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# A Bolza optimal synthesis problem for singular estimate control systems*

by

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**Abstract:** Bolza problem governed by PDE control systems with unbounded controls is considered. The motivating example is fluid structure interaction model with boundary-interface controls. The aim of the work is to provide optimal feedback synthesis associated with well defined gain operator constructed from the Riccati equation.

The dynamics considered is of mixed parabolic-hyperbolic type which prevents applicability of tools developed earlier for analytic semigroups.

It is shown, however, that the control operator along with the generator of the semigroup under consideration satisfy singular estimate referred to as Revisited Singular Estimate (RSE) . This estimate, which measures "unboundedness" of control actions, is a generalization and a weaker form of Singular Estimate (SE) treated in the past literature.

The main result of the paper provides Riccati theory developed for this new class of control systems labeled as RSECS (Revisited Singular Estimate Control Systems). The important feature is that the gain operator, constructed via Riccati operator, is consistent with the optimal feedback synthesis. The gain operator, though unbounded, has a controlled algebraically singularity at the terminal point. This enables one to establish well-posedness of the Riccati solutions and of the optimal feedback representation.

An application of the theoretical framework to boundary control of a fluid-structure interaction model is given.

**Keywords:** Bolza problem, Riccati equation, singular estimate control systems, fluid-structure interaction.

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## 1. Introduction

We consider a boundary control problem governed by $A : D(A) \subset Y \to Y$, a generator of a strongly continuous semigroup $e^{At}$ defined on the Hilbert space $Y$, and an *unbounded* control operator $B : U \to [D(A^*)]'$, where the duality is with respect to the pivot space $Y$. More specifically, we are interested in minimizing a suitable objective functional defined on $L_2([0,T];U)$ for the dynamics driven by the abstract control system with *unbounded* control operator $B$

$$y_t = Ay + Bu, \ in \ [D(A^*)]', \ \ y(0) \in Y. \tag{1}$$

Of particular interest is the optimal feedback synthesis and the associated Riccati equations. This particular framework is motivated by a multitude of examples arising in the context of boundary or point control problems governed by partial differential equations (PDE) (see Bensoussan, Da Prato, Delfour & Mitter, 2007, and Lasiecka & Triggiani, 2000). For these models, the control operators are intrinsically "unbounded" with the action taking values in a larger space $[D(A^*)]'$ than simply the state space $Y$. This, of course, forces a nonstandard formulation of the control problem that is rooted in distribution theory. In response to an increasing importance in engineering applications, the last two decades or so have witnessed rapid developments of new tools and methodologies which can treat *unbounded control actions*. The semigroup framework was found suitable for the formulation of this class of problems since it provides a natural generalization of ODE theory where the dynamics resides in a suitable Hilbert or Banach space. In fact, the very first treatment of boundary control problems defined for the heat equation within the semigroup framework goes back to Balakrishnan (1975). Later on, Lasiecka (1980) provided a unified treatment of regularity and control theoretic properties of unbounded action control problems governed by *analytic* semigroups. The analysis provided in these first developments critically depends on regularity of the dynamics induced by the *analyticity* property of the semigroup under consideration. This, along with singular integrals theory, allowed for a natural extension of the variation of parameters formula taking its values in the basic state space $Y$ (though the nontrivial range of the operator $B$ is never in $Y$). Comprehensive and essentially complete treatment of *unbounded control* optimal control problems governed by analytic semigroups can be found in the books by Bensoussan, Da Prato, Delfour & Mitter (2007) and Lasiecka & Triggiani (2000, vol. I) as well as in the many references therein. In the case of *non-analytic* dynamics, the situation is more complex. In fact, the class of unbounded control operators considered within the framework of general semigroups has been labeled, after an influential review paper by Russell (1978), as "admissible controls". "Admissible controls" can be identified with control operators $B$ and the generators $A$ satisfying the following abstract estimate:

$$\int_0^T \|B^* e^{A^* t} x\|_U^2 dt \le C \|x\|_Y^2, x \in \mathcal{D}(A^*). \tag{2}$$

In Lasiecka & Triggiani (1983) (with generalizations in Lasiecka, Lions & Triggiani, 1986), it was shown that a Dirichlet boundary control action applied to a classical wave equation defined on a bounded domain in $R^n$ is "admissible", i.e. it satisfies the estimate (2). In fact, this PDE regularity result (which amounts to saying that the co-normal derivative on the boundary of the solution to the homogeneous wave equation is bounded in $L_2(\Sigma)$ by initial data in $H_0^1(\Omega) \times L_2(\Omega)$ ) is often referred to as "hidden regularity" or "trace regularity" - see Bensoussan, Da Prato, Delfour & Mitter (1993), Lasiecka & Triggiani (2000), Lions (1988), and references therein. Lasiecka & Triggiani (1986) exploit this trace result in order to build Riccati theory for hyperbolic PDEs with Dirichlet boundary controls. It has turned out that the trace estimate (2) is valid for many PDEs with unbounded control actions, including Schrodinger and plate equations with boundary or point controls (see Lasiecka & Triggiani (1991 and 2000, vol. II)). These and other developments led to the axiomatization of the properties discovered in these hyperbolic-like PDE's with an eye on some generalizations at the abstract - semigroup level, particularly in the direction of infinite dimensional system theory; see Salamon (1989), Weiss (1989), Staffans (2005) and the review paper by Jacob & Partingon (2004). On the other hand, trace estimates alone, particularly within the context of hyperbolic equations, turn out to be insufficient for a complete and general Riccati theory with unbounded control actions. Indeed, while there are several results in this area, e.g. Lasiecka & Triggiani (2000), Barbu, Lasiecka & Triggiani (2000), Camurdan (1995), the results often depend on a particular setup and particular properties of the dynamics considered. Even more, it is known that in the purely hyperbolic case, the optimal feedback synthesis may not be consistent with the Riccati feedback synthesis generated by a solution to the Riccati equation (see Weiss & Zwart, 1998). This raises an obvious concern and questions on how to characterize certain classes of dynamics so that the classical optimal feedback control theory admits natural generalizations that include consistency of feedback representations via solutions to a Riccati equation. It was in this context, with critical inspiration by Balakrishnan (1975), that the class of *Singular Estimate Control Systems* (SECS) was introduced in Avalos & Lasiecka (1996) and Lasiecka (2002). This class, on the one hand, captures very basic mathematical properties of the controlled dynamics and, on the other hand, responds to technological needs arising in *coupled dynamics* which combine parabolic and hyperbolic components. Mathematical classification of this class of systems is defined by requesting that the pair $(A, B)$ satisfy the so called *Singular Estimate*

$$\|e^{At}B\|_{\mathcal{L}(U,Y)} \leq \frac{c}{t^\gamma}, \quad t \in (0,1]. \tag{3}$$

The above class of systems (SECS) is a natural generalization of control systems governed by *analytic* semigroups with relatively bounded control operators. On the other hand, the estimate is representative of a large class of controlled dynamics, which are of hyperbolic-parabolic type, hence non-analytic. Canonical examples of systems displaying the Singular Estimate (SE) (3) are coupled PDE structures with parabolic-hyperbolic interactive dynamics. These include structural acoustic interactions, thermoelastic interactions, magneto-structure interactions; see Avalos & Lasiecka (1996), Lasiecka (2002), Bucci & Lasiecka (2004), Bucci, Lasiecka & Triggiani (2002), Aquistapace, Bucci & Lasiecka (2005), Bucci (2007, 2008), Lasiecka & Triggiani (2004) and Lasiecka (2004). A comprehensive theory of optimal feedback synthesis pertaining to (SECS systems) can be found in Lasiecka (2002, 2004) and Lasiecka & Tuffaha (2008). It turns out, however, that there are other interesting PDE systems studied which fall a little short of the condition required in order to be in the SECS class (i.e. $\gamma = 1$). On the other hand, appropriate transfer functions corresponding to these systems do satisfy the singular estimate. Examples of such systems include: fluid-structure interaction systems with boundary - force - control, some thermoelastic plates with either boundary or point controls (see Bucci, Lasiecka & Triggiani, 2002; Aquistapace, Bucci & Lasiecka, 2005). The goal of this paper is to present a theory that covers this new class of problems satisfying the so called Revisited Singular Estimate (RSECS), which is characterized by a singular estimate imposed on the "input-output" map (7), rather than the "input-state" map, as in (3). As we shall see, this extension is critical, from both mathematical and application points of view. Indeed, on the application side, the extended class encompasses new control models arising in fluid-structure interactions (see Lasiecka & Tuffaha, 2009) and, on the mathematical side, the new difficulties include not only singular behavior of the feedback at the terminal point, but also the lack of time invariance of evolution processes (so critical to the notion of the feedback) that are no longer evolving within the original state space $Y$. In order to cope with these, new functional analytic apparatus and spaces capturing arising singularities are introduced.

## 2. The problem and the main result

We consider a Bolza optimal control problem formulated for a system satisfying the Revisited Singular Estimate (RSE). Bolza control problems are of particular mathematical interest, owing to the fact that a final state penalization with *unbounded controls* leads to unbounded state operators, which holds even in the analytic case. This translates into singularities in both the control and the associated gain operator, which are exhibited at the terminal state. The above feature requires careful analysis of the singularities and the terminal time blow-up associated with the control action.

