \qquad (3.13a)$$

It is well known that (Lasiecka and Triggiani, 2000c, Ch. 3, Lem. 3.3.1.1)

$$N^*\tilde{\mathcal{A}_N} f = \begin{cases} 0 & \text{on } \Gamma_1; \\ f|_{\Gamma_0} & \text{on } \Gamma_0; \end{cases} \text{ initially for } f \in \mathcal{D}(\mathcal{A}_N), \text{ extended to } f \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}), \tag{3.13b}$$

where $N^* : L_2(\Omega) \to L_2(\Gamma)$ is the adjoint of the bounded operator $N : L_2(\Gamma) \to L_2(\Omega)$, $(Ng, v)_{L_2(\Omega)} = (g, N^*v)_{L_2(\Gamma)}$.
[In fact, if $f \in \mathcal{D}(\mathcal{A}_N)$ and $g \in L_2(\Gamma)$, then Green's second theorem with (3.11) and (3.12) yield $((\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\Gamma_0}$ being the respective L_2-inner products):

$$\begin{aligned} -(N^*\tilde{\mathcal{A}_N} f, g)_\Gamma &= ((\Delta - I)f, Ng)_\Omega = \\ &= \cancel{(f, (\Delta - I)h)_\Omega} + \cancel{(\frac{\partial}{\partial\nu} f, h)_\Gamma} - (f, \frac{\partial}{\partial\nu} f)_\Gamma \\ &= -(f, g)_{\Gamma_0}, \qquad g \in L_2(\Gamma) \end{aligned} \tag{3.14}$$

and (3.13b) follows for $f \in \mathcal{D}(\mathcal{A}_N)$. Next, extend its validity to all $f \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, as $\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$ is dense in $\mathcal{D}(\mathcal{A}_N)$.]

Second-order model. In the above notation, the PDE-problem (2.1a–g) can be rewritten abstractly as (Lasiecka and Triggiani, 2000d, p. 888)

$$\begin{cases} z_{tt} = -\mathcal{A}_N z - (\mathcal{A}_N + I)N(w_t|_{\Gamma_0}) & \text{on } [\mathcal{D}(\mathcal{A}_N)]'; & (3.15a) \\ B_\gamma w_{tt} + Aw - B\theta - N^*(\mathcal{A}_N + I)z_t = \delta u & \text{on } L_2(\Gamma_0); & (3.15b) \\ \theta_t + B\theta + Bw_t = 0 & \text{on } L_2(\Gamma_0), & (3.15c) \end{cases}$$

where $[\mathcal{D}(\mathcal{A}_N)]'$ is the dual space of $\mathcal{D}(\mathcal{A}_N)$ with respect to $L_2(\Omega)$ as a pivot space. In fact, as to (3.15a), we obtain from (2.1a) via (3.11), (3.12):

$$\begin{aligned} z_{tt} &= \Delta z = (\Delta - I)z + z = (\Delta - I)(z - Ng) + z = \Delta(z - Ng) + Ng \\ &= -\mathcal{A}_N(z - Ng) + Ng \in L_2(\Omega) \end{aligned} \tag{3.16}$$

since $(z - Ng) \in \mathcal{D}(\mathcal{A}_N)$. Extending now the original operator $\mathcal{A}_N$ in (3.12), by isomorphism, as: continuous $L_2(\Omega) \to [\mathcal{D}(\mathcal{A}_N)]'$, while retaining the same symbol, the above equation (3.16) yields

$$z_{tt} = -\mathcal{A}_N z + (\mathcal{A}_N + I)Ng \in [\mathcal{D}(\mathcal{A}_N)]' \tag{3.17}$$

from which (3.15a) follows with $g \equiv -w_t|_{\Gamma_0}$, $g \equiv 0$ on Γ_1, as dictated by (2.1b-c). To obtain (3.15b) from (2.1c), we have invoked (3.1b) for B_γ, (3.2) for A and (3.13b) for the Dirichlet trace on Γ_0.

First-order model. The corresponding first-order model is (Lasiecka and Triggiani, 2000d, p. 888)

$$\dot{y} = \mathbb{A}y + \mathcal{B}u; \quad y(t) = [z(t), z_t(t), w(t), w_t(t), \theta(t)];$$
$$y(0) = y_0 = [z_0, z_1, w_0, w_1, \theta_0] \in Y_\gamma \tag{3.18}$$

$$\mathbb{A} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ -\mathcal{A}_N & 0 & 0 & -(\mathcal{A}_N + I)N(\,\cdot\,|_{\Gamma_0}) & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & B_\gamma^{-1}N^*(\mathcal{A}_N + I) & -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & 0 & 0 & -B & -B \end{bmatrix},$$

$$\mathcal{B}u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\delta u \\ 0 \end{bmatrix}, \tag{3.19}$$

with the space Y_γ defined as (Lasiecka and Triggiani, 2000d, p. 889)

$$Y_\gamma \equiv \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \times L_2^0(\Omega) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B_\gamma^{\frac{1}{2}}) \times L_2(\Gamma_0); \tag{3.20}$$
$$(x_1, x_2)_{\mathcal{D}(B_\gamma^{\frac{1}{2}})} = ((I + \gamma B)x_1, x_2)_{L_2(\Gamma_0)}. \tag{3.21}$$

Characterization of $\mathcal{D}(\mathbb{A})$ Next, we characterize the domain of $\mathbb{A}$, $\mathcal{D}(\mathbb{A})$. To this end, let $y = [y_1, y_2, y_3, y_4, y_5] \in \mathcal{D}(\mathbb{A})$. We require that $\mathbb{A}y \in Y_\gamma$ that is, invoking (3.19) for $\mathbb{A}$ and (3.20) for Y_γ, we require that

$$\mathbb{A}y = \begin{cases} y_2 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) & \text{(3.22a)} \\ -(\mathcal{A}_N+I)^{\frac{1}{2}}\left[(\mathcal{A}_N+I)^{\frac{1}{2}}y_1+(\mathcal{A}_N+I)^{\frac{1}{2}}N(y_4|_{\Gamma_0})\right]+y_1 \in L_2(\Omega) & \text{(3.22b)} \\ y_4 \in \mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B) & \text{(3.22c)} \\ B_\gamma^{-1}N^*(\mathcal{A}_N + I)y_2 - B_\gamma^{-1}Ay_3 + B_\gamma^{-1}By_5 \in \mathcal{D}(B_\gamma^{\frac{1}{2}}) & \text{(3.22d)} \\ -By_4 - By_5 \in L_2(\Gamma_0) & \text{(3.22e)} \end{cases}$$

where the penultimate line (3.22d) is rewritten equivalently as

$$B_\gamma^{-\frac{1}{2}}N^*(\mathcal{A}_N + I)y_2 - (B_\gamma^{-\frac{1}{2}}A^{\frac{1}{4}})A^{\frac{3}{4}}y_3 + B_\gamma^{-\frac{1}{2}}By_5 \in L_2(\Gamma_0). \tag{3.23}$$

With $y_2 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) = H^1(\Omega)/\mathbb{R}$ as in (3.22a), see (3.12), we have $N^*(\mathcal{A}_N+I)y_2 = y_2|_{\Gamma_0} \in H^{\frac{1}{2}}(\Gamma_0)$, and $N^*(\mathcal{A}_N + I)y_2 = 0$ on Γ_1, by (3.13b). Moreover, since

$(B_\gamma^{-\frac{1}{2}}A^{\frac{1}{4}})$ is an isomorphism on $L_2(\Gamma_0)$ by (3.5), we see that (3.23) requires $y_3 \in \mathcal{D}(A^{\frac{3}{4}})$. Finally, (3.22e), where $By_4 \in L_2(\Gamma_0)$ by (3.22c), requires $y_5 \in \mathcal{D}(B)$. Thus, the domain $\mathcal{D}(\mathbb{A})$ of $\mathbb{A} : Y_\gamma \supset \mathcal{D}(\mathbb{A}) \to Y_\gamma$ is characterized as follows:

$$\mathcal{D}(\mathbb{A}) = S_1 \times \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{3}{4}}) \times S_4 \times \mathcal{D}(B) \tag{3.24a}$$

$$S_1 \times S_4 = \Big\{ y_1 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}), y_4 \in \mathcal{D}(A^{\frac{1}{2}}) : [\tilde{\mathcal{A}_N}^{\frac{1}{2}} y_1 + \tilde{A}_N^{\frac{1}{2}} N(y_4|_{\Gamma_0}) \in \mathcal{D}(\tilde{A}_N^{\frac{1}{2}}) \Big\} \tag{3.24b}$$

$$= \left\{ y_1 \in H^2(\Omega),\ y_4 \in H_0^2(\Gamma_0),\ \ \frac{\partial}{\partial\nu} y_1\Big|_{\Gamma_1} = 0,\ \ \frac{\partial}{\partial\nu} y_1\Big|_{\Gamma_0} = y_4 \right\} \tag{3.24c}$$

(passage from (3.24b) to (3.24c) uses (3.11) and (3.12) – in particular $N(y_4|_{\Gamma_0}) \in H^2(\Omega)$, since $y_4|_{\Gamma_0} \in H^{\frac{3}{2}}(\Gamma_0)$). *A-fortiori,* we have

$$\mathcal{D}(\mathbb{A}) \subset H^2(\Omega) \times \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{3}{4}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(B). \tag{3.25}$$

The domain $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$ of $\mathbb{A}^{\frac{1}{2}}$. In the definition of $\mathcal{D}(\mathbb{A})$ in (3.24) above, the first and the third component space variables are coupled, while the remaining variables are uncoupled. Thus, by interpolation between (3.20) for Y_γ and (3.24) for $\mathcal{D}(\mathbb{A})$, we obtain:

$$\mathcal{D}(\mathbb{A}^{\frac{1}{2}}) = V_1 \times \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}}) \times \mathcal{D}(A^{\frac{5}{8}}) \times V_4 \times \mathcal{D}(B^{\frac{1}{2}}). \tag{3.26}$$

Moreover, interpolating—this time componentwise—between the RHS of Y_γ in (3.20) and the RHS of (3.25), we readily obtain [to complement (3.26)]

$$\begin{aligned}
\mathcal{D}(\mathbb{A}^{\frac{1}{2}}) &= [\mathcal{D}(\mathbb{A}), Y_\gamma]_{\frac{1}{2}} \\
&\subset [[\text{RHS of } (3.25)], [\text{RHS of } (3.20)]]_{\frac{1}{2}} \\
&= H^{\frac{3}{2}}(\Omega)/\mathbb{R} \times \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}}) \times \mathcal{D}(A^{\frac{5}{8}}) \times \mathcal{D}(A^{\frac{3}{8}}) \times \mathcal{D}(B^{\frac{1}{2}}) &&(3.27a) \\
&= H^{\frac{3}{2}}(\Omega)/\mathbb{R} \times H^{\frac{1}{2}}(\Omega) \times \Big[H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0)\Big] \times H_{00}^{\frac{3}{2}}(\Gamma_0) \times H_0^1(\Gamma_0). &&(3.27b)
\end{aligned}$$

For the top line, recall Lasiecka and Triggiani (2000c, p. 5). In obtaining (3.27a), we have also invoked (3.5) in $\mathcal{D}(B_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}})$; while (3.6),(3.4) and (3.10) are also used in obtaining (3.27b). With reference to the first and fourth components of $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$ in (3.26), the following reverse containment holds true:

$$\left\{ \begin{bmatrix} z_1 \\ z_4 \end{bmatrix} : z_1 \in H^{\frac{3}{2}}(\Omega)/\mathbb{R},\ \Delta z_1 \in [H^{\frac{1}{2}}(\Omega)]' = H^{-\frac{1}{2}}(\Omega);\ z_4 \in H_{00}^{\frac{3}{2}}(\Gamma_0), \ \frac{\partial}{\partial\nu} z_1\Big|_{\Gamma_1} = 0,\ \ \frac{\partial}{\partial\nu} z_1\Big|_{\Gamma_0} = -z_4 \right\} \subset V_1 \times V_4. \tag{3.27c}$$

This result is established in Appendix B, Proposition B.1 (which, in particular, shows that $\frac{\partial}{\partial\nu} z_1\big|_\Gamma$ is well-defined in $H^{-\varepsilon}(\Gamma) \quad \forall\varepsilon > 0$). Combining (3.27b) with (3.27c) in (3.26), we obtain that

$$\begin{cases} \mathcal{D}(\mathbb{A}^{\frac{1}{2}}) \text{ is topologically equivalent to} \\ H^{\frac{3}{2}}(\Omega)/\mathbb{R} \times H^{\frac{1}{2}}(\Omega) \times \left[H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0)\right] \times H_{00}^{\frac{3}{2}}(\Gamma_0) \times H_0^1(\Gamma_0). \end{cases} \tag{3.27d}$$

The domain $\mathcal{D}(\mathbb{A}^2)$ of $\mathbb{A}^2$. In Step 7 of the proof of the main result of the present paper, Theorem 4.3, we shall need to identify the third space component of $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$, the domain of the power $\mathbb{A}^{\frac{3}{2}}$. This will be accomplished by first identifying the third component space of $\mathcal{D}(\mathbb{A}^2)$, the domain of $\mathbb{A}^2$, and then interpolating with the third component space of $\mathcal{D}(\mathbb{A})$ in (3.24). As noted, the third component spaces of these domains are not coupled with other component spaces. The following result will be critical to obtain the optimal regularity result of Theorem 4.3.

PROPOSITION 3.1 *(a) With reference to the operator $\mathbb{A}$ in* (3.19), (3.24) *we have the following characterization for the third component space of $\mathcal{D}(\mathbb{A}^2)$:*

$$\mathcal{D}(\mathbb{A}^2) = ① \times ② \times (A^{-1}\mathcal{H}^\perp) \times ④ \times ⑤ \tag{3.28}$$

$$(A^{-1}\mathcal{H}^\perp) = \mathcal{D}(A^{\frac{1}{2}} B_\gamma^{-1} A) \tag{3.29}$$

where, as in Lasiecka and Triggiani (2001, eqns. (2.5),(2.6), p. 448)

$$\mathcal{H} \equiv \left\{h \in L_2(\Gamma_0) : (1-\gamma\Delta)h = 0 \text{ in } H^{-2}(\Gamma_0)\right\} = \mathcal{N}\left\{(1-\gamma\Delta)\right\} \tag{3.30}$$

$$\mathcal{H}^\perp \equiv \left\{f \in L_2(\Gamma_0) : (f,h)_{L_2(\Gamma_0)} = 0, \quad \forall h \in \mathcal{H}\right\} \tag{3.31}$$

$$L_2(\Gamma_0) = \mathcal{H} + \mathcal{H}^\perp \qquad \textit{orthogonal sum} \tag{3.32}$$

(b) The following interpolation result holds true:

$$\left[A^{-1}\mathcal{H}^\perp, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = \mathcal{D}(A^{\frac{7}{8}}). \tag{3.33}$$

As a consequence, regarding the third component space of $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$, we have:

$$\mathcal{D}(\mathbb{A}^{\frac{3}{2}}) = ① \times ② \times \mathcal{D}(A^{\frac{7}{8}}) \times ④ \times ⑤. \tag{3.34}$$

REMARK 3.1 *A result such as* (3.28) *for $\mathcal{D}(\mathbb{A}^2)$ was already noted in Lasiecka and Triggiani (2001, Lemma 4.2, p. 466), in the context of the Kirchhoff elastic plate equation with clamped boundary alone (in arbitrary dimension). It is such an elastic component that is responsible for the same result* (3.28) *now, in the context of its coupling with the thermal component as well as with the acoustic* *chamber. As noted in Lasiecka and Triggiani (2001, Lemma 4.2, eqn (4.11), p. 466)*

$$\mathcal{H}^{\perp} \cong \tilde{L}_2(\Gamma_0), \quad \text{so that} \quad A^{-1}\mathcal{H}^{\perp} \cong A^{-1}\tilde{L}_2(\Gamma_0). \tag{3.35}$$

In fact, the space $\tilde{L}_2(\Gamma_0)$, defined in (5.18)-(5.21) *is isometrically isomorphic (congruent) to the factor (or quotient) space $L_2(\Gamma_0)/\mathcal{H}$. The subspace $\mathcal{H}$ is precisely the "invisible" subspace of the operator $A^{-\frac{1}{2}}B_\gamma \in \mathcal{L}(L_2(\Gamma_0))$, see Lasiecka and Triggiani (2001, Lemma 2.1(ii) and remark just below it).*

Proof of Proposition 3.1 (a) With $y = [y_1, y_2, y_3, y_4, y_5] \in Y_\gamma$, we require that $\mathbb{A}^2 y = \mathbb{A}(\mathbb{A}y) \in Y_\gamma$ in (3.20) where $\mathbb{A}y$ is given in (3.22). Our intent in (3.28) is simply to identify the third space component of $\mathcal{D}(\mathbb{A}^2)$ – thus, below, we shall chase and track down only the third coordinate y_3. Accordingly, by (3.19) and (3.22) we obtain

$$\mathbb{A}(\mathbb{A}y) =$$

$$\begin{bmatrix} 0 & I & 0 & 0 & 0 \\ -\mathcal{A}_N & 0 & 0 & -\tilde{\mathcal{A}}_N N(\,\cdot\,|_{\Gamma_0}) & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & B_\gamma^{-1}N^*\tilde{\mathcal{A}}_N & -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & 0 & 0 & -B & -B \end{bmatrix} \begin{bmatrix} y_2 \\ \cdots \\ \cdots \\ -B_\gamma^{-1}Ay_3 \\ \quad + \cdots \\ \cdots \end{bmatrix} \tag{3.36}$$

$$= \begin{bmatrix} \cdots \\ -\tilde{\mathcal{A}}_N^{\frac{1}{2}}\left[\tilde{\mathcal{A}}_N^{\frac{1}{2}}y_2 - \tilde{\mathcal{A}}_N^{\frac{1}{2}}N(B_\gamma^{-1}Ay_3)\right] + y_2 + \cdots \\ -B_\gamma^{-1}Ay_3 \\ \cdots \\ BB_\gamma^{-1}Ay_3 + \cdots \end{bmatrix} \tag{3.37}$$

where $\cdots$ refer to terms *not* involving y_3 and we have used $\tilde{\mathcal{A}}_N = (\mathcal{A}_N + I)$. Via Y_γ in (3.20) we require that:

$$B_\gamma^{-1}Ay_3 \in \mathcal{D}(A^{\frac{1}{2}}); \text{ that is, } A^{\frac{1}{2}}B_\gamma^{-1}Ay_3 \in L_2(\Gamma_0) \tag{3.38}$$

$$BB_\gamma^{-1}Ay_3 \in L_2(\Gamma_0); \text{ that is, } y_3 \in \mathcal{D}(A) \tag{3.39}$$

in addition to the second term in (3.37) being in $L_2(\Gamma_0)$. With $y_3 \in \mathcal{D}(A)$ by (3.39), we see that $B_\gamma^{-1}Ay_3 \in \mathcal{D}(B_\gamma) \subset H^2(\Gamma_0)$ and the second term is well defined concerning the coordinate y_3. But the critical element is the one in (3.38): recalling Lasiecka and Triggiani (2001, eqn. (2.17) p. 450; Lemma 4.2, eqn. (4.11) p. 466), we require

$$Ay_3 \in \mathcal{H}^{\perp}, \text{ or } y_3 \in \mathcal{D}(A^{\frac{1}{2}}B_\gamma^{-1}A) = A^{-1}\mathcal{H}^{\perp} \cong A^{-1}\tilde{L}_2(\Gamma_0). \tag{3.40}$$

Thus, via (3.39) and (3.40), conclusion (3.28) is established, as desired.

(b) To prove the interpolation result (3.33), we shall use the setting of Lions and Magenes (1972, Section 14.3, pp. 96-98). This is an interpolation result between subspaces; that is, between spaces subject to additional constraints. To this end, let (we use the notation of Lions and Magenes, 1972, Section 14.3):

$$X = \mathcal{D}(A) \subset \Phi, \quad \mathcal{X} = \mathcal{H}^{\perp} = \bar{\mathcal{X}} \subset \Psi, \quad \delta = A \tag{3.41}$$

so that we may equivalently rewrite $A^{-1}\mathcal{H}^{\perp}$ as

$$\begin{aligned} A^{-1}\mathcal{H}^{\perp} = (X)_{\delta,\mathcal{X}} &= \{x \in X : \delta x \in \mathcal{X}\} \\ &= \left\{x \in \mathcal{D}(A) : Ax \in \mathcal{H}^{\perp}\right\}. \end{aligned} \tag{3.42}$$

Similarly, we set

$$Y = \mathcal{D}(A^{\frac{3}{4}}) = \Phi, \quad \mathcal{Y} = \left[\mathcal{D}(A^{\frac{1}{4}})\right]' = \bar{\mathcal{Y}} \equiv \Psi, \quad \delta = A, \quad \delta \in \mathcal{L}(\Phi;\Psi) \tag{3.43}$$

so that $\delta \in \mathcal{L}(X;\bar{\mathcal{X}}) \cap \mathcal{L}(Y;\bar{\mathcal{Y}})$ as well, and we may equivalently rewrite $\mathcal{D}(A^{\frac{3}{4}})$ as

$$\begin{aligned} \mathcal{D}(A^{\frac{3}{4}}) = (Y)_{\delta,\mathcal{Y}} &= \{y \in Y : \quad \delta y \in \mathcal{Y}\} \\ &= \left\{y \in \mathcal{D}(A^{\frac{3}{4}}) : \quad Ay \in \left[\mathcal{D}(A^{\frac{1}{4}})\right]'\right\}. \end{aligned} \tag{3.44}$$

In (3.43), (3.44), $[\ \cdot\ ]'$ denotes duality with respect to $L_2(\Gamma_0)$ as a pivot space. Then, our original object $\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}}$ is accordingly equivalently rewritten via (3.42) and (3.43) as

$$\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = [(X)_{\delta,\mathcal{X}}, (Y)_{\delta,\mathcal{Y}}]_{\frac{1}{2}}. \tag{3.45}$$

Finally, to verify the remaining assumption in Lions and Magenes (1972, Eqn (14.23)(iii)), we take $\mathcal{G} = A^{-1}$, $\chi \in \bar{\mathcal{X}} + \bar{\mathcal{Y}} = [\mathcal{D}(A^{\frac{1}{4}})]'$, and $r = 0$. We can now appeal to Lions and Magenes (1972, Theorem 14.3, p. 97) to get

$$\left[A^{-\frac{1}{2}}\mathcal{H}^{\perp}, \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = [(X)_{\delta,\mathcal{X}}, (Y)_{\delta,\mathcal{Y}}]_{\frac{1}{2}} = \left([X,Y]_{\frac{1}{2}}\right)_{\delta,[\mathcal{X},\mathcal{Y}]_{\frac{1}{2}}}. \tag{3.46}$$

But from (3.41) and (3.43), we compute

$$[X,Y]_{\frac{1}{2}} = \left[\mathcal{D}(A), \mathcal{D}(A^{\frac{3}{4}})\right]_{\frac{1}{2}} = \mathcal{D}(A^{\frac{7}{8}}) \tag{3.47}$$

as desired. Via (3.46) and (3.47), our sought after conclusion (3.33) will be established, as soon as we verify that the required constraint

$$\delta\left([X,Y]_{\frac{1}{2}}\right) \in [\mathcal{X},\mathcal{Y}]_{\frac{1}{2}} \tag{3.48}$$

is automatically satisfied. Via (3.47) and $\delta = A$, $\mathcal{X} = \mathcal{H}^{\perp}$ and $\mathcal{Y} = [\mathcal{D}(A^{\frac{1}{4}})]'$ in (3.41),(3.43), we re-write the terms in (3.48) explicitly as

$$\delta\left([X,Y]_{\frac{1}{2}}\right) = A\mathcal{D}(A^{\frac{7}{8}}) = [\mathcal{D}(A^{\frac{1}{8}})]' \tag{3.49}$$

$$[\mathcal{X},\mathcal{Y}]_{\frac{1}{2}} = \left[\mathcal{H}^{\perp}, [\mathcal{D}(A^{\frac{1}{4}})]'\right]_{\frac{1}{2}} = \left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]'_{\frac{1}{2}} \tag{3.50}$$

where in the last step we have invoked the duality result (Lions and Magenes, 1972, Theorem 6.2, p. 29). In conclusion, via (3.49) and (3.50), verifying the validity of statement (3.48) means establishing that

$$\left[\mathcal{D}(A^{\frac{1}{8}})\right]' \subset \left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]'_{\frac{1}{2}} \tag{3.51}$$

where $(\mathcal{H}^{\perp})'$ denotes duality with respect to the $L_2(\Gamma_0)$-topology. In turn, (3.51) is equivalent to

$$\left[\mathcal{D}(A^{\frac{1}{4}}), (\mathcal{H}^{\perp})'\right]_{\frac{1}{2}} \subset \mathcal{D}(A^{\frac{1}{8}}) = \left[\mathcal{D}(A^{\frac{1}{4}}), L_2(\Gamma_0)\right]_{\frac{1}{2}} \tag{3.52}$$

which is plainly true. Thus, the required condition (3.48) has been verified. In conclusion, (3.46), (3.47), (3.48) cumulatively establish the validity of (3.33). Finally, interpolating for the third component space between (3.24) for $\mathcal{D}(\mathbb{A})$ and (3.28) for $\mathcal{D}(\mathbb{A}^2)$, and invoking (3.33) yields (3.34) at once. Proposition 3.1 is proved. ∎

4. Statement of main regularity results

In this section, we collect the statement of the relevant regularity results of system (2.1a-g). These include a restatement of the main direct regularity result, Theorem 2.1, this time expressed also in terms of the domain of fractional powers of the relevant operators introduced in Section 3. A dual version will also be stated explicitly, though not in the optimal spaces for the sake of shortness.