### 2.1. Formulation of the problem

We study an abstract Bolza control problem that models PDE control systems with point or boundary control. Let $Y$ the state space and $U$ the control space be all Hilbert spaces, and consider the quadratic Bolza control problem in which the goal is to minimize the cost functional

$$J(u, y, s, y_s) = \int_s^T \|u(t)\|_U^2 \, dt + \|Gy(T)\|_Z^2 \tag{4}$$

on the time interval $[s, T]$ over all $u \in L_2([s, T]; U)$ subject to the dynamics satisfying the abstract differential equation

$$y_t = Ay + Bu \quad \text{on} \quad [\mathcal{D}(A^\star)]' \tag{5}$$

$$y(s) = y_s \quad \text{in} \quad Y. \tag{6}$$

The operators $A$, $B$, and $G$ are all linear operators satisfying the conditions

Assumption 1 *(a) $A$ is a generator of a strongly continuous semigroup on a Hilbert space $Y$ denoted by $e^{At}$ .*

*(b) $B$ is a linear operator from $U \to [\mathcal{D}(A^\star)]'$ such that $R(\lambda, A)B \in \mathcal{L}(U, Y)$, for some $\lambda \in \rho(A)$, where $R(\lambda, A)$ is the resolvent of $A$. Without loss of generality, we can assume that $\lambda = 0$ and hence $A^{-1}B \in \mathcal{L}(U; Y)$.*

*(c) There exists a parameter $\gamma \in (0, 1)$ such that*

$$\|Ge^{At}Bu\|_Z \leq \frac{C}{t^\gamma}\|u\|_U \tag{7}$$

*for all $0 < t \leq 1$.*

*(d) $G$ is a bounded linear operator from $Y$ to $Z$ another Hilbert space.*

We first define the control-state operator $L_s : L_2([s, T]; U) \to C([s, T]; [D(A^*)]')$ as

$$(L_s u)(t) \equiv A \int_s^t e^{A(t-\tau)} A^{-1} B u(\tau) d\tau. \tag{8}$$

By Assumption (1)(b) this operator is linear and bounded within the topologies indicated above with the values in a dual space typically related to some distributions. We also define the pointwise $GL_{sT}$ as

$$GL_{sT} u \equiv (GL_s u)(T) \tag{9}$$

from $D(GL_{sT}) \subset L_2([s, T]; U)$ to $Z$. It is known (see Lasiecka, 2002, 2004) that under Assumption (1)(b) we have $H_0^1([0, T]; U) \subset D(L_{sT})$. Therefore, the operator $L_{sT}$ is densely defined and also closed (note that $A^{-1}L_{sT}$ is bounded). The operator $L_{sT}$, describing terminal action of the control, plays a critical role in the study of the Bolza problem. The first difficulty encountered is that this operator is *not bounded* on the control space $L_2([s, T]; U)$. Thus, the functional cost $J(u, y, s, y_s)$ is *not bounded*. This feature is in sharp contrast with the SECS theory for problems *without final state penalization*. The *necessary and sufficient* condition, for the existence of a unique minimizer to the problem is the condition that

Assumption 2 *The operator $GL_{sT}$ is closed from $L_2([s,T];U) \to Z$.*

Remark 1 *We note that Assumption* (2) *is always satisfied when $G$ is invertible $Z \to Y$. On the other hand, it is known that the lack of closability leads to counterexamples to the very existence of the optimal control (see Flandoli, 1984, and Lasiecka & Triggiani, 2000).*

## 2.2. Main result

In the theorem that follows, we formulate our result in regards to existence and regularity of the optimal control and state.

Theorem 1 *Under the set of conditions in Assumption* (1) *and Assumption* (2)*, we have that for any initial state $y_s \in Y$ there exists a unique optimal control $u^0(t,s,y_s) \in L_2([s,T];U)$ and optimal trajectory $y^0(t,s,y_s) \in C([s,T];[\mathcal{D}(A^\star)]')$ such that $J(u^0,y^0,s,y_s) = min_{u \in L_2([s,T],U)} J(u,y(u),s,y_s)$. Moreover, the optimal solutions satisfy*

*(a) The optimal control $u^0(t)$ is continuous on $[s,T)$ with values in $U$ but has a singularity of order $\gamma$ at the terminal time. More specifically, we have*

$$\|u^0(t,s,y_s)\|_U \le \frac{C}{(T-t)^\gamma}, \quad s \le t < T. \tag{10}$$

*(b) The observed optimal output $Gy^0(t)$ is continuous on $[s,T]$. In particular, we have*

$$\|Gy^0(t,s,y_s)\|_Z \le \|y_s\|_Y, \quad s \le t < T. \tag{11}$$

The following Theorem provides a characterization of the Riccati operator $P(t)$ as a value function. The main result of this paper, formulated below, asserts that the feedback operator is well defined on $[0,T)$ and exhibits controlled singularity at the point $t = T$. This property allows one to prove well-posedness of the Riccati equation and of the optimal synthesis.

Theorem 2 *Under the assumptions of Theorem 1, we have*

*(a) With $J(u^0,y^0,s,y_s) \equiv \min_{u \in L_2([s,T],U)} J(u,y(u),s,y_s)$ we have that there exists a self-adjoint positive operator $P(t) \in \mathcal{L}(Y)$ with $t \in [0,T)$ such that $\langle P(t)x, x\rangle_Y = J(u^0,y^0,s,x)$.*

*(b) (i) $P(t)$ is continuous on $[0,T]$ and $P(t) \in \mathcal{L}(Y;C([0,T];Y))$.*

*(ii) The feedback operator $B^*P(t) \in \mathcal{L}(Y;C([s,T),U))$ exhibits the singularity*

$$\|B^*P(t)x\|_U \le \frac{C\|x\|_Y}{(T-t)^\gamma}, \quad 0 \le t < T. \tag{12}$$

*(c) The optimal control $u^0$ is given by the feedback formula*

$$u^0(t,s;y_s) = -B^*P(t)y^0(t,s;y_s), \quad s \le t < T. \tag{13}$$

*(d)* *$P(t)$ satisfies the differential Riccati equation*

$$\langle P_t x, y\rangle_Y + \langle P(t)x, Ay\rangle_Y + \langle P(t)Ax, y\rangle_Y = \langle B^*P(t)x, B^*P(t)y\rangle_U. \quad (14)$$

*with for all $t < T$,* $\;x, y \in \mathcal{D}(A)$*, in addition to the condition*

$$\lim_{t\to T} P(t)x = G^*Gx \quad \forall x \in Y. \quad (15)$$

*(e)* *The solution of the Riccati equation above is unique in the class of positive and self-adjoint operators $P(t)$ satisfying* (12) *with $\gamma < \frac{1}{2}$.*

As an illustration of the abstract theory described above we shall present in Section 8 an example of fluid-structure interaction with boundary controls, which exhibits all the properties postulated by the theory. This particular model serves as a prime example of hyperbolic-parabolic coupling, therefore not analytic, and for which more standard SE estimate (3) fails. Yet, as shown later, the Revisited Singular Estimate (RSE) (7) is satisfied, and hence the conclusions of Theorem 2 are applicable.

### 2.3. Discussion of the results

- The results above are known in the case when $A$ generates an analytic semigroup and $A^{-\gamma}B \in \mathcal{L}(U;Y)$, (see Lasiecka & Triggiani, 2000, and Bensoussan, Da Prato, Delfour & Mitter, 2007, and references therein). In such case, the standard singular estimate $\|e^{At}Bu\|_Y \leq Ct^{-\gamma}$ holds by the virtue of analyticity and the theory of fractional powers of closed operators. In fact, the singularity index exhibited is exactly the same.
- In the more general case of an arbitrary $C_0$-semigroup and functional cost (4) with $G = 0$ and time distributed observation $\int_0^T \|Ry(t)\|_Y^2 dt$, the corresponding results of Theorems 1 and 2, under $SE$ estimate (3) have been proved in Avalos & Lasiecka (1996) – in the particular case of a structural acoustic model. More general abstract treatment is given in Lasiecka (2002, 2004) and Lasiecka & Triggiani (2004). In this case, there is no singularity of the control and gain operator exhibited at the terminal time. In fact, the control is continuous over the entire time of existence.
- Problems that include Bolza penalization are more delicate. This is due to potential singularity of feedback control (see Lasiecka & Triggiani, 2000, Flandoli, 1984, and Frankowska & Ochal, 2005), as manifested even in the case of analytic dynamics (see Bensoussan, Da Prato, Delfour & Mitter, 2007, Lasiecka & Triggiani, 2000, and Flandoli, 1984). The first treatment of Bolza problem with unbounded coefficients and within the non-analytic setting has been presented in Lasiecka & Tuffaha (2008). Lasiecka & Tuffaha (2008) provide full well-posedness theory for Riccati equations along with feedback synthesis derived under the assumption of SE estimate (3).

In view of the above, the novel contribution of this article is the fact that the system under consideration is not required to satisfy the singular estimate (SE) but the (RSE), which is not only much weaker requirement, but also a correct setup for important PDE applications - see Section 8. Mathematical consequences of this relaxation of the assumption are substantial. The main new issue encountered is that the state is no longer defined as a trajectory in $Y$. In fact, the state $y$ takes its values in an extended space only. This, in turn, raises the issue of the correct definition of the optimal evolution. For the latter it is essential that the state be invariant on some coherent space. This is no longer true under weaker RSE hypotheses. As a consequence, a careful definition of the evolution and transition property is necessary. For this, special topological structures will be introduced. On the other hand, the relaxation of the singular estimate is motivated by important applications, such as fluid-structure interaction control systems with boundary force control (see Lasiecka & Tuffaha, 2009). Indeed, in that case classical SE "fails by an $\epsilon$". This is due to a loss of maximal parabolic regularity at the level of $L_\infty$ spaces. On the other hand, one can show that the RSE does hold, and hence solve the problem. The remainder of the paper is devoted to the proof of both theorems.

REMARK 2 *We note that further generalizations of SECS systems - without terminal state Bolza penalization - became available and were considered in Acquistapace, Bucci & Lasiecka (2005). These apply to models where the singular estimate is satisfied only locally - for some components of the system. The remaining part of the system is required to satisfy certain $L_p$ "admissibility". For such class of control systems, it is shown in Acquistapace, Bucci & Lasiecka (2005) that the gain operator $B^*P(t)$ is densely defined only as an intrinsically unbounded operator. This property suffices for the construction of a coherent Riccati theory with an optimal feedback operator, which is consistent with the Riccati gain operator. Applications of this theory to thermoelasticity and fluid-structure interactions are given, respectively, in Acquistapace, Bucci & Lasiecka (2005) and Bucci & Lasiecka (2009).*

## 3. Existence, uniqueness and characterization of the optimal control

The first step in our analysis is to establish the existence of a unique optimal control in the space $L_2([0,T];U)$ to the problem defined in (5) and (4).