Homogeneous problem. We begin by recalling a known well-posedness result, Lebiedzik (2000), on the homogeneous problem (2.1), with forcing term $u \equiv 0$. Its proof is based on Lumer-Phillips Theorem.

PROPOSITION 4.1 *(Lebiedzik, 2000) The operator* $\mathbb{A}$ *in* (3.19), (3.24) *is the infinitesimal generator of a strongly continuous semigroup of contractions* $e^{\mathbb{A}t}$ *on the space* Y_γ.

More precisely, for $z = [z_1, z_2, z_3, z_4, z_5] \in \mathcal{D}(\mathbb{A})$, one obtains

$$(\mathbb{A}z, z)_{Y_\gamma} = -(Bz_5, z_5)_{L_2(\Gamma_0)} \leq 0 \tag{4.1}$$

(after cancellation of eight pair-wise terms). Thus, equation (4.1) shows, in particular, that the operator obtained from $\mathbb{A}$ in (3.19) after replacing the term

$-B$ by 0 in its a_{55} entry becomes then skew-adjoint on Y_γ (as the right-hand side of (4.1) then becomes zero).

Input $u \to$ solution y. As a consequence of Proposition 4.1, the solution to the abstract problem (3.18) —ultimately, of (2.1a–g)—can be written as

$$y(t) = e^{\mathbb{A}t} y_0 + (Lu)(t); \quad (Lu)(t) = \int_0^t e^{\mathbb{A}(t-\tau)} \mathcal{B} u(\tau)\, d\tau. \tag{4.2}$$

Control operator $\mathcal{B} : U \to [\mathcal{D}(\mathbb{A}^*)]'$.

PROPOSITION 4.2 *The operator $\mathcal{B}$ in* (3.19) *is not bounded from U to Y_γ, but rather $\mathcal{B}:$ continuous $U \to [\mathcal{D}(\mathbb{A}^*)]'$.*

The proof will be given in Appendix A.

Optimal regularity of non-homogeneous problem (3.18) **or** (2.1)**.** The following is the main result of the present paper. It rephrases Theorem 2.1.

THEOREM 4.3 *(Regularity of L) With reference to problem* (3.18) *or* (2.1)*, or* (4.2)*, let*

$$u \in L_2(0,T). \tag{4.3}$$

(i) For each $0 < T < \infty$, the input-solution operator L, defined in (4.2)*, satisfies the property*

$$L: \text{ continuous } L_2(0,T) \to C([0,T]; Z_\gamma), \tag{4.4a}$$

where

$$Z_\gamma \equiv H^{\frac{3}{2}}(\Omega)/\mathbb{R} \times H^{\frac{1}{2}}(\Omega) \times [H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0)] \times H_{00}^{\frac{3}{2}}(\Gamma_0) \times H_0^1(\Gamma_0) \tag{4.4b}$$

where we recall from (3.1b), (3.5),(3.6), (3.10), (3.12)*:*

$$\mathcal{D}(\mathcal{A}_N^{\frac{3}{4}}) \equiv H^{\frac{3}{2}}(\Omega)/\mathbb{R}; \quad \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}}) \equiv H^{\frac{1}{2}}(\Omega); \tag{4.5}$$

$$\mathcal{D}(A^{\frac{5}{8}}) \equiv H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0)], \quad \mathcal{D}(A^{\frac{3}{8}}) \equiv H_{00}^{\frac{3}{2}}(\Gamma_0) \tag{4.6}$$

$$\mathcal{D}(A^{\frac{1}{4}}) = \mathcal{D}(B^{\frac{1}{2}}) = \mathcal{D}(B_\gamma^{\frac{1}{2}}) \equiv H_0^1(\Gamma_0). \tag{4.7}$$

As noted in (3.27d), the space Z_γ is topologically equivalent to $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$, a consequence of Proposition B.1 in Appendix B (see (3.27c) and (3.27b)). In PDE terms, we have that with zero initial conditions $[z_0, z_1, w_0, w_1, \theta_0] = 0$, the corresponding solution of the original PDE-model (2.1) *satisfies*

$$u \in L_2(0,T) \to [z(t), z_t(t), w(t), w_t(t), \theta(t)] \in C([0,T]; Z_\gamma), \tag{4.8}$$

continuously, where the space Z_γ is defined in (4.4b)*. Moreover, from* (4.8) *and* (4.6)*, whereby $A^{\frac{3}{8}} w_t \in C([0,T]; L_2(\Gamma_0))$, it readily follows from a standard semigroup result, Lasiecka and Triggiani (1990, Proposition 0.1, p. 4) that* *the regularity of the parabolic component θ can be further improved to read via (3.15c):*

$$\begin{aligned}\theta(t) &= -\int_0^t e^{-B(t-\tau)} B w_t(\tau) d\tau \\ &= -\int_0^t B^{\frac{1}{4}} e^{-B(t-\tau)} (B^{\frac{3}{4}} A^{-\frac{3}{8}}) A^{\frac{3}{8}} w_t(\tau) d\tau \qquad (4.9a) \\ &\in C([0,T]; \mathcal{D}(B^{\frac{3}{4}-\varepsilon})) \cap L_p(0,T; \mathcal{D}(B^{\frac{3}{4}})), \qquad (4.9b) \\ &\equiv C([0,T]; H^{\frac{3}{2}-\varepsilon}(\Gamma_0)) \cap L_p(0,T; H_{00}^{\frac{3}{2}}(\Gamma_0)). \qquad (4.9c)\end{aligned}$$

for $\varepsilon > 0$, $1 < p < \infty$, as the operator ( ) in (4.9a) is bounded on $L_2(\Gamma_0)$ by (3.6), (3.7).

(ii) Duality results as in Lasiecka and Triggiani (2000d, Ch. 7, Thm. 7.2.1) can be given. For simplicity (and lack of space), we shall only state those that are dual to the regularity expressed by (4.8) *with, however, the optimal space Z_γ in* (4.4b) *replaced by the space of finite energy Y_γ in* (3.20). *The following abstract trace regularity property holds true: For each $0 < T < \infty$, there exists a constant $C_T > 0$ such that*

$$\int_0^T |\mathcal{B}^* e^{\mathbb{A}^* t} y|_U^2 \leq C_T ||y||_{Y_\gamma}^2, \qquad (4.10)$$

for all y first in $\mathcal{D}(\mathbb{A}^*)$, next extended to all of Y_γ.

With reference to (4.10), we have

$$\mathbb{A}^* = \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ \mathcal{A}_N & 0 & 0 & -\mathcal{A}_N N(\,\cdot\,|_{\Gamma_0}) & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & B_\gamma^{-1} N^* \mathcal{A}_N & B_\gamma^{-1} A & 0 & -B_\gamma^{-1} B \\ 0 & 0 & 0 & B & -B \end{bmatrix}, \ \mathcal{D}(\mathbb{A}^*) = \mathcal{D}(\mathbb{A}); \qquad (4.11a)$$

$$\mathcal{B}^* \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = y_4|_{x=x_0}, \quad y = [y_1, y_2, y_3, y_4, y_5] \in \mathcal{D}(\mathbb{A}^*). \qquad (4.11b)$$

In PDE terms the meaning of (4.10) is the following, by virtue of (4.11) and (4.12). Let $u \equiv 0$ in Eqn. (2.1d). Then, the corresponding homogeneous *problem (2.1a–g) satisfies the estimate*

$$\int_0^T |w_t(t,x;y_0)|_{x=x_0}|^2\, dt \le C_T \|y_0\|^2_{Y_\gamma}, \tag{4.12}$$

$y_0 = [z_0, z_1, w_0, w_1, \theta_o]$, where $w_t(\,\cdot\,; y_0)|_{x=x_0}$ is the velocity w_t evaluated at the point $x = x_0 \in \Gamma_0$ of the elastic component w of problem (2.1a–g) due to the I.C. y_0 with $u \equiv 0$.

The proof of Theorem 4.3 is given in Section 5. The regularity result in Theorem 4.3 makes the point-control problem (2.1a–g) fit the abstract dynamical properties of Lasiecka and Triggiani (2000d, Ch. 9).

Remark 4.1 *If in (2.1f) we replace the clamped boundary conditions for w with the corresponding hinged boundary conditions $w = \Delta w = 0$ on $(0,T] \times \partial\Gamma_0$, then the counterpart of Theorem 4.3* holds true *in this case as well. In fact, this case is even easier to handle at the thermoelastic level (on Γ_0) alone: Compare the hinged case in Triggiani (2007a)(in dimension 1, 2, 3) against the more challenging treatment of the clamped case in Triggiani (2007b) (in dimension 1 and 3) and, above all, in Triggiani (2008) (in dimension 2, the physically significant case for the structural acoustic thermoelastic problem).*

Remark 4.2 *In view of Theorem 4.3 – the basic regularity result, we could set the subsequent optimal control problem in the space $C([0,T]; Z_\gamma)$, topologically equivalent to $C([0,T]; \mathcal{D}(\mathbb{A}^{\frac{1}{2}}))$. However, we shall confine ourselves to set up the subsequent optimal control problem in the basic finite energy space $C([0,T]; Y_\gamma)$, which is physically significant.*

A corresponding optimal control problem with quadratic cost functional. Related Riccati differential equation. Next, we will use the well-posedness result of Theorem 4.3 to analyze the following optimal control problem for system (2.1) over the time interval $[s,T]$, $0 \le s < T < \infty$) in the setting of Lasiecka and Triggiani (2000d, Ch. 9):

Minimize over all $u \in L_2(s,T;U)$ the cost functional

$$J_s(u,y) = \int_s^T \left[|Ry(t)|^2_Z + |u(t)|^2_U\right]\, dt + |Gy(T)|^2_{Z_f}, \tag{4.13}$$

where Z and Z_f are Hilbert (output) spaces and the observation operators R and G satisfy the following assumptions (Lasiecka and Triggiani, 2000d, pp. 766–7):

(i) $\mathrm{R} \in \mathcal{L}(\mathrm{Y}_\gamma; \mathrm{Z})$, $\qquad \mathrm{G} \in \mathcal{L}(\mathrm{Y}_\gamma; \mathrm{Z_f})$. (4.14)

(ii) The map $R^*Re^{\mathbb{A}t}\mathcal{B}$ can be extended as a map

$$R^*Re^{\mathbb{A}t}\mathcal{B}: \text{ continuous } U \to L_1(0,T;Y_\gamma): \tag{4.15a}$$

$$\int_0^T |R^* R e^{\mathbb{A}t} \mathcal{B} u|_{Y_\gamma} \, dt \le c_T |u|_U, \quad u \in U. \tag{4.15b}$$

(iii) The operator G has the properties:

$$\mathcal{B}^* e^{\mathbb{A}^* t} G^* G \in \mathcal{L}(Y_\gamma; U); \quad \text{and} \quad \sup_{0 \le t \le T} |\mathcal{B}^* e^{\mathbb{A}^* t} G^* G|_{\mathcal{L}(Y_\gamma;U)} < \infty, \tag{4.16a}$$

so that

$$\sup_{0 \le t \le T} |\mathcal{B}^* e^{\mathbb{A}^* t} G^* G x|_{\mathcal{L}(Y_\gamma;U)} \le c_T |x|_{Y_\gamma}, \quad x \in Y_\gamma. \tag{4.16b}$$

In (4.13), $y(t) = y(t, s; y_0)$ is the solution of equation (3.18) with initial condition $y(s) = y_0$, that is,

$$y(t, s; y_0) = e^{\mathbb{A}(t-s)} y_0 + (L_s u)(t) \in C([s, T]; Y_\gamma); \tag{4.17}$$

$$(L_s u)(t) \quad = \quad \int_s^t e^{\mathbb{A}(t-\tau)} \mathcal{B} u(\tau) \, d\tau \tag{4.18a}$$

$$: \text{continuous } L_2([s, T]; U) \to C([s, T]; Y_\gamma). \tag{4.18b}$$

The regularity guaranteed *a-fortiori* by Theorem 4.3 (see Remark 4.2) and the assumptions on R and G (4.14)–(4.16) give directly (Lasiecka and Triggiani, 2000d, Ch. 9, Thms. 9.2.1 and 9.2.2, pp. 773–776) that