### 3.1. Preliminary results and definitions

In this section, we provide a functional analytic framework for studying the problem and we collect a number of properties that are known already. With operator $L_s$ defined in (8), it is well known (see Lasiecka, 2002, 2004, and Lasiecka & Triggiani, 2004) that the trajectory $y$ due to the input $u$ and initial condition $y_s$ is given by

$$y(t, s; y_0) = e^{A(t-s)}y_0 + (L_s u)(t). \tag{16}$$

In addition, due to the singular estimate assumption, the operator $GL_s$ is bounded from $L_p([s,T];U) \to L_p([s,T];Z)$ for $p > 1$, which can be easily established using Young's inequality. We introduce the adjoint operators to $GL_s : L_2([s,T];U) \to L_2([s,T];Z)$ defined with respect to the $L_2$ topology as

$$(GL_s)^\star f(t) = \int_t^T B^\star e^{A^\star(s-t)} G^\star f(s) ds \tag{17}$$

and bounded from $L_2([s,T];Z)$ to $L_2([s,T];U)$ uniformly with respect to $s$. The adjoint of $GL_{sT} : \mathcal{D}(GL_{st}) \subset L_2([s,T];U) \to Z$ is given by

$$(L_{sT}^\star G^\star z)(t) = B^\star e^{A^\star(T-t)} G^\star z \tag{18}$$

from $D(L_{sT}^\star G^\star) \subset Z$ to $L_2([s,T];u)$. Next, consider the composition $GL_{sT}$ which is densely defined, since $G$ is bounded and is closable by assumption (2) and this enables us to define a new Hilbert space $V([s,T];U)$ as the closure of $\mathcal{D}(GL_{sT})$ when equipped with the inner product

$$\langle u, v\rangle_{V([s,T];U)} = \langle u, v\rangle_{L_2([s,T];U)} + \langle GL_{sT}u, GL_{sT}v\rangle_Z. \tag{19}$$

Let $[V([s,T];U)]'$ be the dual of $V([s,T];U)$ with respect to the pivot space $L_2([s,T];U)$. Therefore, we have the relation

$$V([s,T];U) \subset L_2([s,T];U) \subset [V([s,T];U)]', \tag{20}$$

and the inequality

$$\|u\|_{[V([s,T];U)]'} \le \|u\|_{L_2([s,T];U)} \le \|u\|_{V([s,T];U)}. \tag{21}$$

### 3.2. Existence and uniqueness of the optimal control

To show the existence of a minimizer $u$ in $L_2([s,T];U)$, it suffices to show that $J$ as a functional is weakly lower semi-continuous. For the latter, it suffices to establish convexity and lower semi-continuity. That $J$ is convex follows immediately from the linearity of the operators $L$ and $GL_{sT}$ as well as the convexity of the square of the norm. As for lower semi-continuity, one only needs to show that the terminal time penalization term $\|Gy(T)\|_Z$ is lower semi-continuous in $u$, since otherwise the quadratic cost is indeed continuous. This can be established via Assumption (2) pertaining to the closability of $GL_{sT}$, which is crucial to guarantee existence of the optimal control. Here the argument is the same as in Lasiecka & Triggiani (2000), and hence it will not be repeated.

### 3.3. Characterization of the solution to the optimal control

Having introduced the space $V([s,T];U)$, we alter the problem to minimizing the cost functional (4) over all $u \in V([s,T];U)$ instead of $u \in L_2([s,T];U)$. By standard optimization theory, this new problem has a unique optimal solution $u^0$ since $J(u,y,s,y_s)$ is continuous, and strictly convex in $u$ with respect to the $V$ norm. Since $u^0 \in V([s,T];U)$ we have that $GL_{sT}u^0 \in Z$ and $L_{sT}G^*GL_{st}u^0 \in [V(s,T);U)]'$. This observation allows us to consider the optimization problem on a smaller space $V([s,T];U)$, on which operator $GL_{sT}$ is bounded. Applying the maximum principle to this "new" optimization problem (5) and (4) we obtain the following expression for the optimal control (see Lasiecka & Triggiani, 2000, p. 27, for more details)

$$\begin{aligned} -u^0(\cdot,s;y_s) &= L^\star_{sT}G^\star G\big(e^{A(T-s)}y_s + L_{sT}u^0\big) \\ &= L^\star_{sT}G^\star G y^0(T) \ \in \ [V([s,T];U)]'. \end{aligned} \tag{22}$$

The explicit characterization of optimal control is given in the following Proposition

PROPOSITION 1 *The unique solution $u^0(\cdot,s,y_s) \in V([s,T];U) \subset L_2([s,T];U)$ minimizing the cost functional $J(u,y,s,y_s)$ defined in (4) admits the representation*

$$-u^0(t,s;y_s) = \Lambda^{-1}_{sT}\big(L^\star_{sT}G^\star G e^{A(T-s)}y_s\big) \tag{23}$$

*where $\Lambda_{sT} \equiv I + L^\star_{sT}G^\star GL_{sT}$. In addition, we have the estimate*

$$\|u^0(\cdot,s,y_s)\|_{V([s,T],U)} \le C\|y_s\|_Y \tag{24}$$

*where the constant $C$ can be made uniform in $[s,T]$ for $s \in [0,T]$.*

*Proof.* This follows from (22) after we assert the needed invertibility of the operator $\Lambda_{sT}$, which is accomplished in the next Lemma.

LEMMA 1 *We have the following regularity properties pertaining to the operators $GL_{sT}$, $L^\star_{sT}G^\star$ and $\Lambda_{sT}$*

*(i) The operators $GL_{sT}$ and $L^\star_{sT}G^\star$ satisfy*

$$\|GL_{sT}\|_{\mathcal{L}(V([s,T];U);Z)} = \|L^\star_{sT}G^\star\|_{\mathcal{L}(Z;[V([s,T];U)]')} \le 1.$$

*(ii) The operator $\Lambda_{sT} = I + L^\star_{sT}G^\star GL_{sT}$ is bounded from $V([s,T];U)$ onto $[V([s,T];U)]'$, and we have the estimate*

$$\|\Lambda_{sT}u\|_{[V([s,T];U)]'} \le C\|u\|_{V([s,T];U)}$$

*uniformly in $0 < s < T$.*

*(iii) The inverse operator* $\Lambda_{sT}^{-1}$ *exists and is bounded from* $[V([s,T];U)]'$ *to* $V([s,T];U)$*. In particular, we have the estimate*

$$\|\Lambda_{sT}^{-1}u\|_{V([s,T];U)} \leq C\|u\|_{[V([s,T];U)]'}.$$

*Proof.* Part (i) and (ii) are immediate from the definition of the space $V$ and its corresponding norm, while part (iii) follows from the Lax-Milgram Theorem since $\Lambda$ is continuous and coercive with respect to the space $V$. ■

Now notice that $L_{sT}^{\star}G^{\star}Ge^{AT}y_s$ is an element of $[V([s,T];U)]'$ since for all $\phi \in V = \overline{\mathcal{D}(GL_{sT})}$, we have

$$\langle L_{sT}^{\star}G^{\star}Ge^{AT}y_s, \phi\rangle = \langle Ge^{AT}y_s, GL_{sT}\phi\rangle_Z,$$

which is well defined. Hence, the existence of the inverse of $\Lambda_{sT}$ from the dual space of $V$ to $V$, as stated in Lemma 1 part (iii), enables us to give sense to the expression for the optimal control

$$-u^0(t,s,y_s) = \Lambda_{sT}^{-1}\big(L_{sT}^{\star}G^{\star}Ge^{A(T-s)}y_s\big). \tag{25}$$

We also note that by (19) we have

$$||u||^2_{V([s,T];U)} = \|\Lambda_{sT}^{1/2}u\|^2_{L_2((s,T);U)}. \tag{26}$$

## 4. Regularity of the optimal solutions

In the following proposition, we establish continuity of the optimal control on the interval $[s,T)$ with a possible singularity of order $\gamma$ at the final time $T$. To describe this behavior, we define a Banach space which is suitable for capturing the singularities in optimal solutions (see Da Prato & Ichikawa, 1985).

Definition 1 *We define the normed space* $C_\gamma([s,T];H)$ *by*

$$C_\gamma([s,T];H) = \{f \in C([s,T);H) : \sup_{t\in[s,T]}(T-t)^\gamma\|f(t)\|_H < \infty\}$$

*equipped with the norm*

$$\|f\|_{C_\gamma([s,T];H)} = \sup_{t\in[s,T]}(T-t)^\gamma\|f(t)\|_H.$$

It is immediate to notice that (18) and the dual version of RSE imply

$$L_{sT}^{*}G^{*} \in \mathcal{L}(Z, C_\gamma([s,T];U)). \tag{27}$$

The regularity of the optimal state $y^0$ can be then inferred from the relation

$$y(t,s;y_0) = e^{A(t-s)}y_0 + (L_s u)(t).$$

The next proposition contains the regularity properties of the optimal state and control.

Proposition 2 *(a) The optimal control $u^0$ satisfies the estimate*

$$\|u^0(\cdot, s; y_0)\|_{C_\gamma([s,T];U)} \le C\|y_0\|_Y,$$

*where $C$ is independent of the initial time $s$.*

*(b) The optimal state is continuous in time and satisfies*

$$\|y^0(\cdot, s, y_0)\|_{C([s,T];[\mathcal{D}(A^\star)]')} \le C\|y_0\|_Y,$$

*where $C$ is independent of the initial time $s$.*

*(c) The observed optimal state $Gy^0$ satisfies the following continuity property for any $x \in Y$:*

$$\lim_{s\to T} \quad Gy^0(T, s, x) = Gx \tag{28}$$

*and for $\gamma \in [0, 1/2)$, we also have*

$$\lim_{t\to T} Gy^0(t, s, x) \to Gy^0(T, s, x), s < t \le T. \tag{29}$$

The remainder of this section is devoted to the proof of Proposition 2.

### 4.1. Preliminary results

We first recall the expression (22) for the optimal control

$$-u^0 = L^\star_{sT} G^\star G y^0(T) \tag{30}$$

in order to prove the following regularity results

Lemma 2 *(i) The optimal control $u^0$ satisfies the estimate*

$$\|u^0(\cdot, s; y_0)\|_{L_2([s,T];U)} \le \|u^0(\cdot, s; y_0)\|_{V([s,T];U)} \le C\|y_0\|_Y$$

*where $C$ does not depend on $s$.*

*(ii) The observed optimal state at the final time $T$ satisfies the estimate*

$$\|Gy^0(T, s; y_0)\|_Z \le C\|y_0\|_Y$$

*where $C$ does not depend on $s$.*

*Proof.* (i) This follows immediately from relation (21) and application of Lemma 1 part (iii) to the expression for the control in equation (25).