Theorem 4.4 (Optimal control problem) *With reference to the optimal control problem* (4.13) *for system* (3.18) *[the abstract version of (2.1a–g)] for initial data $y(s) = y_0 \in Y_\gamma$, we have, with $L_{sT} u = (L_s u)(T)$:*

(i) There exists a unique optimal pair $\{u^0(t, s; y_0), y^0(t, s; y_0)\}$ satisfying the optimality condition

$$u^0(t, s; y_0) = -\{L_s^* R^* R y^0(\cdot, s; y_0)\}(t) - \{L_{s_T}^* G^* G y^0(T, s; y_0)\}(t) \in L_2(s, T; U), \tag{4.19}$$

and given explicitly in terms of the data of the problem by the following representation formulas

$$u^0(t, s; y_0) = -\{\Lambda_{sT}^{-1} [L_s^* R^* R e^{A(\,\cdot\, - s)} y_0 + L_{sT}^* G^* G e^{A(T-s)} y_0]\}(t) \tag{4.20a}$$

$$\in L_2(s, T; U); \tag{4.20b}$$

$$\Lambda_{sT} = I_s + L_s^* R^* R L_s + L_{sT}^* G^* G L_{sT} \in \mathcal{L}(L_2(s, T; U)); \tag{4.21}$$

$$\|\Lambda_{sT}^{-1}\|_{\mathcal{L}(L_2(s,T;U))} \le 1, \tag{4.22}$$

where I_s is the identity operator on $L_2(s, T; U)$,

$$y^0(t, s; y_0) = e^{A(t-s)} y_0 + \{L_s u^0(\,\cdot\,, s; y_0)\}(t) \in C([s, T]; Y_\gamma), \tag{4.23}$$

which becomes explicit upon substituting (4.20a) into (4.23);

(ii) The optimal pair satisfies the estimates (here $U \equiv \mathbb{R}$),

$$\begin{cases} u^0(\,\cdot\,, s; y_0) \in L_\infty(s,T), \;\; u^0(t,s;y_0) \in U \equiv \mathbb{R} & (4.24\text{a}) \\ \text{for all } t\text{; moreover, } u^0(\,\cdot\,,s;y_0) \in C([s,T]) \text{ if } G = 0; & (4.24\text{b}) \end{cases}$$

$$\sup_{0\le s\le T} |u^0(\,\cdot\,,s;y_0)|_{L_\infty(s,T)} \;\; \le \;\; C_T\|y_0\|_{Y_\gamma}; \tag{4.25}$$

$$\sup_{0\le s\le T} \|y^0(\,\cdot\,,s;y_0)\|_{C([s,T];Y_\gamma)} \;\; \le \;\; C_T\|y_0\|_{Y_\gamma}. \tag{4.26}$$

(iii) The operator

$$\Phi(t,s)y_0 \equiv y^0(t,s;y_0): \;\; Y_\gamma \to C([s,T];Y_\gamma) \tag{4.27}$$

(strong continuity in the first variable) is an evolution operator satisfying the following properties:

(a) (transition property of optimal solution)

$$\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s), \quad 0 \le s \le \tau \le t \le T, \;\; \Phi(t,t) = I; \tag{4.28}$$

(b) (transition property of optimal control)

$$u^0(t,\tau;\Phi(\tau,s)x) = u^0(t,s;x) \;\text{ a.e. in } t, \quad 0 \le s \le \tau \le t \le T, \;\; x \in Y_\gamma; \tag{4.29}$$

(c) for $0 < t \le T$ fixed

$$\text{the map } s \to \Phi(t,s)x \text{ is continuous on } Y_\gamma, \;\forall x \in Y_\gamma;\; 0 \le s \le t; \tag{4.30}$$

(strong continuity in the second variable).

(iv) There exists an operator $P(t) \in \mathcal{L}(Y_\gamma)$, $0 \le t \le T$ defined explicitly in terms of the data by

$$\begin{aligned} P(t)x \;\; &= \;\; \int_t^T e^{\mathbb{A}^*(\tau-t)}R^*Ry^0(\tau,t;x)\,d\tau \\ &\quad + e^{\mathbb{A}^*(T-t)}G^*Gy^0(T,t;x), \;\; x \in Y_\gamma, \;\; 0 \le t \le T && (4.31\text{a}) \\ &\quad : \text{ continuous } Y_s \to C([0,T];Y_\gamma), && (4.31\text{b}) \end{aligned}$$

where $P(t)$ is positive, self-adjoint, $P(t) = P^*(t) \in \mathcal{L}(Y_\gamma)$, and we have that

$$u^0(t,s;y_0) = -\mathcal{B}^*P(t)y^0(t,s;y_0) \in L_2(s,T;U) = L_2(s,T) \tag{4.32}$$

for the optimal pair u^0 and y^0.

(v) *The operator $P(t)$ defined in (4.31) satisfies the following additional regularity properties:*

$$V(t) \equiv \mathcal{B}^* P(t) \in \mathcal{L}(Y_\gamma; U), \qquad 0 \le t \le T; \tag{4.33}$$

$$V(t) \equiv \mathcal{B}^* P(t): \ \textit{continuous } Y_\gamma \to L_\infty(0,T;U); \tag{4.34}$$

$$\begin{aligned}&\mathcal{B}^* P(\cdot) e^{\mathbb{A}(\cdot - s)} \mathcal{B}: \ \textit{continuous } U \to L_2(s,T;U) \textit{ for any } s \\ &\qquad\qquad \textit{with norm that may be taken independent of } s.\end{aligned} \tag{4.35}$$

(vi) *$P(t)$ is a solution to the following operator Differential Riccati Equation:*

$$\begin{cases} (P_t(t)x, y)_{Y_\gamma} &= -(P(t)x, \mathbb{A}y)_{Y_\gamma} - (P(t)\mathbb{A}x, y)_{Y_\gamma} \\ &\quad -(Rx, Ry)_Z + (B^* P(t)x, B^* P(t)y)_U & (4.36a) \\ P(T) &= G^* G, \quad \textit{for } x, y \in \mathcal{D}(\mathbb{A}). & (4.36b) \end{cases}$$

Finally, $P(t)$ in (4.31) is the only positive, self-adjoint solution to satisfy properties (4.31b), (4.33)–(4.35).

5. Proof of Theorem 4.3

We will prove the regularity of L by uncoupling the wave and thermoelastic equations in system (2.1). In other words, following the strategy of the purely elastic case in Camurdan and Triggiani (1999), Lasiecka and Triggiani (2000d, Section 10, p. 884), as well as Triggiani (2007a,b, 2008), we shall prove the theorem by first looking at the thermoelastic and wave components separately. At this stage, a direct semigroup argument (based on a *loss* of regularity aready achieved of the term ξ_t in (5.30)) readily produces the weaker result (5.38) in Remark 5.2. To obtain the optimal regularity result of Theorem 4.3, this last step requires two non-trivial ingredients, pointed out in Remark 5.1 and Orientation, below.

Step 1. (*Uncoupled nonhomogeneous thermoelastic equations on* Γ_0) Let ψ, h be the solution to the following (uncoupled) thermoelastic system:

$$\begin{cases} \psi_{tt} - \gamma\Delta\psi_{tt} + \Delta^2\psi + \Delta h = \delta(x_0)u(t) & \text{on } \ \Sigma_0, & (5.1a) \\ h_t - \Delta h - \Delta\psi_t = 0 & \text{on } \ \Sigma_0, & (5.1b) \\ \psi = 0; \quad \dfrac{\partial}{\partial\nu}\psi = 0; \quad h = 0 & \text{on } \ (0,T] \times \partial\Gamma_0, & (5.1c) \\ \psi(0,\cdot) = 0; \ \psi_t(0,\cdot) = 0; \ h(0,\cdot) = 0 & \text{in } \Gamma_0. & (5.1d) \end{cases}$$

The optimal regularity result for this problem in $\dim \Gamma_0 = 2$ with zero initial conditions is given by Triggiani (2008).

Theorem 5.1 *Triggiani (2008, Theorem 1.2) Let* $n =$ *dim* $\Gamma_0 = 2$ *and assume* (4.3)*:* $u \in L_2(0,T)$ *for the problem* (5.1)*. Then, continuously, the following interior regularity holds true:*

$$\begin{cases} \psi \in C\left([0,T];\ \mathcal{D}(A^{\frac{5}{8}}) \equiv H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0)\right); & (5.2a) \\ \psi_t \in C\left([0,T];\ \mathcal{D}(A^{\frac{3}{8}}) = H_{00}^{\frac{3}{2}}(\Gamma_0)\right), & (5.2b) \\ h \in L_p\left(0,T;\ \mathcal{D}(B^{\frac{3}{4}}) \equiv H_0^{\frac{3}{2}}(\Gamma_0)\right) \\ \qquad \cap\, C\left([0,T];\ \mathcal{D}(B^{\frac{3}{4}-\frac{\epsilon}{2}}) \equiv H_0^{\frac{3}{2}-\epsilon}(\Gamma_0)\right),\ 1 < p < \infty. & (5.2c) \end{cases}$$

Moreover, still continuously in $u \in L_2(0,T)$*, the following boundary regularity of the elastic component holds true:*

$$\Delta\psi|_{\partial\Gamma_0} \in L_2(0,T;\ L_2(\partial\Gamma_0)). \tag{5.2d}$$

Step 2 (*Uncoupled nonhomogeneous wave equation on* Ω) Next, let ϕ denote the solution to the following (uncoupled) wave mixed problem:

$$\begin{cases} \phi_{tt} = \Delta\phi & \text{on } (0,T] \times \Omega \equiv Q, & (5.3a) \\ \dfrac{\partial}{\partial\nu}\phi = g & \text{on } (0,T] \times \Gamma \equiv \Sigma, & (5.3b) \\ \phi(0,\cdot) = 0;\ \phi_t(0,\cdot) = 0 \text{ in } \Omega. & & (5.3c) \end{cases}$$

Sharp regularity theory for this problem is given in Lasiecka and Triggiani (1990, 1991a, 1994, 2000d, Ch. 8, p. 755), Tataru (1998). Following Lasiecka and Triggiani (2000d, Theorem 9.10.3.2, p. 893) with $\alpha = \frac{2}{3}$ by Tataru (1998), we have (here dim $\Omega = 3$, but this is *not* important, as long as dim $\Omega \geq 2$):

Theorem 5.2 *For the problem* (5.3)*, we have*
(i) (interior regularity) Let

$$g \in H^1(0,T;\ L_2(\Gamma)) \cap C\left([0,T];\ H^{\frac{1}{6}}(\Gamma)\right), \quad g|_{t=0} = 0. \tag{5.4}$$

Then, continuously,

$$[\phi, \phi_t, \phi_{tt}] \in C\left([0,T];\ H^{\frac{5}{3}}(\Omega) \times H^{\frac{2}{3}}(\Omega) \times H^{-\frac{1}{3}}(\Omega)\right). \tag{5.5}$$

(ii) (boundary regularity) Let

$$g \in H^1(\Sigma) \equiv L_2(0,T; H^1(\Gamma)) \cap H^1(0,T; L_2(\Gamma)), \quad g|_{t=0} = 0. \tag{5.6}$$

Then, continuously,

$$\phi|_\Sigma \in H^{\frac{4}{3}}(\Sigma) = L_2(0,T; H^{\frac{4}{3}}(\Gamma)) \cap H^{\frac{4}{3}}(0,T; L_2(\Gamma)). \tag{5.7}$$

Remark 5.1 *We note that by Lions and Magenes (1972, Thm. 3.1, p. 19), we have*

$$g \in H^1(\Sigma) \Rightarrow g \in C([0,T]; [H^1(\Gamma), L_2(\Gamma)]_{\frac{1}{2}} = H^{\frac{1}{2}}(\Gamma)). \tag{5.8}$$

Thus, hypothesis (5.6) implies hypothesis (5.4).

Step 3. Next, with ψ_t provided by Theorem 5.1, (5.2b) for the thermoelastic problem (5.1a–d), we consider the following mixed problem in Ω:

$$\begin{cases} \xi_{tt} = \Delta\xi & \text{on } Q, & (5.9a) \\ \dfrac{\partial}{\partial\nu}\xi = 0 & \text{on } \Sigma_1, & (5.9b) \\ \dfrac{\partial}{\partial\nu}\xi = -\psi_t & \text{on } \Sigma_0, & (5.9c) \\ \xi(0,\cdot) = 0;\ \xi_t(0,\cdot) = 0 \text{ in } \Omega. & & (5.9d) \end{cases}$$

In this step, we seek to apply Theorem 5.2 to the mixed problem (5.9a–d) in ξ. In the notation of (5.3b), we then have

$$g = \begin{cases} 0 & \text{on } \Sigma_1 \\ -\psi_t & \text{on } \Sigma_0 \end{cases}, \text{ so that } g|_{t=0} = \begin{cases} 0 & \text{on } \Gamma_1 \\ -\psi_t|_{t=0} & \text{on } \Gamma_0 \end{cases} \equiv 0, \tag{5.10}$$

after recalling the initial condition (5.1d) for $\psi_t|_{t=0}$ on Γ_0. Having verified in (5.10) the compatibility condition required by (5.6), in order to apply Theorem 5.2 on the mixed problem (5.9a–d) in ξ, it remains to verify the validity of the assumption of regularity in (5.6) for g defined in (5.10); that is, that

$$\psi_t \in H^1(\Sigma_0) \equiv L_2(0,T;H^1(\Gamma_0)) \cap H^1(0,T;L_2(\Gamma_0)). \tag{5.11}$$

One half of condition (5.11) is provided at once (*a-fortiori*) by Theorem 5.1, Eqn. (5.2b) (as dim $\Gamma = 2$ in our case), namely

$$\psi_t \in C\left([0,T];\ \mathcal{D}(A^{\frac{3}{8}}) = H_{00}^{\frac{3}{2}}(\Gamma_0)\right) \subset C([0,T];H^1(\Gamma_0)) \subset L_2(0,T;H^1(\Gamma_0)). \tag{5.12}$$

Thus, in order to satisfy (5.11), it remains to verify that

$$\psi_t \in H^1(0,T;L_2(\Gamma_0)). \tag{5.13}$$

But showing this is equivalent to showing that

$$\psi_{tt} \in L_2(0,T;L_2(\Gamma_0)), \tag{5.14}$$

which is the next task. (This task is challenging in the case of *clamped* boundary conditions, as opposed to the case of *hinged* boundary conditions.)

Step 4. *(Proof of (5.14))* To this end, we return to the ψ-problem in (5.1a-d) and rewrite it abstractly, recalling the operators B, B_γ in (3.1a–b) and A in (3.2). We thus obtain the abstract form of problem (5.1a–d), that is

$$B_\gamma \psi_{tt} + A\psi - Bh = \delta u. \tag{5.15}$$

Applying $A^{-\frac{1}{2}}$ on (5.15) gives

$$A^{-\frac{1}{2}} B_\gamma \psi_{tt} = -A^{-\frac{1}{2}} A\psi + A^{-\frac{1}{2}} Bh + A^{-\frac{1}{2}} \delta u. \tag{5.16}$$

Our goal is now to establish that the right-hand side of (5.16) satisfies the following regularity property

$$\text{RHS of (5.16)} \in L_2(0,T;L_2(\Gamma_0)). \tag{5.17}$$

Once (5.17) is established, we then obtain via (5.16), (5.17), that

$$A^{-\frac{1}{2}} B_\gamma \psi_{tt} \in L_2(0,T;L_2(\Gamma_0)). \tag{5.18}$$

Now, according to the results of Lasiecka and Triggiani (2001, Proposition 2.3, p. 453), one has

$$A^{-\frac{1}{2}} B_\gamma f \in L_2(\Gamma_0) \Longleftrightarrow f \in \tilde{L}_2(\Gamma_0), \tag{5.19}$$

so that (5.18) implies

$$\psi_{tt} \in L_2(0,T;\tilde{L}_2(\Gamma_0)), \tag{5.20}$$

and (5.14) is *a-fortiori* established. In fact (Lasiecka and Triggiani, 2001), the space $\tilde{L}_2(\Gamma_0)$ can be characterized in two ways:

(i) as the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ with respect to the space $\mathcal{D}(B_\gamma^{\frac{1}{2}})$ as a pivot space, endowed with the norm

$$\|f\|^2_{\mathcal{D}(B_\gamma^{\frac{1}{2}})} = (B_\gamma^{\frac{1}{2}} f, B_\gamma^{\frac{1}{2}} f)_{L_2(\Gamma_0)} = ((I+\gamma B)f, f)_{L_2(\Gamma_0)}; \tag{5.21}$$

(ii) isometric to the factor space $L_2(\Gamma_0)/\mathcal{H}$, with $\mathcal{H}$ defined in (3.30). Thus, characterization (ii) says that (5.20) implies *a-fortiori* the desired regularity property (5.14) for ψ_{tt}. The appearance of the space $\tilde{L}_2(\Gamma_0)$ is a pathological fact due to the clamped boundary conditions.

It remains to show (5.17), after which the proof of (5.14), hence of (5.11) is complete. To this end, we examine individually the regularity property of each component on the RHS of (5.16).

As to the first term in the RHS of (5.16), we have, recalling (5.2a) and (3.8):

$$-A^{-\frac{1}{2}} A\psi = -A^{\frac{1}{2}}\psi \in C([0,T];\mathcal{D}(A^{\frac{1}{8}}) \equiv H_{00}^{\frac{1}{2}}(\Gamma_0)) \subset L_2(0,T;L_2(\Gamma_0)). \tag{5.22}$$

As to the second term in the RHS of (5.16), we have

$$A^{-\frac{1}{2}}Bh \in L_p(0,T;L_2(\Gamma_0)),\ 1<p<\infty,\ \text{in particular } p=2. \tag{5.23}$$

In fact, (5.23) follows by virtue of the following two properties:

$$h \in L_p(0,T;L_2(\Gamma_0)),\ \textit{a-fortiori} \text{ from (5.2c)}; \tag{5.24}$$

$$A^{-\frac{1}{2}}B \in \mathcal{L}(L_2(\Gamma_0)),\ \text{since } BA^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Gamma_0)). \tag{5.25}$$

Indeed, $\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Gamma_0) \subset \mathcal{D}(B) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0)$ by (3.1a), (3.5), and the closed graph theorem then yields $BA^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Gamma_0))$.

Then, with B and A self-adjoint, the adjoint $A^{-\frac{1}{2}}B \in \mathcal{L}(L_2(\Gamma_0))$ as well, and (5.25) is established.

Then, (5.24) and (5.25) imply (5.23).

The third term on the RHS of (5.16) is analyzed in (A.4) of Appendix A. Thus

$$A^{-\frac{1}{2}}\delta u \in L_2(0,T;L_2(\Gamma_0)),\ \text{for } u \in L_2(0,T). \tag{5.26}$$

Invoking (5.22), (5.23), (5.26) on the RHS of (5.16), we obtain

$$A^{-\frac{1}{2}}B_\gamma \psi_{tt} = \text{RHS of (5.16)}\ \in L_2(0,T;L_2(\Gamma_0)). \tag{5.27}$$

Thus, (5.27) proves (5.17), hence (5.18), hence (5.20), finally (5.14), as desired. This concludes Step 4.

By (5.12) and (5.14), we conclude that $\psi_t \in H^1(\Sigma_0)$, and (5.11) is established. This concludes Step 3.

Step 5. Next, we return to the mixed problem (5.9a–d) in ξ. This is precisely the same as the mixed problem (5.3a–c) in ϕ, with the Neumann boundary datum g defined by (5.10): $g \equiv 0$ in Σ_1, $g = -\psi_t$ in Σ_0. We seek to apply Theorem 5.2(i),(ii). To this end, with the aforementioned g, the required assumptions (5.4) for interior regularity and (5.6) for boundary regularity, have already been verified in the preceding analysis:

$$\begin{cases} g \ \in \ H^1(\Sigma);\ g|_{t=0} = 0 \text{ by (5.11) on } \Gamma_0 \text{ and (5.10)}; & (5.28a) \\ g \ \in \ C([0,T];H^{\frac{1}{6}}(\Gamma))\ \textit{a-fortiori} \text{ from (5.12) on } \Gamma_0 \text{ and (5.10)} & (5.28b) \end{cases}$$

(recall also Remark 5.1). Hence, via (5.28a–b), we are authorized to apply Theorem 5.2 in its entirety to the ξ-problem (5.9a–d) and obtain the following interior and boundary regularity results, corresponding to (5.5) and (5.7), respectively: the interior regularity

$$[\xi,\xi_t,\xi_{tt}] \in C\left([0,T];\ H^{\frac{5}{3}}(\Omega) \times H^{\frac{2}{3}}(\Omega) \times H^{-\frac{1}{3}}(\Omega)\right), \tag{5.29}$$

and boundary regularity

$$\xi \in H^{\frac{4}{3}}(0,T;L_2(\Gamma_0)), \text{ or } \xi_t \in H^{\frac{1}{3}}(0,T;L_2(\Gamma_0)). \tag{5.30}$$

Step 6. (*coupled system*) With the regularity of ξ_t given by (5.30), we consider the following coupled system in the variables $\{\zeta, \upsilon, q\}$:

$$\text{chamber } \Omega : \begin{cases} \zeta_{tt} = \Delta\zeta & \text{on } (0,T] \times \Omega \equiv Q; & (5.31\text{a}) \\ \dfrac{\partial}{\partial\nu}\zeta = 0 & \text{on } (0,T] \times \Gamma_1 \equiv \Sigma_1; & (5.31\text{b}) \\ \dfrac{\partial}{\partial\nu}\zeta = -\upsilon_t & \text{on } (0,T] \times \Gamma_0 \equiv \Sigma_0; & (5.31\text{c}) \end{cases}$$

$$\text{wall } \Gamma_0 : \begin{cases} \upsilon_{tt} - \gamma\Delta\upsilon_{tt} + \Delta^2\upsilon + \Delta q - \zeta_t = \xi_t & \text{on } \Sigma_0; & (5.31\text{d}) \\ q_t - \Delta q - \Delta\upsilon_t = 0 & \text{on } \Sigma_0; & (5.31\text{e}) \\ \upsilon = 0; \quad \dfrac{\partial}{\partial\nu}\upsilon = 0; \quad q = 0 & \text{on } (0,T] \times \partial\Gamma_0 & (5.31\text{f}) \end{cases}$$

$$\text{I.C.} \quad \zeta(0,\cdot) = 0;\ \zeta_t(0,\cdot) = 0 \text{ in } \Omega;\ \upsilon(0,\cdot) = 0;\ \upsilon_t(0,\cdot) = 0;\ q(0,\cdot) = 0 \text{ in } \Gamma_0. \tag{5.31g}$$

Notice that problem (5.31) is obtained by setting

$$\zeta(t,x) \equiv z(t,x) - \xi(t,x), \quad \upsilon(t,x) \equiv w(t,x) - \psi(t,x),\ q(t,x) = \theta(t,x) - h(t,x), \tag{5.32}$$

with $\{z, w\}$ the wave solution in (2.1a–c) and the elastic solution in (2.1d), respectively; with ξ solution of the wave problem (5.9a–d); with ψ solution of the elastic problem (5.1a–d), finally, with θ and h the thermal terms in (2.1d–e) and (5.1a–b). Differentiating formally from (5.32) and invoking the respective problems (2.1), (5.9), (5.1) yields problem (5.31). The coupled problem (5.31) in the new variables $\{\zeta, \upsilon, q\}$ is the same as the original coupled problem (2.1) in $\{z, w, \theta\}$, except for the fact that the point control term δu in (2.1d) on Σ_0 is now replaced by the general nonhomogenous term ξ_t in (5.31d) on Σ_0, for which we already have the regularity noted in (5.30).

Thus, problem $\{\zeta, \upsilon, q\}$ is expected to be easier to analyze than problem $\{z, w, \theta\}$, the advantage of the change of variables. In fact, this is genuinely the case when it comes to obtaining the weaker regularity result (5.38) of Remark 5.2 (after accepting the ε loss of regularity of ξ_t, see (5.41) below.) To this end, we shall obtain regularity results for problem $\{\zeta, \upsilon, q\}$ by semigroup methods, just by using the s.c. semigroup asserted by Proposition 4.1 generated by the operator $\mathbb{A}$ in (3.19). Instead, in contrast, to obtain the optimal regularity result of Theorem 4.3, (4.10a), it is necessary to overcome the two additional difficulties pointed out in Remark 5.2 below.