(ii) From (16) we have

$$Gy^0(T) = Ge^{A(T-s)}y_0 + GL_{sT}u^0.$$

The first term indeed satisfies the inequality by the boundedness of $G : Y \to Z$ and the basic properties of the semigroup $e^{A(T-s)}$. As for the second term, this inequality follows by applying Lemma 1(i) as well as part (i) of this lemma. ∎

LEMMA 3 *With reference to the operators $GL_s$ and $L_s^\star G^\star$, defined in* (8) *and* (17)*, respectively, with* $0 < \gamma < 1$*, we have*

*(i) For $r$ such that $r+\gamma < 1$, the operator $GL_s$ is continuous $C_r([s,T];U) \to C([s,T];Z)$ and satisfies the estimate*

$$\|GL_s u\|_{C([s,T];Z)} \leq \frac{C_{T,\gamma}}{1-\gamma-r}\|u\|_{C_r([s,T];U)}$$

*where the constant $C$ is independent of $s$.*

*(ii) For $r$ such that $r+\gamma \geq 1$, the operator $GL_s$ is continuous from $C_r([s,T];U) \to C_{r+\gamma-1+\epsilon}([s,T];Z)$ and satisfies the estimate*

$$\|GL_s u\|_{C_{r+\gamma-1+\epsilon}([s,T];Z)} \leq C_{T,\gamma}\|u\|_{C_r([s,T];U)}$$

*where $C$ again is independent of $s$.*

*(iii) For $0 \leq r < 1$, the operator $L_s^\star G^\star$ is continuous from $C_r([s,T];Z) \to C_{r+\gamma-1}([s,T];U)$ and satisfies the estimate*

$$\|L_s^\star G^\star y\|_{C_{r+\gamma-1}([s,T];U)} \leq C_{T,\gamma}\|y\|_{C_r([s,T];Z)}$$

*where the constant $C$ is independent of $s$.*

*Proof.* See Lasiecka & Triggiani (2000) p. 35-37 for the proof. The only difference is that the estimate in the first step comes from the singular estimate (3) satisfied by the assumption on $GL_s$ and not from the analyticity of the semigroup. ■

LEMMA 4 *The operator $L_{sT}^\star G^\star G y^0(T)$ — considered as an operator acting on initial data $y_0 \in Y$ into $[V([s,T],U)]'$; see* (22)*— also satisfies*

$$\|L_{sT}^\star G^\star G y^0(T)\|_{C_\gamma([s,T];U)} \leq C\|y_0\|_Y$$

*with $C$ a constant independent of $s$.*

*Proof.* We first write the explicit expression for the operator to obtain

$$\|L_{sT}^\star G^\star G y^0(T)\|_{C_\gamma([s,T];U)} = \|B^\star e^{A^\star(T-\cdot)} G^\star G y^0(T)\|_{C_\gamma([s,T];U)}.$$

Applying the singular estimate (7) in Assumption (1), we have

$$\|L_{sT}^\star G^\star G y^0(T)\|_{C_\gamma([s,T];U)} \leq \sup_{t\in[s,T]}(T-t)^\gamma \frac{C}{(T-t)^\gamma}\|G y^0(T)\|_Y \leq C\|y_0\|_Y$$

where the last inequality follows from Lemma 2 part (ii) and thus $C$ does not depend on $s$. ■

### 4.2. Proof of Proposition 2

*Proof.* (a) This follows, after some algebraic manipulations, from the expression for the optimal control in equation (22) and Lemma 4.

(b) This follows immediately from (16) due to the fact that $L_s : L_2([s,T];U) \to C([s,T];[\mathcal{D}(A^\star)]')$.

(c) The continuity in (28) follows from the arguments that are identical to Proposition 1.4.22, p. 29 in Lasiecka & Triggiani (2000). For the second continuity property we consider $0 \leq \gamma < \frac{1}{2}$. From the optimal dynamics in (16) we have

$$\|Gy^0\|_{C([s,T];Z)} = \|Ge^{A(\cdot - s)}y_0 + GL_s u^0\|_{C([s,T];Z)}.$$

The first term clearly satisfies the desired estimate. By applying Lemma 3(i) with $r = \gamma$ to the second term we conclude that $Gy^0 \in C([s,T];Z)$. ∎

## 5. The optimal state and control as evolutions

One of the difficulties of the Bolza problem is the fact that the state (either final or transient trajectory) does not have values in $Y$, but rather in a larger space $[D(A^*)]'$. On the other hand, the notion of an evolution and the associated transition property require some type of invariance of subsets of initial data. It turns out that it is possible to establish such invariance for all transient times up to the terminal time $T$. We begin with the definition of the "evolution" operator. The issue here (related to the singular Bolza problem) is that a natural definition $\Phi(t,s)x = y^0(t,s;x), x \in Y, s < t$ runs into an immediate difficulty due to the loss of invariance of the state space $Y$. Indeed, the state resides a priori in $[D(A^*)]'$, which is larger than $Y$. In order to overcome the difficulty, the evolution must be defined on properly calibrated spaces, which depend also on evolution time. Indeed, we introduce a family of sets $H_s$ parameterized with respect to the parameter $s \in [0,T]$ and consisting of points located on backward trajectories:

$$H_s \equiv span\{e^{A(s-z)}x,\, e^{A(s-z)}Bw,\, L_z u(s) : z \leq s,\, x \in Y, w \in U, u \in L_2([z,s];U)\}.$$

We shall show that these subspaces describe the evolution $H_s \to H_t$, when $s \to t$. Indeed, with the above notation we introduce the evolution operator.

Definition 2 *Define an evolution on the trajectory of* $y$ *by*

$$\Phi(t,s)x \equiv y^0(t,s;x) \in C([s,T];[\mathcal{D}(A^\star)]')$$

*where* $y^0(s,s;x) = x \in H_s$.

Our goal is to show that with the above definition and $x \in H_s$, we have $\Phi(t,s;x) \in H_t$. We will show that this evolution on the optimal trajectory indeed satisfies the transition properties of an evolution, and this allows for a meaningful feedback characterization of the optimal control in terms of the optimal trajectory via the Riccati operator $P(t)$ introduced in the next section. We must first verify that the optimal control defines an evolution in the sense of $u^0(t,s;x) = u^0(t,z;y^0(z,s;x))$ for which it is necessary to give sense to the optimal control $u^0$ when the initial condition is a special element of $[\mathcal{D}(A^\star)]'$. Note that the optimal trajectory $y^0(t,s;x)$ is only an element of $[\mathcal{D}(A^\star)]'$ even when the initial condition $x \in Y$.

In fact, the existence of a unique optimal control $u^0 \in V([s,t];U)$ for the functional (4) still holds when the initial condition $y_s \in H_s$, since, as before, $J$ is weakly lower semi-continuous in $u$ while the term $Ge^{A(T-s)}y_s$ belongs to $Z$ by the singular estimate assumption, so that $J$ is always finite for $u \in V([s,T];U)$. In the next proposition, we characterize the optimal control $u^0 \in V([s,T];U) \subset L_2([s,T];U))$ for the special choices of initial data, which are outside the state space but in $H_s$. We begin with initial data defined in the range of the operator $B$. This is possible due to the validity of the RSE estimate (7).

Proposition 3 *Given an initial state $y_s \in Range(B) \subset [\mathcal{D}(A^\star)]'$, there exists a unique optimal control $u^0(t,s;y_s) \in V([s,T];U) \subset L_2([s,T];U)$. Moreover, the following singular estimate holds*

$$\|u^0(t,s;Bw)\|_{V([s,T];U)} \le C\frac{\|w\|_U}{(T-s)^\gamma}$$

*for all $w \in U$.*

*Proof.* By virtue of RSE (7), we have

$$\|Ge^{A(T-s)}Bw\|_Z \le C\frac{\|w\|_U}{(T-s)^\gamma} \tag{31}$$

and from Lemma 1

$$\|L^*_{sT}G^*Ge^{A(T-s)}Bw\|_{[V([s,T];U)]'} \le C\frac{\|w\|_U}{(T-s)^\gamma}.$$

Part (iii) of Lemma 1, along with the representation of the optimal control, given by Proposition 1, lead to the desired inequality. ∎

Our next step is to extend the action of optimal control to points lying on pieces of trajectories with values in the extended dual space. The corresponding result is formulated below.

Proposition 4 *Let $z \le s < t \le T$, then*
*(a) For all $w \in U$, we have $u^0(\cdot,s;e^{A(s-z)}Bw) \in V([s,T];U)$ and*

$$\|u^0(\cdot,s;e^{A(s-z)}Bw)\|_{V([s,T];U)} \le \frac{\|w\|_U}{(T-z)^\gamma}.$$

*(b) For all $u \in L_2([z,s];U)$, we have $u^0(\cdot,s;(L_zu)(s)) \in V([s,T];U)$ and*

$$\|u^0(\cdot,s;(L_zu)(s))\|_{V([s,T];U)} \leq C\max\{(T-s)^{1/2-\gamma},(T-s)^{1/2-\gamma}\}\|u\|_{L_2([z,s];U)}.$$

*Proof.* Let $y_s \in Y$ denote the initial state corresponding to the process originating at time $s$. Appealing to the expression for the optimal control in (22) and rewriting the expression for the corresponding optimal control $u^0(t,s;y_s)$ we have

$$(I + L^\star_{sT}G^\star GL_{sT})u^0 = -L^\star_{sT}G^\star Ge^{A(T-s)}y_s. \tag{32}$$

Therefore, we have

$$u^0 = -(I + L^\star_{sT}G^\star GL_{sT})^{-1}L^\star_{sT}G^\star Ge^{A(T-s)}y_s. \tag{33}$$

By estimating the norm of $u^0$, we get

$$\begin{aligned}\|u^0\|_{V([s,T];U)} &\leq \|(I + L^\star_{sT}G^\star GL_{sT})^{-1}L^\star_{sT}G^\star Ge^{A(T-s)}y_s\|_{V([s,T];U)}\\ &\leq \|L^\star_{sT}G^\star Ge^{A(T-s)}y_s\|_{[V([s,T];U)]'}\end{aligned} \tag{34}$$

where we used the inequality (21) and the fact that $I + L^\star_{sT}G^\star GL_{sT}$ is an isomorphism from the space $V$ to the dual space $V'$ (see Lemma 1). Since $L^\star_{sT}G^\star$ is bounded when acting from the space $Z$ to the space $[V([s,T];U)]'$, we have

$$\|u^0\|_{L_2([s,T];U)} \leq \|u^0\|_{V([s,T];U)} \leq \|Ge^{A(T-s)}y_s\|_Z. \tag{35}$$

The above expression is a basis for the extension of the action of the optimal control to points on trajectories defined by the membership in $H_s$. We need to verify that the action of optimal control is well defined on the extended trajectories. This verification is carried out first by regularizing the appropriate elements so that they belong to $Y$ and then passing to the limit via the density argument. Since the regularization is a standard step in semigroup theory, we focus on the crux of the matter which is proper extension. Setting $y_s = e^{A(s-z)}Bw$ and applying the singular estimate condition yields

$$\begin{aligned}\|u^0\|_{L_2([s,T];U)} &\leq \|Ge^{A(T-z)}Bw\|_Z\\ &\leq \frac{C}{(T-z)^\gamma}\|w\|_U.\end{aligned} \tag{36}$$

This verifies that $u^0$ belong to $V([s,T];U)$ (note that $z \leq s < T$ ) when the initial state is of the form $y_s = e^{A(s-z)}Bw$ with $w \in U$. The first claim in the Proposition is proved. For the second claim, we consider the initial state $y_s = (L_zu)(s)$ for some $u \in L_2([z,s];U)$ and $z < s$. Proceeding as before, we set $y_s = (L_z u)(s)$ in the inequality (35) to obtain

$$\begin{aligned}\|u^0\|_{V([s,T];U)} &\leq \int_z^s \|Ge^{A(T-s)}e^{A(s-\tau)}Bu(\tau)\|_Z \, d\tau \\ &\leq \int_z^s \frac{C}{(T-\tau)^\gamma}\|u(\tau)\|_U \, d\tau \\ &\leq C\max\{(T-s)^{-\gamma+1/2}, (T-z)^{-\gamma+1/2}\}\|u\|_{L_2([z,s];U)}, \quad (37)\end{aligned}$$

where we used the singular estimate condition (7) as well as Hölder's inequality to arrive at the desired estimate. ∎

As a corollary, we shall derive similar extension properties for the corresponding optimal state. These will follow from Proposition 2 and Lemma 2. The key to this argument is showing that $H_s$ evolves into $H_t$ when time $s$ evolves into $t$. Thus, subspaces $H_s$ describe the evolution of the optimal process $H_s \to H_t$ when $s \to t$ .

Corollary 1 *Given initial state $y_s \in H_s$ and $z \leq s$, we have the following regularity properties pertaining to the optimal state*
*(a) The optimal state $y^0(\cdot, s; y_s) \in C([s,T]; [\mathcal{D}(A^\star)]')$.*
*(b) The optimal state $y^0(t, s; y_s) \in H_t$ for every $s < t < T$, $y_s \in H_s$ .*
*(c) The observed final optimal state $Gy^0(T, s; y_s) \in Z$.*
*(d) The optimal control $u^0(\cdot, s; y_s) \in C_\gamma([s,T]; U)$.*

*Proof.* (a) Since $y^0(t, s; y_s) = e^{A(t-s)}y_s + L_s u^0(\cdot, s; y_s)$, the regularity of $y^0$ immediately follows from the fact that $L_s$ is bounded from $L_2([s,T];U) \to C([s,T]; [\mathcal{D}(A^\star)]')$ and that $u^0 \in L_2([s,T];U)$ by Proposition 4.

(b) Appealing once more to (16) and setting $y_s = ae^{A(s-z)}Bw + be^{A(s-z)}y + c(L_z g)(s)$, for some $w \in U$, $y \in Y$, and $g \in L_2([z,s];U)$, we can express the optimal state as

$$\begin{aligned} y^0(t,s,y_s) &= ae^{A(t-z)}y + be^{A(t-s)}e^{A(s-z)}Bw + be^{A(t-s)}\int_z^s e^{A(s-\tau)}Bg(\tau)\,d\tau \\ &\quad + c\int_s^t e^{A(t-\tau)}Bu^0(\tau,s;y_s)\,d\tau \\ &= ae^{A(t-z)}y + be^{A(t-z)}Bw + b\int_z^s e^{A(t-\tau)}Bg(\tau)\,d\tau \\ &\quad + c\int_s^t e^{A(t-\tau)}Bu^0(\tau,s;y_s)\,d\tau. \\ &= ae^{A(t-z)}y + be^{A(t-z)}Bw + c(L_z\hat{u})(t) \end{aligned}$$

where

$$\hat{u} \equiv \begin{cases} bg(\xi), & \xi \in (z,s) \\ cu^0(\xi,s,y_s), & \xi \in (s,t) \end{cases} .$$

By Proposition 4, we have $\hat{u} \in L_2([z,t];U)$. Hence, $y^0(t,s;y_s) \in H_t$ when $y_s \in H_s$ as claimed.

(c) The observed final state is given by

$$Gy^0([T,s];y_s) = Ge^{A(T-s)}y_s + GL_{sT}u^0(\cdot,s;y_s).$$

The second term is indeed in $Z$, since $u^0$ belongs to the space $V([s,T];U)$ by Proposition 4 while $GL_{sT} : V([s,T];U) \to Z$ is bounded. As for the first term, we will consider two cases separately. The first case is $y_s = e^{A(s-z)}Bw$, in which case we have

$$\|Ge^{A(T-s)}y_s\|_Z \leq \frac{C}{(T-z)^\gamma}\|w\|_U,$$

where we used the estimate as in (36). This confirms $Gy^0(T) \in Z$ for he first case. As for the second case, in which we have $y_s = \int_z^s e^{A(s-\tau)}Bg(\tau)\,d\tau$, we similarly estimate the term $Ge^{A(T-s)}y_s$ to get

$$\|Ge^{A(T-s)}y_s\|_Z \leq C_T\|g(\cdot)\|_{L_2([z,s];U)}$$

where we again used the same estimate as in (37). Hence, $Gy^0(T,s;y_s) \in Z$.

(d) Recalling the expression for the optimal control in (22), we have

$$\begin{aligned}
\|u^0\|_{C_\gamma([s,T];U)} &= \|L_{sT}^\star G^\star Gy^0(T)\|_{C_\gamma([s,T];U)} \\
&\leq \|B^\star e^{A^\star(T-\cdot)}G^\star Gy^0(T)\|_{C_\gamma([s,T];U)} \\
&\leq \|B^\star e^{A^\star(T-\cdot)}G^\star Gy^0(T)\|_{C_\gamma([s,T];U)} \\
&\leq \sup_{t\in[s,T]}(T-t)^\gamma \frac{C}{(T-t)^\gamma}\|Gy^0(T,s;y_s)\|_Z \\
&\leq C\|Gy^0(T,s;y_s)\|_Z
\end{aligned}$$

where we again used the singular estimate assumption. By part (iii) of this Lemma, we indeed have $Gy^0(T;s;y_s) \in Z$ given $y_s \in H$, while $u^0$ is indeed continuous on $[s,T)$ by the membership of $B^\star e^{A^\star(T-t)}G^\star$ in $C_\gamma([s,T],U)$. ∎

We are now ready to investigate the evolution property of both the optimal control and the optimal trajectory. The following Lemma is critical.

PROPOSITION 5 *For $s$ fixed, and initial state $x = x_s \in H_s \subset [\mathcal{D}(A^\star)]'$, the optimal pair satisfy certain transition properties pointwise. In particular,*

*(a) The optimal control $u^0$ defines an evolution*

$$u^0(t,\tau;\Phi(\tau,s,x)) = u^0(t,s;x), \quad \textit{in} \quad V([s,t];U) \tag{38}$$

*for all $0 \leq s \leq \tau \leq t < T$.*

(b) *The operator* $\Phi(t, s)$ *given in Definition* (2) *indeed satisfies the evolution property*

$$\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)x, \quad in \quad H_t$$

*for all* $0 \leq s \leq \tau \leq t < T$.

(c) *The observed optimal trajectory* $G\Phi(t, s)$ *also satisfies the evolution property*

$$G\Phi(T, s)x = G\Phi(T, \tau)\Phi(\tau, s)x, \quad in \quad Z$$

*for all* $0 \leq s \leq \tau < T$.

*Proof.* (a) By Lemma 4, the optimal control $u^0(t, s; x)$, corresponding to the initial state $x \in H_s$ exists and the corresponding optimal state trajectory also takes values in the space $H_t$ by Corollary 1. Therefore, one can make sense of the expression $u^0(t, z, y^0(z, s; x))$, which signifies the optimal control minimizing the functional $J$ over the interval $[z, T]$ given an initial state $y^0(z, s; x)$. The optimal control does indeed satisfy the evolution property almost everywhere in $\tau$ and $t$ i.e. $u^0(t, s; x) = u^0(t, z, y^0(z, s; x))$ with $y^0(s, s; x) = x$ as a result of the Bellman optimality principle (see Proposition 1.4.3.1 in Lasiecka & Triggiani, 2000). The result can be boosted to a pointwise result on $[s, T)$ via the continuity of the optimal control on $[s, T)$ established in Proposition 2(a).

(b) The evolution property of the optimal solution $y^0 \in C([0, T]; H \subset [\mathcal{D}(A^\star)]')$ can then be demonstrated via the optimal dynamics in (16). In particular, we have for all $T > t > z > s$

$$\begin{aligned}
\Phi(t, z)\Phi(z, s)x &= e^{A(t-z)}\Phi(z, s)x + \int_z^t e^{A(t-\tau)}Bu^0(\tau, z, \Phi(z, s, x))\, d\tau \\
&= e^{A(t-z)}\left(e^{A(z-s)}x + \int_s^z e^{A(z-\tau)}Bu^0(\tau, s, x)\, d\tau\right) \\
&\qquad + \int_z^t e^{A(t-\tau)}Bu(\tau, z, \Phi(z, s, x))\, d\tau \\
&= e^{A(t-s)}x + \int_s^z e^{A(t-\tau)}Bu^0(\tau, s, x)\, d\tau \\
&\qquad + \int_z^t e^{A(t-\tau)}Bu^0(\tau, s, x)\, d\tau
\end{aligned}$$

where we used the evolution property of the optimal control $u$. Therefore,

$$\Phi(t, z)\Phi(z, s)x = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu^0(\tau, s, x)\, d\tau = \Phi(t, s)x.$$

This establishes that $\Phi(t, z)\Phi(z, s)x = \Phi(t, s)x$ for $s \leq z \leq t < T$.

(c) The result follows from part (a) and the fact that $GL_{sT} \in \mathcal{L}(V([s, T]; U) \to Z)$, and $u^0 \in V([s, T]; U)$ (see the detailed argument in Proposition 1.4.3.1 in Lasiecka & Triggiani, 2000). ∎

## 6. The Riccati operator and a feedback characterization of the optimal control

We define the Riccati operator $P(t)$ in terms of the optimal state evolution operator $\Phi(t,s)$ as

DEFINITION 3

$$P(t)x = e^{A^\star(T-t)}G^\star G\Phi(T,t)x \tag{39}$$

*a linear operator from $Y$ to $L_\infty([0,T];Y)$.*

The two propositions stated below provide more information about $P(t)$ and its connection with the optimization problem.