Recalling the operators $\mathcal{A}_N$, N, B_γ, B, A, $N^*\mathcal{A}_N$ from Section 2, and proceeding as in obtaining the abstract version (3.15a–c) of the original $\{z, w, \theta\}$-problem in (2.1a–d), we see likewise that the abstract version of the $\{\zeta, \upsilon, q\}$-problem (5.31) is given by

$$\begin{cases} \zeta_{tt} = -\mathcal{A}_N\zeta - (A_N + I)N(\upsilon_t|_{\Gamma_0}) & \text{on} \quad [\mathcal{D}(\mathcal{A}_N)]'; & (5.33\text{a}) \\ B_\gamma\upsilon_{tt} + A\upsilon - Bq + N^*(\mathcal{A}_N + I)\zeta_t = \xi_t & \text{on} \quad L_2(\Gamma_0); & (5.33\text{b}) \\ q_t + Bq + B\upsilon_t = 0 & \text{on} \quad L_2(\Gamma_0). & (5.33\text{c}) \end{cases}$$

Thus, recalling (3.18) and (3.19), we see that the corresponding first-order system of (5.33a–c) is given by

$$\frac{\partial}{\partial t}\begin{bmatrix} \zeta \\ \zeta_t \\ \upsilon \\ \upsilon_t \\ q \end{bmatrix} = \mathbb{A}\begin{bmatrix} \zeta \\ \zeta_t \\ \upsilon \\ \upsilon_t \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\xi_t \\ 0 \end{bmatrix}. \tag{5.34}$$

So, according to Proposition 4.1, the solution of (5.34) is given by

$$\eta(t) = \begin{bmatrix} \zeta(t) \\ \zeta_t(t) \\ \upsilon(t) \\ \upsilon_t(t) \\ q(t) \end{bmatrix} = \int_0^t e^{\mathbb{A}(t-\tau)}\begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\xi_t(\tau) \\ 0 \end{bmatrix} d\tau. \tag{5.35}$$

Next, we need to establish the regularity of $\eta(t)$.

STEP 7. PROPOSITION 5.3 *With Y_γ as in* (3.20) *and η as in (5.35), we have*

$$\eta(t) = [\zeta(t), \zeta_t(t), \upsilon(t), \upsilon_t(t), q(t)] \in C([0, T]; \mathcal{D}(\mathbb{A}^{\frac{1}{2}})). \tag{5.36}$$

Thus, recalling (3.27b),

$$\begin{bmatrix} \zeta(t) \\ \zeta_t(t) \\ \upsilon(t) \\ \upsilon_t(t) \\ q(t) \end{bmatrix} = C([0, T]; \begin{bmatrix} H^{\frac{3}{2}}(\Omega)/\mathbb{R} \\ \mathcal{D}(\mathcal{A}_N^{\frac{1}{4}}) = H^{\frac{1}{2}}(\Omega) \\ \mathcal{D}(A^{\frac{5}{8}}) = H^{\frac{5}{2}}(\Gamma_0) \times H_0^2(\Gamma_0) \\ \mathcal{D}(A^{\frac{3}{4}}) = H_{00}^{\frac{3}{2}}(\Gamma_0) \\ \mathcal{D}(B^{\frac{1}{2}}) = H_0^1(\Gamma_0) \end{bmatrix}). \tag{5.37}$$

REMARK 5.2 *The proof of the sharp* (optimal) *regularity result* (5.36) *runs into additional technical difficulties. However,* (5.36) *is critical to obtain the final sought-after result of Theorem 4.3, equation (4.4a). To overcome these, we need two new ideas that are pointed out and described in the* **Orientation** *below. To emphasize this point, we shall proceed as follows. We shall first show the* weaker*, non-optimal result*

$$\eta(t) = [\zeta(t), \zeta_t(t), \upsilon(t), \upsilon_t(t), q(t)] \in C([0,T]; Y_\gamma). \tag{5.38}$$

At the present stage, conclusion (5.38) *can be readily reached, by accepting a loss of regularity of one unit in Sobolev regularity, as displayed in* (5.41) *below. Then, returning in Step 8 to the structural acoustic problem* (3.18), (2.1)*, the readily achieved regularity* (5.38) *will then automatically imply the result*

$$u \in L_2(0,T) \Rightarrow [z(t),\ z_t(t),\ w(t),\ w_t(t),\ \theta(t)] \in C([0,T]; Y_\gamma) \tag{5.39}$$

continuously. Such a result is still satisfactory in that Y_γ is the space of finite energy for the problem (2.1)*, and the natural state space of the solution semigroup $e^{\mathbb{A}t}$, describing the evolution of the free dynamics (i.e. due only to the initial data). Having noted this positive feature, we should, however, also point out that the regularity result* (5.39) *is only ε more regular than the result that could be obtained at the outset by a direct application of the semigroup formula* (2.6)*. All this was already elaborated in the comments made after equation* (2.6)*. We now pass to the proof of the weaker regularity result* (5.38)*.*

Direct proof of the weaker regularity result (5.38)*.* The *a priori* regularity of ξ_t given by (5.30) yields preliminarily

$$B_\gamma^{-1}\xi_t \in H^{\frac{1}{3}}(0,T;\mathcal{D}(B)). \tag{5.40}$$

The direct proof of the weaker (yet physically satisfactory) regularity result (5.38) in the finite energy space Y_γ is based on acceptance of the following *loss* of regularity:

$$B_\gamma^{-1}\xi_t \in H^{\frac{1}{3}}(0,T;\mathcal{D}(B)) \subset L_2(0,T;\mathcal{D}(B_\gamma^{\frac{1}{2}})), \tag{5.41}$$

where a serious loss takes place in the space variable by one unit in Sobolev space regularity: from $H^2(\Gamma_0)$ to $H^1(\Gamma_0)$. Using this fact in (5.35) and the definition of Y_γ in (3.20), in particular, its fourth component space, readily gives (5.38). ∎

The loss of regularity in (5.38) for the variables $[\zeta, \zeta_t, \upsilon, \upsilon_t, q]$ then propagates also to the original variables $[z, z_t, w, w_t, \theta]$, producing the weaker result (5.39).

Proof of Proposition 5.3, Eqn. (5.36)*: sharp regularity*

Orientation To improve upon the proof given above (of the weaker result (5.38) for $\eta(t)$), it is necessary to drastically modify the argument by injecting two new ideas expressed in points (a) and (b) below.

(a) First, with reference to the variation of parameters formula (5.35), we would like to take full advantage of the original regularity result (5.40), at least in space, that is in the form $B_\gamma^{-1}\xi_t \in L_2(0,T;\mathcal{D}(B))$, and *seek to obtain* that

$$\left[0,\ 0,\ 0,\ B_\gamma^{-1}\xi_t,\ 0\right] \in L_2(0,T;\mathcal{D}(\mathbb{A}^{\frac{1}{2}})). \tag{5.42}$$

Unfortunately, there are technical difficulties in establishing (5.42), since the active component B_γ^{-1} occurs on the fourth component space; as we have seen in characterizing $\mathcal{D}(\mathbb{A})$ in (3.24a,b,c), the first and fourth coordinates are *coupled*. Thus, it does not appear to be a trivial matter to characterize exactly the fourth component space of $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$ (such a result seems to be unknown, even in the case of the *elastic equation* per se). See, however, the topological equivalence in (3.27d). With reference to (5.40) and (3.27d), whose fourth component is $H_{00}^{\frac{3}{2}}(\Omega)$, we note that $\mathcal{D}(B) \not\subset H_{00}^{\frac{3}{2}}(\Omega)$, an obstacle to invoking (5.35) directly. Inspiration for overcoming this difficulty comes from (A.1), (A.3) of Appendix A. Accordingly, we have

$$\mathbb{A}^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\xi_t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -A^{-1}\xi_t \\ 0 \\ 0 \end{bmatrix} \tag{5.43}$$

and then re-write (5.35), via(5.43) as

$$\begin{aligned} \eta(t) = \begin{bmatrix} \zeta(t) \\ \zeta_t(t) \\ \upsilon(t) \\ \upsilon_t(t) \\ q(t) \end{bmatrix} &= \int_0^t \mathbb{A}e^{\mathbb{A}(t-\tau))}\mathbb{A}^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\xi_t(\tau) \\ 0 \end{bmatrix} d\tau \\ &= \int_0^t \mathbb{A}e^{\mathbb{A}(t-\tau))}\begin{bmatrix} 0 \\ 0 \\ -A^{-1}\xi_t(\tau) \\ 0 \\ 0 \end{bmatrix} d\tau \end{aligned} \tag{5.44}$$

where, by (5.30), we have

$$A^{-1}\xi_t \in H^{\frac{1}{3}}(0,T;\mathcal{D}(A)); \tag{5.45}$$

thus transferring the active term in (5.44) from the fourth to the third component space. While the coupling between the fourth and first component spaces presents challenges when it comes to characterizing $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$ [or $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$], by contrast, the third components of the domain of fractional powers of $\mathbb{A}$ are dealt with directly for $\mathcal{D}(\mathbb{A})$ in (3.24a), for $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$ in (3.26). Moreover, they involve the operator A (and its fractional powers) – for which the term in (5.45) is then a suitable and promising starting point for the subsequent analysis. In the present case, with reference to (5.44) and (5.45), we need to characterize the *third* component space of $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$ (as opposed to the *fourth* component space of $\mathcal{D}(\mathbb{A}^{\frac{1}{2}})$). To this end, the dependence of (5.45) in terms of $\mathcal{D}(A)$ helps.

For instance – consistently with the direct proof of the weaker regularity result (5.38) given above – if we accept the loss of space regularity

$$A^{-1}\xi_t \in H^{\frac{1}{3}}(0,T;\mathcal{D}(A))) \subset L_2(0,T;\mathcal{D}(A^{\frac{3}{4}})) \tag{5.46}$$

(that is, a loss of $\frac{1}{4}$ in terms of domains of fractional powers of A, which translates – again – into a loss of one unit in Sobolev regularity from $H^2(\Gamma_0)$ to $H^3(\Gamma_0)$ (see (3.4)), then again (5.44) would imply at once the weaker regularity result $\eta(t) \in C([0,T];Y_\gamma)$ in (5.38), by (3.24a) and (5.44).

Our present goal, instead, is to exploit the weaker topological loss

$$A^{-1}\xi_t \in H^{\frac{1}{3}}(0,T;\mathcal{D}(A))) \subset L_2(0,T;\mathcal{D}(A^{\frac{7}{8}})) \tag{5.47}$$

which will amount to identifying the third component space of $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$.

(b) Indeed, seeking to characterize the third component space of $\mathcal{D}(\mathbb{A}^{\frac{3}{2}})$ does run into technical difficulties, due to the pathology that the third component space of $\mathcal{D}(\mathbb{A}^2)$ in (3.28) is only $A^{-1}\mathcal{H}^{\perp}$, and not the full space $\mathcal{D}(A)$ (thus an algebraic, not a topological, loss takes place here). To handle this obstacle, the technical argument of Proposition 3.1b, leading to (3.34) is needed: the third component space $\mathcal{D}(A^{\frac{7}{8}})$ in (3.34) would have been a straightforward interpolation result between $\mathcal{D}(A)$ and $\mathcal{D}(A^{\frac{3}{4}})$ for $\mathcal{D}(\mathbb{A})$, if $\mathcal{D}(A)$ – not $A^{-1}\mathcal{H}^{-1}$ – had been the third component space of $\mathcal{D}(\mathbb{A}^2)$.

Proceeding with the proof of Proposition 5.3, equation (5.36), we return to (5.44) with active term given by (5.45). We then appeal to Proposition 3.1b, equations (3.33) and (3.34) and obtain that the statement (5.47) does indeed imply as a consequence, that

$$\left[0,\ 0, -A^{-1}\xi_t,\ 0,\ 0\right] \in L_2(0,T;\mathcal{D}(\mathbb{A}^{\frac{3}{2}})). \tag{5.48}$$

Then, (5.48), used in (5.44) yields $\eta \in C([0,T];\mathcal{D}(\mathbb{A}^{\frac{1}{2}}))$, as desired. Proposition 5.3, equation (5.36) is proved. ∎

Remark 5.3 *The trick, exhibited in (5.43), (5.44), works also in the purely thermo-elastic case* ($n = 2$) *of Triggiani (2008), thus reproducing, through a simpler proof, the interior regularity result of this paper (Lebiedzik and Triggiani, 2010). The latter yielded, however, both* interior *and* boundary *optimal regularity results.*

Step 8. We now return to the (z, w, θ)-problem (2.1 from the (ζ, υ, q)-problem (5.31)), and use relations in (5.32). Thus,

$$\begin{aligned} &z(t,x) = \zeta(t,x) + \xi(t,x); \quad w(t,x) = \upsilon(t,x) + \psi(t,x); \\ &\theta(t,x) = h(t,x) + q(t,x). \end{aligned} \tag{5.49}$$

Thus, we can obtain the regularity of (z, w, θ) from (ζ, υ, q) via that of $\{\psi, h\}$ obtained in Theorem 5.1, and of ξ obtained in (5.29), (5.30). First, we have, from (5.36),(5.37) of Proposition 5.3:

$$\begin{bmatrix} \zeta \\ \zeta_t \\ \upsilon \\ \upsilon_t \\ q \end{bmatrix} \in C\left([0,T]; \begin{bmatrix} H^{\frac{3}{2}}(\Omega)/\mathbb{R} & & \\ \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) & = & H^{\frac{1}{2}}(\Omega) \\ \mathcal{D}(A^{\frac{5}{8}}) & = & H_0^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0) \\ \mathcal{D}(A^{\frac{3}{8}}) & = & H_{00}^{\frac{3}{2}}(\Gamma_0) \\ \mathcal{D}(B^{\frac{1}{2}}) & = & H_0^1(\Gamma_0) \end{bmatrix}\right). \tag{5.50}$$

Moreover, Eqns. (5.29) for $\{\xi, \xi_t\}$ and Eqn. (5.2a–c) for $\{\psi, \psi_t, h\}$ give, respectively,

$$\begin{bmatrix} \xi \\ \xi_t \end{bmatrix} \in C\left([0,T]; \begin{bmatrix} H^{\frac{5}{3}}(\Omega) \\ H^{\frac{2}{3}}(\Omega) \end{bmatrix}\right);$$

$$\begin{bmatrix} \psi \\ \psi_t \\ h \end{bmatrix} \in C\left([0,T]; \begin{bmatrix} \mathcal{D}(A^{\frac{5}{8}}) & = & H^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0) \\ \mathcal{D}(A^{\frac{3}{8}}) & = & H_{00}^{\frac{3}{2}}(\Gamma_0) \\ & & H_0^{\frac{3}{2}-\epsilon}(\Gamma_0) \end{bmatrix}\right). \tag{5.51}$$

Finally, using in (5.49) the previous regularity statements, given by (5.50) and (5.51), we obtain

$$\begin{bmatrix} z \\ z_t \\ w \\ w_t \\ \theta \end{bmatrix} \in C\left([0,T]; \begin{bmatrix} H^{\frac{3}{2}}(\Omega)/\mathbb{R} & & \\ \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) & = & H^{\frac{1}{2}}(\Omega) \\ \mathcal{D}(A^{\frac{5}{8}}) & = & H_0^{\frac{5}{2}}(\Gamma_0) \cap H_0^2(\Gamma_0) \\ \mathcal{D}(A^{\frac{3}{8}}) & = & H_{00}^{\frac{3}{2}}(\Gamma_0) \\ \mathcal{D}(B^{\frac{1}{2}}) & = & H_0^1(\Gamma_0) \end{bmatrix}\right) \tag{5.52}$$

and Theorem 4.3 is proved. ∎

Acknowledgements. The authors wish to thank the two reviewers of the originally submitted version: one, for a query that stimulated the authors to carry out their investigation beyond the space of finite energy and into the space of optimal regularity of the input–solution map L; the other, for suggesting to provide a less terse exposition. The research of Catherine Lebiedzik was partially supported by the National Science Foundation under grant DMS-0708076. The research of Roberto Triggiani was partially supported by the National Science Foundation under grant DMS-0104305 and the AFOSR under grant FA9550-09-1-0459.

Appendix A: Proof of Proposition 4.2

As in Lasiecka and Triggiani (2000d, bottom of p. 890), given $u \in U \equiv R$, we seek $g \in \mathcal{D}(\mathbb{A}))$ such that $\mathcal{B}u = \mathbb{A}g$. By (3.19), we verify that

$$\mathcal{B}u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_\gamma^{-1}\delta u \\ 0 \end{bmatrix} \tag{A.1}$$

$$= \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ -\mathcal{A}_N & 0 & 0 & -\tilde{\mathcal{A}}_N N(\cdot|_{\Gamma_0}) & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & -B_\gamma^{-1}N^*\tilde{\mathcal{A}}_N & -B_\gamma^{-1}A & 0 & B_\gamma^{-1}B \\ 0 & 0 & 0 & -B & -B \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -A^{-1}\delta u \\ 0 \\ 0 \end{bmatrix}. \tag{A.2}$$

Hence, we deduce that the sought-after vector g is given by

$$\mathbb{A}^{-1}\mathcal{B}u = g = \begin{bmatrix} 0 \\ 0 \\ -A^{-1}\delta u \\ 0 \\ 0 \end{bmatrix}, \tag{A.3}$$

where $\mathbb{A}^{-1} \in \mathcal{L}(Y_\gamma)$. Thus, by (A.2) and the third component space $\mathcal{D}(A^{\frac{1}{2}})$ of Y_γ in (3.20), we see that in order to establish Proposition 4.2, we need to show that

$$g \in Y_\gamma \quad \text{or} \quad A^{-1}\delta u \in \mathcal{D}(A^{\frac{1}{2}}) \text{ or } A^{-\frac{1}{2}}\delta u \in L_2(\Gamma_0). \tag{A.4}$$

Indeed, we shall show below the stronger result that

$$A^{-1}\delta u = A^{-\frac{3}{4}+\frac{\epsilon}{4}} A^{-(\frac{1}{4}+\frac{\epsilon}{4})}\delta u \in \mathcal{D}\left(A^{\frac{3}{4}-\frac{\epsilon}{4}}\right) \subset \mathcal{D}(A^{\frac{1}{2}}),$$
$$\textit{a-fortiori } A^{-\frac{1}{2}}\delta u \in L_2(\Gamma_0), \tag{A.5}$$

since we shall establish, in fact, that (set theoretically)

$$\delta \in [H^{1+\epsilon}(\Gamma_0)]' \subset \left[\mathcal{D}(A^{\frac{1}{4}+\frac{\epsilon}{4}})\right]' \quad \text{or} \quad A^{-(\frac{1}{4}+\frac{\epsilon}{4})}\delta u \in L_2(\Gamma_0). \tag{A.6}$$

It remains to verify (A.5). To this end, recall that for dim $\Gamma_0 = 2$, Sobolev embedding yields:

$$\delta \in [H^{1+\epsilon}(\Gamma_0)]', \ \epsilon > 0, \ \text{arbitrary}. \tag{A.7}$$

Moreover, we shall establish below the following identification:

$$\mathcal{D}(A^{\frac{1}{4}+\frac{\epsilon}{4}}) = H_0^{1+\epsilon}(\Gamma_0) \subset H^{1+\epsilon}(\Gamma_0). \tag{A.8}$$

In fact, since Lions and Magenes (1972, Thm. 11.6, p. 64 and Thm. 11.7, p. 66) give

$$[H_0^2(\Gamma_0), H_0^1(\Gamma_0)]_\theta = H_0^{(1-\theta)2+\theta\cdot 1}(\Gamma_0) = H_0^{1+\epsilon}(\Gamma_0), \tag{A.9}$$

as long as $(1+\epsilon) \neq (\text{integer}+\frac{1}{2})$, whereby then $2(1-\theta)+\theta = 1+\epsilon$, i.e., $\theta = 1-\epsilon$. Also, from Lions and Magenes (1972, Eqn. (1.4a)) and recalling (3.5), one sees that the above is norm-equivalent to

$$[\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A^{\frac{1}{4}})]_{\theta=1-\epsilon} = \mathcal{D}(A^{\frac{1}{2}(1-\theta)+\frac{1}{4}\theta}) = \mathcal{D}(A^{\frac{1}{4}+\frac{\epsilon}{4}}) = H_0^{1+\epsilon}(\Gamma_0), \tag{A.10}$$

thus giving again (A.9). So, (A.7), (A.6), yield (A.5), as desired. The proof of [(A.5), hence of (A.3), hence of] $\mathcal{B}u = \mathbb{A}g$ is complete. ∎

Appendix B: Proof of containment (3.27c)

We return to equation (3.26):

$$\mathcal{D}(\mathbb{A}^{\frac{1}{2}}) = V_1 \times \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{5}{8}}) \times V_4 \times \mathcal{D}(B^{\frac{1}{2}}). \tag{B.1}$$

The goal of the present Appendix B is to show the following reverse containment (equation (3.27c).

PROPOSITION B.1 *We have*

$$\left\{ \begin{bmatrix} z_1 \\ z_4 \end{bmatrix} : z_1 \in H^{\frac{3}{2}}(\Omega)/\mathbb{R}, \ \Delta z_1 \in [H^{\frac{1}{2}}(\Omega)]' = H^{-\frac{1}{2}}(\Omega); \ z_4 \in H_{00}^{\frac{3}{2}}(\Gamma_0), \right.$$
$$\left. \frac{\partial}{\partial\nu} z_1\Big|_{\Gamma_1} = 0, \ \ \frac{\partial}{\partial\nu} z_1\Big|_{\Gamma_0} = -z_4 \right\} \subset V_1 \times V_4 \tag{B.2}$$

where the normal derivative $\frac{\partial}{\partial\nu}z\big|_\Gamma$ is a-priori well-defined in $H^{-\varepsilon}(\Gamma)$, $\varepsilon > 0$; see Claim #2 below. Recall that $f \in H^{\frac{3}{2}}(\Omega)$ automatically implies $\Delta f \in [H_{00}^{\frac{1}{2}}(\Omega)]'$.

Proof of Proposition B.1. We seek to apply Lions and Magenes (1972, Theorem 14.3, p. 97), whereby extra conditions that define subspaces are preserved under interpolation. The first and fourth space components, $S_1 \times S_4$, of the domain $\mathcal{D}(\mathbb{A})$ of $\mathbb{A}$ readily define a subspace of $H^2(\Omega) \times H_0^2(\Gamma_0)$, see (3.24c). However, the first space component of the space Y_γ in (3.20)—that is, the component $\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \equiv H^1(\Omega)/\mathbb{R}$—is at a too low topological level and does not support the normal derivative on the boundary Γ, as a well-defined operation for each of its members. To remedy this we then introduce the space $\mathbb{Y}_1$ below, $\mathbb{Y}_1 \subset \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, but $\mathbb{Y}_1$ at the same toplogical level as $\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$ for which—in contrast—the normal derivative on Γ is well-defined in $H^{-\frac{1}{2}}(\Gamma)$, see Claim #1 below. To fall into the setting of Lions and Magenes (1972, Section 14.3, p. 96), we introduce the following spaces and operator δ:

$$X \equiv \begin{bmatrix} X_1 \equiv H^2(\Omega) \\ X_4 \equiv H_0^2(\Gamma_0) \end{bmatrix} \subset \Phi; \; X_1 \equiv \left\{ x_1 \in H^2(\Omega) \text{ so that } \Delta x_1 \in L_2(\Omega) \right\} \tag{B.3}$$

$$\mathcal{X} \equiv \begin{bmatrix} L_2(\Omega) \\ 0 \text{ on } \Gamma_1 \\ H_0^2(\Gamma_0) \end{bmatrix} \equiv \bar{\mathcal{X}} \subset \Psi; \quad \delta \in \mathcal{L}(X; \mathcal{X}) \tag{B.4}$$

$$(X)_{\delta,\mathcal{X}} \equiv \left\{ \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \in X : \delta \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \equiv \begin{cases} \Delta x_1 \in L_2(\Omega) & \text{(B.5a)} \\ \dfrac{\partial}{\partial \nu} x_1 \Big|_{\Gamma_1} = 0 & \text{(B.5b)} \\ \dfrac{\partial}{\partial \nu} x_1 \Big|_{\Gamma_0} = -x_4 \in H_0^2(\Gamma_0) & \text{(B.5c)} \end{cases} \right\}$$

so that by equation (3.24c)

$$(X)_{\delta,\mathcal{X}} = \left\{ \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \in X : \; \delta \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \in \mathcal{X} \right\} \equiv S_1 \times S_4; \tag{B.6}$$

$$\mathbb{Y} \equiv \begin{bmatrix} \mathbb{Y}_1 \\ \mathbb{Y}_4 \equiv H_0^1(\Gamma_0) \end{bmatrix} \equiv \Phi; \; \mathbb{Y}_1 = \left\{ y_1 \in H^1(\Omega)/\mathbb{R} : \; \Delta y_1 \in [H^1(\Omega)/\mathbb{R}]' \right\} \tag{B.7}$$

Thus, $\mathbb{Y}_1$ is at the same topological level as $\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, but $\mathbb{Y}_1 \subset \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$, since $y_1 \in \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})$ implies automatically that $\Delta y_1 \in H^{-1}(\Omega)/\mathbb{R}$, where $[H^1(\Omega)/\mathbb{R}]' \subset H^1(\Omega)/\mathbb{R}$.