PROPOSITION 6 *(a) The Riccati operator $P(t) \in \mathcal{L}(Y, L_\infty([0,T];Y))$.*
*(b) The optimal control is given by the feedback formula*

$$u^0(t,s;x) = -B^\star P(t)y^0(t,s;x).$$

*(c) The feedback operator $B^\star P(t)$ satisfies for all $0 \le t < T$ the singular estimate*

$$\|B^\star P(t)x\|_U \le \frac{C}{(T-t)^\gamma}\|x\|_Y \quad \text{for all } x \text{ in } Y.$$

*(d) The feedback operator $B^\star P(t)$ satisfies the singular estimate*

$$\|B^\star P(t)x\|_U \le \frac{C}{(T-t)^\gamma}\|G\Phi(T,t)x\|_Z < \infty, 0 \le t < T, \quad x \in H_t.$$

PROPOSITION 7 *(a) The Riccati operator satisfies for all $0 \le t < T$ the identity*

$$\langle P(t)x, y\rangle_Y = \langle G\Phi(T,t)x, G\Phi(T,t)y\rangle_Z + \int_t^T \langle B^\star P(\tau)\Phi(\tau,t)x, B^\star P(\tau)\Phi(\tau,t)y\rangle_U \, d\tau. \tag{40}$$

*(b) As a consequence of the identity above, we have $P(t) = P^\star(t) \ge 0$.*
*(c) The minimum value for the cost functional $J$ defined in (4) on the interval $[t,T]$ with initial value $y_0 \in Y$ and initial time $t$ is*

$$J^0(u^0, y^0, t, y_0) = \int_t^T \|B^\star P(\tau)\Phi(\tau,t)y_0\|_U^2 \, d\tau + \|G\Phi(T,t)y_0\|_Z^2 = \langle P(t)y_0, y_0\rangle_Y. \tag{41}$$

### 6.1. Proof of Proposition 6

*Proof.* (a) Estimating the $Y$ norm of the Riccati operator $P(t)x$ for any $x \in Y$, we have

$$\begin{aligned}\|P(t)x\|_Y &\le \|e^{A^\star(T-t)}G^\star G\Phi(T,t)x\|_Y \\ &\le C_T\|Gy^0(T,t,x)\|_Z.\end{aligned}$$

We next apply the estimate in Lemma 2(ii) and take the supremum over all $t$ of the left hand side to get the desired result.

(b) This follows directly from equation (22), the expression for the operator $P(t)$, given in Definition (3), and the evolution property of the optimal solution, $y^0$ established in Proposition 5.

(c) Estimating the norm of the feedback operator $B^\star P$ at a given $x \in Y$ we have

$$\|B^\star P(t)x\|_Y \le \|B^\star e^{A^\star(T-t)}G^\star G\Phi(T,t)x\|_Y.$$

We now apply the singular estimate in (7) of assumption (1), to obtain

$$\|B^\star P(t)x\|_Y \le \frac{C}{(T-t)^\gamma}\|Gy(T,t,x)\|_Z.$$

Application of Lemma 2(ii) to the second term produces the desired estimate.

(d) Proceeding in the same way as in part (c), we obtain the desired estimate. By Corollary 1(c), $Gy^0(T,t,x)$ is indeed in the space $Z$ for $x \in H_t$. ∎

### 6.2. Proof of Proposition 7

*Proof.* (a) By Definition (3) we have

$$\langle P(t)x, y\rangle_Y = \langle G^\star G\Phi(T,t)x, e^{A(T-t)}y\rangle_Y, x, y \in Y.$$

We next use the expression for the state trajectory in terms of the control in (16) and in turn express the optimal control $u^0$ as in part (b) of Proposition 6 in terms of the operator $P(t)$. By switching integrals, and noting the definition for $P(z)$, we arrive at the desired expression, which is known as the integral form of the Riccati equation.

(b) From (a), we have $\langle P(t)x, y\rangle = \langle x, P(t)y\rangle$, for all $x, y \in Y$. Therefore, $P(t)$ is self-adjoint. Also, from (a), it is easy to see that $\langle P(t)x, x\rangle \ge 0$.

(c) Using the expression for $J$ in (4) and the expression in part (a) we have

$$\begin{aligned}J^0(u^0, y_0, t, y_0) &= \int_t^T \|B^\star P(\tau)\Phi(\tau,t)y_0\|_U^2\, d\tau + \|G\Phi(T,t)y_0\|_Z^2 \\ &= \langle P(t)y_0, y_0\rangle_Y\end{aligned}$$

as claimed. ∎

## 7. The Riccati equation

In this section, we demonstrate that the operator $P$ indeed satisfies the Riccati equation. To show this, the main issue is to establish weak differentiability of $P(t)$. While such property is relatively straightforward in the case of dynamics governed by analytic semigroups (see Lasiecka & Triggiani, 2000, and Bensoussan, Da Prato, Delfour and Mitter, 2007), the issue is much more subtle in the non-analytic case. The main difficulty to contend with is differentiability of the evolution $\Phi(t,s)$ with respect to both $t$ and $s$, where the obstacle comes from differentiability with respect to the second variable $s$. With the above definitions we are ready to state the main results of this section, which are central for the derivation of the Riccati equation.

PROPOSITION 8 *(a) For every $x \in Y$ and $w \in U$ we have*

$$G\Phi(T,s)x = (I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}Ge^{A(T-s)}x,$$

*and*

$$G\Phi(T,s)Bw = (I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}Ge^{A(T-s)}Bw.$$

*(b) For every $w \in U$, we have*

$$\|G\Phi(T,s)Bw\|_Z \le \frac{C}{(T-s)^{\gamma}}\|w\|_U.$$

*(c) For any $x \in \mathcal{D}(A)$ and $s < T$:*

$$\frac{\partial}{\partial s}G\Phi(T,s)x = -G\Phi(T,s)(Ax - BB^{*}P(s)x) \in Z.$$

THEOREM 3 *The operator $P(t)$ satisfies the differential Riccati equation*

$$\langle \dot{P}(t)x, y\rangle_Y = -\langle P(t)x, Ay\rangle_Y - \langle P(t)Ax, y\rangle_Y + \langle B^{\star}P(t)x, B^{\star}P(t)y\rangle_U \quad (42)$$

*and the condition*

$$P(T)x = G^{\star}Gx$$

for all $0 \le t < T$ and $x, y \in \mathcal{D}(A)$.

Before giving the proofs of Proposition 8 and Theorem 3, we shall need the following preliminary definition and Lemma.

DEFINITION 4 *Define a new Hilbert Space $X$ to be the closure of $\mathcal{D}(L_{sT}^{*}G^{*})$ in $Z$ equipped with the inner product $\langle x, y\rangle_X = \langle x, y\rangle_Z + \langle L_{sT}^{*}G^{*}x, L_{sT}^{*}G^{*}y\rangle_{L_2([s,T];U)}$. Completeness follows immediately from closedness of $L_{sT}^{*}G^{*}$.*

LEMMA 5 *The operator $I + GL_{sT}L_{sT}^{\star}G^{\star}$ is an isomorphism $X \to X'$ and has a bounded inverse.*

*Proof.* Let $f \in X$, we then have

$$\|(I + GL_{sT}L_{sT}^{\star}G^{\star})f\|_{X'} \leq \|(I + GL_{sT}L_{sT}^{\star}G^{\star})f\|_{Z} = \|f\|_{X},$$

where we used the inclusion $X \subset Z \subset X'$ and the definition of the norm in the space $X$. The invertibility of this operator follows from the Lax-Milgram theorem. ∎

### 7.1. Proof of Proposition 8

*Proof.* (a) Fix $x \in Y$ and consider the expression for the action of the optimal evolution on $x$ obtained using optimal dynamics (16) as well as the expression for optimal control from Proposition 6(b)

$$\begin{aligned} G\Phi(T,s)x &= Ge^{A(T-s)}x - \int_s^T Ge^{A(T-z)}BB^{\star}P(z)\Phi(z,s)x\,dz \\ &= Ge^{A(T-s)}x - \int_s^T Ge^{A(T-z)}BB^{\star}e^{A^{\star}(T-z)}G^{\star}G\Phi(T,s)x\,dz \\ &= Ge^{A(T-s)}x - GL_{sT}L_{sT}^{\star}G^{\star}G\Phi(T,s)x \end{aligned}$$

where we also used the evolution property. Thus, for $s < T$

$$G\Phi(T,s)x = \left(I + GL_{sT}L_{sT}^{\star}G^{\star}\right)^{-1}Ge^{A(T-s)}x. \tag{43}$$

Now, by Lemma 5, we have that $(I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}$ exists and is bounded on $Z$. In addition, we can extend the action of the optimal evolution to elements in $x \in H_s$ of the form $x = Bw$ such that $w \in U$. Indeed, given $w \in U$ we have that $Ge^{A(T-s)}Bw \in Z$ via the singular estimate assumption, and thus the expression $G\Phi(T,s)Bw$ is a well defined element of $Z$. So,

$$G\Phi(T,s)Bw = (I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}Ge^{A(T-s)}Bw, s < T.$$

(b) We estimate the term $G\Phi(T,s)Bw$ using the expression obtained in the preceding part as

$$\begin{aligned} \|G\Phi(T,s)Bw\|_Z &\leq \|G\Phi(T,s)Bw\|_X \\ &\leq \|(I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}Ge^{A(T-s)}Bw\|_X \\ &\leq \|Ge^{A(T-s)}Bw\|_{X'} \\ &\leq \|Ge^{A(T-s)}Bw\|_Z \\ &\leq \frac{C}{(T-s)^{\gamma}}\|w\|_U \end{aligned}$$

where we used the fact that $(I + GL_{sT}L_{sT}^{\star}G^{\star})^{-1}$ exists and is bounded from $X' \to X$ by Lemma 5 as well as the singular estimate assumption (7).