It is shown below in Claim #1 that in $\mathbb{Y}_1$, the normal derivative on Γ is well-defined in $H^{-\frac{1}{2}}(\Gamma)$;

$$\mathcal{Y} \equiv \begin{bmatrix} [H^1(\Omega)/\mathbb{R}]' \\ 0 \text{ on } \Gamma_1 \\ H_0^1(\Gamma_0) \end{bmatrix} \equiv \bar{\mathcal{Y}} \equiv \Psi \tag{B.8}$$

$$(\mathbb{Y})_{\delta,\mathcal{Y}} \equiv \left\{ \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} \in \mathbb{Y} : \delta \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} \equiv \left\{ \begin{array}{ll} \Delta y_1 \in [H^1(\Omega)/\mathbb{R}]' & \text{(B.9a)} \\ \left.\dfrac{\partial}{\partial\nu} y_1\right|_{\Gamma_1} = 0 & \text{(B.9b)} \\ \left.\dfrac{\partial}{\partial\nu} y_1\right|_{\Gamma_0} = -y_4 \in H_0^2(\Gamma_0) & \text{(B.9c)} \end{array} \right. \right\}$$

so that recalling also (3.20)

$$\begin{aligned} (\mathbb{Y})_{\delta,\mathcal{Y}} &= \left\{ \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} \in \mathbb{Y} :\ \delta \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} \in \mathcal{Y} \right\} \subset H^1(\Omega)/\mathbb{R} \times H_0^1(\Gamma_0) \\ &\equiv \text{first } \times \text{ fourth components of } Y_\gamma \end{aligned} \tag{B.10}$$

The definitions of Φ, Ψ, $\bar{\mathcal{X}}$, $\bar{\mathcal{Y}}$ satisfy properties in Lions and Magenes (1972, Eq (14.18), (14.23)(i), pp. 96-97).

Claim #1

$$y_1 \in \mathbb{Y}_1 \Rightarrow \left.\frac{\partial}{\partial\nu} y_1\right|_{\Gamma} \quad \text{exists well-defined in } H^{-\frac{1}{2}}(\Gamma) \tag{B.11a}$$

continuously:

$$\left|\left| \left.\frac{\partial}{\partial\nu} y_1\right|_{\Gamma} \right|\right|_{H^{-\frac{1}{2}}(\Gamma)} \leq c||y_1||_{H^1(\Omega)}, \quad y_1 \in \mathbb{Y}_1. \tag{B.11b}$$

Proof of Claim #1 This Claim #1 follows by an application of Green's first theorem, upon integration by parts against a test function $\varphi \in H^1(\Omega)/\mathbb{R}$

$$\int_\Omega \Delta y_1 \varphi \, d\Omega = \int_\Gamma \frac{\partial}{\partial\nu} y_1 \varphi \, d\Gamma - \int_\Omega .\nabla y_1 \cdot \nabla\varphi \, d\Omega. \tag{B.12}$$

With $\Delta y_1 \in [H^1(\Omega)/\mathbb{R}]'$, $y_1 \in H^1(\Omega)/\mathbb{R}$ and $\varphi \in H^1(\Omega)/\mathbb{R}$, the first and last integral terms on Ω in (B.12) are well-defined. Thus, so is the boundary integral term $(\frac{\partial}{\partial\nu} y_1\big|_\Gamma, \varphi_\Gamma)_\Gamma$, where $\varphi|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ runs over all of $H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ by surjectivity of the trace operator, as φ runs over $H^1(\Omega)/\mathbb{R}$. Thus, $\frac{\partial}{\partial\nu} y_1\big|_\Gamma$ is well-defined in $H^{-\frac{1}{2}}(\Gamma)$, as claimed by (B.11a). Moreover, (B.12) implies via the above argument also the quantitative estimate (B.11b), as

$$||\Delta y_1||_{[H^1(\Omega)]'} \leq c||\Delta y_1||_{H^{-1}(\Omega)} \leq C||y_1||_{H^1(\Omega)} \tag{B.13}$$

via $H_0^1(\Omega) \subset H^1(\Omega)$, hence $[H^1(\Omega)]' \subset [H_0^1(\Omega)]' = H^{-1}(\Omega)$, as well as via Lions and Magenes (1972, p. 85). Thus, Claim #1 is established. ∎

Notice that Claim #1 implies, via the spaces Φ and Ψ defined in (B.7), (B.8), that the operator δ defined in (B.5) satisfies the first and third relationships of boundedness below:

$$\delta \in \mathcal{L}(\Phi; \Psi); \quad \delta \in \mathcal{L}(X; \bar{\mathcal{X}}), \quad \delta \in \mathcal{L}(\mathbb{Y}; \bar{\mathcal{Y}}) \tag{B.14}$$

as required by Lions and Magenes (1972, eq. (14.19), (14.23)(ii), pp. 96-97), while the second relationship in (B.14) is plain with $\bar{\mathcal{X}} \equiv \mathcal{X}$ as in (B.4). Finally, we need to check the remaining assumption Lions and Magenes (1972, (14.23)(iii)). To this end, we define the operator $\mathcal{G}$ as the solution z_1 (modulo a constant) of the following elliptic problem:

$$\Delta z_1 = \chi_1 \tag{B.15a}$$

$$z_1 = \mathcal{G}\chi = \mathcal{G}\begin{bmatrix} \chi_1 \\ 0 \\ \chi_2 \end{bmatrix} :\equiv \begin{cases} \left.\dfrac{\partial}{\partial \nu} z_1\right|_{\Gamma_1} = 0 & \text{(B.15b)} \\ \left.\dfrac{\partial}{\partial \nu} z_1\right|_{\Gamma_0} = \chi_2 & \text{(B.15c)} \end{cases}$$

so that recalling (B.3),(B.4),(B.7), and (B.8):

$$\mathcal{G} \in \mathcal{L}(\bar{\mathcal{X}} \equiv \mathcal{X}; X) \text{ and } \mathcal{G} \in \mathcal{L}(\bar{\mathcal{Y}} \equiv \Psi; \mathbb{Y}) \tag{B.16}$$

as required; moreover, we have $\delta\mathcal{G}\chi = \mathcal{X}$:

$$\delta\mathcal{G}\chi = \begin{cases} \chi_1 \\ 0 \\ \chi_2 \end{cases} \qquad \forall \chi = \begin{bmatrix} \chi_1 \\ 0 \\ \chi_2 \end{bmatrix} \in \bar{\mathcal{X}} + \bar{\mathcal{Y}} \equiv \bar{\mathcal{Y}} \subset \Psi, \quad \nu = 0. \tag{B.17}$$

We have thus satisfied all the required assumptions of Lions and Magenes (1972, Theorem 14.3, p. 97). Application of this, then, yields

$$[(X)_{\delta,\mathcal{X}}, (\mathbb{Y})_{\delta,\mathcal{Y}}]_{\frac{1}{2}} = \Big([X, \mathbb{Y}]_{\frac{1}{2}}\Big)_{\delta,[\mathcal{X},\mathcal{Y}]_{\frac{1}{2}}}, \tag{B.18}$$

where, by (B.3) and (B.7), we compute

$$[X, \mathbb{Y}]_{\frac{1}{2}} = \left[\begin{bmatrix} x_1 \in H^2(\Omega), \\ x_4 \in H_0^2(\Gamma_0) \end{bmatrix}, \begin{bmatrix} y_1 \in H^1(\Omega)/\mathbb{R} : \Delta y_1 \in [H^1(\Omega)/\mathbb{R}]' \\ y_4 \in H_0^1(\Gamma_0) \end{bmatrix}\right]_{\frac{1}{2}} \tag{B.19}$$

$$= \begin{bmatrix} x_1 \in H^{\frac{3}{2}}(\Omega) : \ \Delta x_1 \in [H^{\frac{1}{2}}(\Omega)]' \\ x_4 \in H_{00}^{\frac{3}{2}}(\Gamma_0) \end{bmatrix}. \tag{B.20}$$

Moreover, by (B.4) and (B.8), we compute

$$[X, \mathcal{Y}]_{\frac{1}{2}} = \left[\begin{bmatrix} L_2(\Omega) \\ 0 \text{ on } \Gamma_1 \\ H_0^2(\Gamma_0) \end{bmatrix}, \begin{bmatrix} [H^1(\Omega)/\mathbb{R}]' \\ 0 \text{ on } \Gamma_1 \\ H_0^1(\Gamma_0) \end{bmatrix} \right]_{\frac{1}{2}} = \begin{bmatrix} [H^{\frac{1}{2}}(\Omega)]' \\ 0 \text{ on } \Gamma_1 \\ H_{00}^{\frac{3}{2}}(\Gamma_0) \end{bmatrix}. \tag{B.21}$$

To conclude the proof of Proposition B.1 that

$$\left([X, \mathbb{Y}]_{\frac{1}{2}}\right)_{\delta, [\mathcal{X}, \mathcal{Y}]_{\frac{1}{2}}} = \text{ LHS of } (B.2) \tag{B.22}$$

via (B.20) it remains to show that the constraint

$$\delta : [X, \mathbb{Y}]_{\frac{1}{2}} \to [\mathcal{X}, \mathcal{Y}]_{\frac{1}{2}} \tag{B.23}$$

is automatically satisfied. This follows from the following Claim #2:

Claim #2

$$z_1 \in Z_1 \text{ meaning } \begin{array}{c} z_1 \in H^{\frac{3}{2}}(\Omega) : \\ \Delta z_1 \in [H^{\frac{1}{2}}(\Omega)]' \end{array} \Rightarrow$$
$$\frac{\partial}{\partial \nu} z_1 \Big|_{\Gamma} \text{ exists well-defined in } H^{-\varepsilon}(\Gamma),\ \varepsilon > 0 \tag{B.24a}$$

continuously:

$$\left|\left| \frac{\partial}{\partial \nu} z_1 \Big|_{\Gamma} \right|\right|_{H^{-\varepsilon}(\Gamma)} \leq c ||z_1||_{H^{\frac{3}{2}}(\Omega)}, \quad z_1 \in Z_1 \tag{B.24b}$$

Proof of Claim #2 Again, this claim follows by use of Green's first theorem, upon integration by parts against a test function $\varphi \in H^{\frac{1}{2}+\varepsilon}(\Omega) \subset H^{\frac{1}{2}}(\Omega),\ \forall \varepsilon > 0:$

$$\int_{\Omega} \Delta z_1 \varphi \, d\Omega = \int_{\Gamma} \frac{\partial}{\partial \nu} z_1 \varphi \, d\Gamma - \int_{\Omega} \nabla z_1 \cdot \nabla \varphi \, d\Omega. \tag{B.25}$$

With $\Delta z_1 \in [H^{\frac{1}{2}}(\Omega)]' = H^{-\frac{1}{2}}(\Omega)$ (Lions and Magenes, 1972, p. 55) and $\varphi \in H^{\frac{1}{2}}(\Omega)$, the first interior integral term on the LHS of (B.25) is well-defined. Next, $z_1 \in H^{\frac{3}{2}}(\Omega)$ implies $\nabla z_1 \in H^{\frac{1}{2}-\varepsilon}(\Omega)$, while $\nabla \varphi \in H^{-\frac{1}{2}+\varepsilon}(\Omega) = H^{-(\frac{1}{2}-\varepsilon)}(\Omega)$, so that the third interior integral term on the RHS of (B.25) is well-defined. Hence the boundary integral term $(\frac{\partial}{\partial \nu} z_1 \big|_{\Gamma}, \varphi|_{\Gamma})_{\Gamma}$ is well-defined, where $\varphi|_{\Gamma} \in H^{\varepsilon}(\Gamma)$ runs over all of $H^{\varepsilon}(\Gamma)$ by surjectivity of the trace operator as φ runs over all of $H^{\frac{1}{2}+\varepsilon}(\Omega)$. Hence $\frac{\partial}{\partial \nu} z_1 \big|_{\Gamma} \in H^{-\varepsilon}(\Gamma)$ is well-defined as claimed in (B.24a) and then the above argument yields the bound in (B.24b). Claim #2 is established. ∎

Remark 5.4 *In contrast, the space $H^{\frac{3}{2}}(\Omega)$ does not support the normal derivative as a well-defined operation on the boundary.*

With the $\frac{\partial}{\partial\nu}\big|_\Gamma$ well-defined by Claim #2, we see that relation (B.23) for δ is automatically satisfied, by use of (B.20) and (B.21), in view of the definition of δ in (B.5). Thus, relation (B.23) holds true and Proposition B.1 is established. ■

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