(c) Let $x \in \mathcal{D}(A)$, $s < T$, then taking the derivative with respect to $s$ of $G\Phi(T,s)x$ we have

$$\frac{\partial}{\partial s}G\Phi(T,s)x = \\ = -GAe^{A(T-s)}x + Ge^{A(T-s)}BB^\star P(s)x - GL_{sT}L_{sT}^\star G^\star \frac{\partial}{\partial s}G\Phi(T,s)x.$$

The above expression is arrived at by using representation (43) and finite difference quotients approximating the respective derivatives. This procedure is carried out with full details in Aquistapace, Bucci & Lasiecka (2005). The same arguments apply to the present context.
Rearranging the equation we get

$$\Big(I + GL_{sT}L_{sT}^\star G^\star\Big)\frac{\partial}{\partial s}G\Phi(T,s)x = -Ge^{A(T-s)}(A - BB^\star P(s))x.$$

The operator $(I+GL_{sT}L_{sT}^\star G^\star)$ is bounded from $X = \mathcal{D}(L_{sT}^\star G^\star) \subset Z$ to $X'$, and, moreover, has a bounded inverse from $X'$ to $X$ by Lemma 5. Hence,

$$\frac{\partial}{\partial s}G\Phi(T,s)x = -\big(I + GL_{sT}L_{sT}^\star G^\star\big)^{-1}Ge^{A(T-s)}(A - BB^\star P(s))x. \quad (44)$$

Since $x \in \mathcal{D}(A)$ and $G$ is bounded, the term $Ge^{A(T-s)}Ax$ belongs to the space $Z \subset X'$, while $Ge^{A(T-s)}BB^\star P(s)x$ satisfies the estimate

$$\begin{aligned} \|Ge^{A(T-s)}BB^\star P(s)x\|_Z &\le \frac{C}{(T-s)^\gamma}\|B^\star P(s)x\|_U \\ &\le \frac{C}{(T-s)^{2\gamma}}\|x\|_Y, \end{aligned}$$

which implies $Ge^{A(T-s)}BB^\star P(s)x \in Z \subset X'$. Therefore, we conclude that $\frac{\partial}{\partial s}G\Phi(T,s)x \in X \subset Z$. Moreover, by the previous part of this lemma, we have

$$(I + GL_{sT}L_{sT}^\star G^\star)^{-1}Ge^{A(T-s)}BB^\star P(s)x = G\Phi(T,s)BB^\star P(s)x$$

and

$$(I + GL_{sT}L_{sT}^\star G^\star)^{-1}Ge^{A(T-s)}Ax = G\Phi(T,s)Ax.$$

In conclusion, we rewrite (44) to get

$$\frac{\partial}{\partial s}G\Phi(T,s)x = -G\Phi(T,s)(A - BB^\star P(s))x$$

for all $x \in D(A)$ and $s < T$ as desired. ∎

### 7.2. Derivation of the Riccati equation and proof of Theorem 3

*Proof.* Given $t < T$ and $x, y \in \mathcal{D}(A)$ we have

$$\langle P(t)x, y\rangle_Y = \langle A^{\star -1} e^{A^\star (T-t)} G^\star G \Phi(T,t)x, Ay\rangle_Y .$$

We next differentiate $P$ with respect to $t$ using the results in Proposition 8 (b) and (c) to get

$$\begin{aligned}\langle \dot{P}(t)x, y\rangle_Y = &-\langle e^{A^\star (T-t)} G^\star G \Phi(T,t)x, Ay\rangle_Y \\ &\qquad + \langle A^{\star -1} e^{A^\star (T-t)} G^\star \frac{\partial}{\partial t} G \Phi(T,t)x, Ay\rangle_Y \\ = &-\langle P(t)x, Ay\rangle_Y - \langle A^{\star -1} e^{A^\star (T-t)} G^\star G \Phi(T,t) A_p(t)x, Ay\rangle_Y ,\end{aligned}$$

where for $x \in D(A)$ we define $A_p(t)x \equiv Ax - BB^* P(t)x$. Since $x \in D(A)$ by Proposition 6 (c), we infer that $A_p(t)x \in H_t$.

Note that $\langle P(t)x, Ay\rangle_Y$ is also defined by boundedness of $P(t)$ on $Y$ for fixed $t$ by Proposition 6(a), and

$$\langle e^{A^\star (T-t)} G^\star G \Phi(T,t) A_p(t)x, y\rangle_Y = \langle G\Phi(T,t) A_p(t)x, G e^{A(T-t)} y\rangle_Z ,$$

which is well defined since $G\Phi(T,t)A_p(t)x \in Z$ by Proposition 8 (c). This establishes that $\dot{P}(t)$ is a well defined operator acting from $\mathcal{D}(A) \to [\mathcal{D}(A)]'$. We proceed with derivation to obtain

$$\langle \dot{P}(t)x, y\rangle_Y = -\langle P(t)x, Ay\rangle_Y - \langle P(t)Ax, y\rangle_Y + \langle B^\star P(t)x, B^\star P(t)y\rangle_U$$

where the right hand side is well defined for all $x, y \in \mathcal{D}(A)$ and $t < T$. ∎

### 7.3. The limit of $P(t)$ as $t$ goes to $T$

Proposition 9 *The Riccati operator $P(t)$ satisfies*

$$\lim_{t \to T} P(t)x = G^\star G x.$$

*Proof.* From the definition of $P(t)$, we must show

$$\lim_{t \to T} \| e^{A^\star (T-t)} G^\star G y^0(T,t,x) - G^\star G x\|_Y \to 0.$$

Rewriting the expression via (16) as

$$\begin{aligned}&\| e^{A^\star (T-t)} G^\star G e^{A(T-t)} x + e^{A^\star (T-t)} G^\star G L_{tT} u^0(\cdot,t,x) - G^\star G x\|_Y \\ &\quad \le \| e^{A^\star (T-t)} G^\star G L_{tT} u^0(\cdot,t,x)\|_Y + \| e^{A^\star (T-t)} G^\star G e^{A(T-t)} x - G^\star G x\|_Y\end{aligned}$$

we see that the second term goes to 0 as $t \to T$ by the strong continuity of the semigroup and boundedness of $G$. The main task is to show

$$\lim_{t\to T} \|u(\cdot, t, y_0)\|_{V([t,T];U)} = 0$$

and then appeal to the boundedness of $GL_{tT} : V \to Z$. The argument is identical to that given in the proof of Proposition 1.4.2.2 in Lasiecka & Triggiani (2000). ■

In the case when $\gamma < 1/2$, uniqueness of solutions to Riccati equations can be shown by the same argument as in Lasiecka & Triggiani (2000). However, uniqueness when $\gamma \geq 1/2$ is an open question.

## 8. An optimal boundary control problem of a fluid-structure interaction system

As an example illustrating the abstract theory presented in this paper we consider a model of fluid-structure interaction with control action implemented on the boundary, or more precisely on the interface between the fluid and the structure. This particular model exhibits a parabolic-hyperbolic interaction of dynamics, and as such it is not driven by an analytic semigroup.

The physical model under consideration has been extensively treated in the literature (see Du, Gunzburger, Hou & Lee, 2003; Moubachir & Zolesio, 2006; Barbu, Grujic, Lasiecka & Tuffaha, 2007, and Lions, 1969) and it describes the elastic motion of a solid fully immersed in a viscous incompressible fluid. The mathematical model consists of a linearized Navier-Stokes equation defined on an open domain $\Omega_f$, coupled with an elastic equation defined on another domain $\Omega_s$, with boundary conditions matching velocities and normal stresses on the boundary $\Gamma_s$, which separates the two open domains $\Omega_f$ and $\Omega_s$. It will be assumed that the solid is subject to small but rapid oscillations (see Du, Gunzburger, Hou & Lee, 2003).

Let $\Omega \in \mathbb{R}^3$ be a bounded domain with an interior region $\Omega_s$ and an exterior region $\Omega_f$. The boundary $\Gamma_f$ is the outer boundary of the domain $\Omega$ while $\Gamma_s$ is the boundary of the region $\Omega_s$, which also borders the exterior region $\Omega_f$ and where the interaction of the two systems takes place. Let $u$ be a function defined on $\Omega_f$, representing the velocity of the fluid while the scalar function $p$ represents the pressure. Additionally, let $w$ and $w_t$ be the displacement and the velocity functions of the solid $\Omega_s$. We also denote by $\nu$ the unit outward normal vector with respect to the domain $\Omega_s$. The boundary-interface control is represented by $g \in L_2([0,T]; L_2(\Gamma_s))$ and is active on the boundary $\Gamma_s$. We work under the assumption of small but rapid oscillations of the solid body, hence the interface $\Gamma_s$ is assumed static; see Du, Gunzburger, Hou & Lee (2003), Moubachir & Zolesio (2006), and Barbu, Grujic, Lasiecka & Tuffaha (2007) for more modeling details.

Given a control $g \in L_2([0,T]; L_2(\Gamma_s))$, the functions $(u, w, w_t, p)$ satisfy the system

$$\begin{cases} u_t - \text{div } \epsilon(u) + Lu + \nabla p = 0 & \text{in } Q_f \equiv \Omega_f \times [0,T] \\ \text{div } \quad u = 0 & \text{in } Q_f \equiv \Omega_f \times [0,T] \\ w_{tt} - \text{div } \sigma(w) = 0 & \text{in } Q_s \equiv \Omega_s \times [0,T] \\ u(0,\cdot) = u_0 & \text{in } \Omega_f \\ w(0,\cdot) = w_0, w_t(0,\cdot) = w_1 & \text{in } \Omega_s \\ w_t = u & \text{on } \Sigma_s \equiv \Gamma_s \times [0,T] \\ u = 0 & \text{on } \Sigma_f \equiv \Gamma_f \times [0,T] \\ \sigma(w)\cdot\nu = \epsilon(u)\cdot\nu - p\nu - g & \text{on } \Sigma_s \equiv \Gamma_s \times [0,T]. \end{cases} \tag{45}$$

The elastic stress tensor $\sigma$ and the strain tensor $\epsilon$, respectively, are given by

$$\sigma_{ij}(u) = \lambda \sum_{k=1}^{k=3} \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u) \quad \text{and} \quad \epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé constants. The term $Lu$ is a linearization of the convective term in Navier-Stokes $(u \cdot \nabla)u$ and is defined as

$$u(x,t) \to (Lu)(x,t) = (\nabla v(x)) \cdot u + (v \cdot \nabla)u. \tag{46}$$

where $v$ is a time independent smooth vector function $\in [C^\infty(\Omega_f)]^n$ with the property div $v = 0$.

The control problem to be considered is of Bolza type. In particular, we wish to minimize the following functional

$$J(u,g) = \int_0^T ||g(t)||^2_{L_2(\Gamma_s)}\, ds + ||u(T,\cdot)||^2_{L_2(\Omega_f)}, \tag{47}$$

over all $g \in L_2([0,T]; \Gamma_s)$.
Throughout the paper, we denote the energy space for the system by

$$Y \equiv H \times H^1(\Omega_s) \times L_2(\Omega_s)$$

where

$$H \equiv \{u \in L_2(\Omega_f) : \text{div } \; u = 0, u \cdot \nu|_{\Gamma_f} = 0\}.$$

The control space $U$ coincides with $L_2(\Gamma_s)$. With the above notation, the observation operator $G : Y \to H$ becomes $G(u, w, v) \equiv (u, 0, 0)$, so $Z \equiv H$ and $G \in \mathcal{L}(Y, Z)$.

In order to be able to apply the result of our main Theorem 2 we represent the solution to (45) as an abstract equation of the form

$$y_t = Ay + Bg, \quad y_0 \in Y \tag{48}$$

where

$$A = \begin{pmatrix} \mathcal{A} - L & \mathcal{A}N\sigma() \cdot \nu & 0 \\ 0 & 0 & I \\ 0 & \text{div } \sigma(\cdot) & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} \mathcal{A}N \\ 0 \\ 0 \end{pmatrix}. \tag{49}$$

Here $\mathcal{A} : V \equiv \{v \in H \cap H^1(\Omega_f), u|_{\Gamma_f} = 0\} \to V'$ is defined by

$$(\mathcal{A}u, \phi) = -(\epsilon(u), \epsilon(\phi)), \quad \forall \phi \in V. \tag{50}$$

The operator $\mathcal{A}$ can be also considered as an unbounded operator from $H$ into itself with $D(\mathcal{A}) = \{u \in V \cap H^2(\Omega_f)\}$. It is well known (see Temam, 1975, 1977) that $\mathcal{A}$ generates an analytic semigroup on $H$. The same remains true for $\mathcal{A} - L$, since $L$ is a relatively bounded perturbation.
In addition, the Neumann map $N : L_2(\Gamma_s) \to H$ is defined by

$$Ng = h \Longleftrightarrow \{(\epsilon(h), \epsilon(\phi)) = \langle g, \phi \rangle, \forall \phi \in V\}. \tag{51}$$

It follows from the Lax-Milgram Theorem that the map $\mathcal{A} \in \mathcal{L}(V \to V')$ and the map $N$ enjoys the property

$$N \in \mathcal{L}(H^{-1/2}(\Gamma_s) \to V \subset H^1(\Omega_f)). \tag{52}$$

It was shown in Barbu, Grujic, Lasiecka & Tuffaha (2007) that the operator $A$ given by (49) and defined on $\mathcal{D}(A) \subset H \to H$ with

$$\mathcal{D}(A) = \{y \in H : u \in V, A(u + N\sigma(w) \cdot \nu) - Lu \in H; z \in H^1(\Omega_s),$$
$$\text{div } \sigma(w) \in L_2(\Omega_s); z|_{\Gamma s} = u|_{\Gamma_s}\}$$

indeed generates a strongly continuous semigroup $e^{At}$ on $Y$. However, this semigroup resulting from hyperbolic-parabolic coupling is *not analytic*. For initial conditions, which are more regular (in the domain of the generator) the corresponding solutions enjoy additional smoothness properties as shown in Barbu, Grujic, Lasiecka & Tuffaha (2008), and Kukavica, Tuffaha, & Ziane (2009).

In verifying the validity of conditions assumed in Assumption (1) we note that the property (d) in Assumption (1) is straightforward. This simply follows from the fact that $G$ acts as the projection operator on the first component on the vector $(u, w, v)$ along with the fact that $A$ is invertible on $Y$. At this point we note that the control operator is *not bounded* since $\mathcal{A}N = \mathcal{A}^{1/4+\epsilon}\mathcal{A}^{3/4-\epsilon}N$ and only $\mathcal{A}^{3/4-\epsilon}N \in \mathcal{L}(U, H)$. However, the third condition (c) in Assumption (1) is valid on the strength of Proposition 4.4 in Lasiecka & Tuffaha (2009).

The main technical issue is the validity of the RSE Estimate (7), which is verified by exploiting the parabolic-hyperbolic interaction in the system. The smoothing effect resulting from parabolicity of $e^{\mathcal{A}t}$ is propagated through the hyperbolic part of the dynamics and leads to the singular estimate. This follows from Theorem 5.1 in Lasiecka & Tuffaha (2009), which states that

$$||e^{At}Bg||_{H_{-\alpha}} \leq \frac{C}{t^{1/4+\epsilon}}||g||_{L_2(\Gamma_s)}, \quad 0 < t \leq 1, \tag{53}$$

where $H_{-\alpha} = L_2(\Omega_f) \times H^{1-\alpha}(\Omega_s) \times H^{-\alpha}(\Omega_s)$ and $\alpha$ can be taken an arbitrary small positive constant.

In order to infer (RSE) estimate (7) it suffices to consider the projection of the estimate (53) onto the first component. This leads to the estimate

$$||Ge^{At}Bg||_{L_2(\Omega)} \leq \frac{C}{t^{1/4+\epsilon}}||g||_{L_2(\Gamma_s)}, \quad 0 < t \leq 1, \tag{54}$$

which is equivalent to (7). Thus, all parts of Assumption (1) and Assumption (2) have been verified with $\gamma = 1/4+\epsilon$. Thus, the results claimed by Theorem 1, Theorem 2 are applicable to the model presented above.

REMARK 3 *The proof of* (53) *given in Lasiecka & Tuffaha (2009), is a technical PDE based proof. It relies on three main ingredients: (i) sharp trace regularity of solutions to the wave equation (see Lasiecka, Lions & Triggiani, 1986), (ii)'hidden regularity' of normal stresses corresponding to solutions of fluid-structure interaction (see Barbu, Grujic, Lasiecka & Tuffaha, 2007) and (iii) maximal boundary parabolic regularity exhibited by analytic semigroups (see Lasiecka, 1980, and Bensoussan, Da Prato, Delfour & Mitter, 2007).*

REMARK 4 *We notice that due to the fact that the estimate* (53) *requires* $\alpha > 0$, *the original version of Singular Estimate* (3) *does not hold. Thus, the relaxation of the Singular Estimate assumption (SE) to the Revisited Singular Estimate (RSE) in* (7) *is essential for solving the optimal feedback control problem associated with this fluid-structure interaction.*

REMARK 5 *The estimate* (53) *in fact implies that Theorems* (1) *and* (2) *are also applicable to this model when the objective functional* $J$ *is of the general form*

$$J(u,g) = \int_0^T ||g(t)||^2_{L_2(\Gamma_s)}\, ds + ||u(T,\cdot)||^2_{L_2(\Omega_f)} + ||w(T,\cdot)||^2_{H^{1-\alpha}(\Omega_s)} + ||w_t(T,\cdot)||^2_{H^{-\alpha}(\Omega_s)}, \tag{55}$$

*where* $\alpha$ *is any strictly positive constant.*

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## References


Acquistapace, P., Bucci, F. and Lasiecka, I. (2005) Optimal boundary control and Riccati theory for abstract dynamics motivated by hybrid systems of PDEs. *Advances in Differential Equations* **10**, 1389–1436.

Acquistapace, P., Bucci, F. and Lasiecka, I. (2005) A trace regularity result for thermoelastic equations with application to optimal boundary control. *J. Math. Anal. Appl* **310** (1), 262–277.

Avalos, G. and Lasiecka, I. (1996) Differential Riccati equation for the active control of a problem in structural acoustics. *J. Optim. Theory Appl.* **91** (3), 695–728.

Balakrishnan, A.-V. (1975) *Applied Functional Analysis*. Springer-Verlag.

Barbu, V., Lasiecka, I. and Triggiani, R. (2000) Extended Algebraic Riccati equations in the Abstract Hyperbolic Case. *Nonlinear Analysis* **40**, 105–129.

Barbu, V., Grujic, Z., Lasiecka, I. and Tuffaha, A. (2007) Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model. *Contemp. Math.*, **440**, Amer. Math. Soc., Providence, RI, 55–82.

Barbu, V., Grujic, Z., Lasiecka, I. and Tuffaha, A. (2008) Smoothness of weak solutions to a nonlinear fluid-structure interaction model. *Indiana Journal of Mathematics* **57** (3), 1173–1207.

Bensoussan, A., Da Prato, G., Delfour, M.-C. and Mitter, S.-K. (2007) *Representation and Control of Infinite Dimensional Systems*. Birkhauser, 2nd ed.

Bucci, F., Lasiecka, I. and Triggiani, R. (2002) Singular estimates and uniform stability of coupled systems of Hyperbolic / Parabolic PDEs. *Abstract and Applied Analysis* **7** (4), 169–237.

Bucci, F. and Lasiecka, I. (2004) Singular estimates and Riccati theory for thermoelastic plate models with boundary thermal control. *Dynamics of Continuous, Discrete and Impulsive Systems* **11**, 545–568.

Bucci, F. (2007) Control-theoretic properties of structural acoustic models with thermal effects, I. Singular estimates. *J. Evol. Equ.* **7**, 387–414.

Bucci, F. (2008) Control-theoretic properties of structural acoustic models with thermal effects, II. Trace regularity results. *Appl. Math.* **35**(3), 305–321.

Bucci, F. and Lasiecka, I. (2009) Optimal boundary control with critical penalization for a PDE model of fluid-solid interactions. *Calculus of Variations and PDE's.*

Camurdan, M. (1995) Control problems in noise reduction. The case of two coupled hyperbolic equations. *Proceedings of SPIE Conference*, Bellingham, WA.

Da Prato, G. and Ichikawa, A. (1985) Riccati Equations with unbounded coefficients. *Ann. Math. Pura Appl* **140**, 209–221.

DU, Q., GUNZBURGER, M.-D., HOU, L.-S. and LEE, J. (2003) Analysis of a Linear Fluid-Structure Interaction Problem. *Discrete Continuous Dynamical Systems* **9** (3), 633–650.

FLANDOLI, F. (1984) Riccati equations arising in boundary control problems with distributed parameters. *SIAM J. Control* **22**, 76–86.

FRANKOWSKA, H. and OCHAL, A. (2005) On singularities of value function for Bolza optimal control problem. *Journal of Math. Anal. and Appl.* **306**, 2.

JACOB, B. and PARTINGTON, J.-R. (2004) Admissibility of control and observation operators for semigroups: A survey. In: J.A. Ball, J.W. Helton, M. Klaus and L. Rodman, eds., *Current Trends in Operator Theory and Its Applications*, *Proceedings of the IWOTA 2002, Oper. Theory Adv. Appl.* **149**, Birkhuser, 199-221.

KUKAVICA I., TUFFAHA, A. and ZIANE, M. (2009) Strong Solutions to a Nonlinear Fluid Structure Interaction System. *J. Differential Equations* **247** (5), 1452–1478.

LASIECKA, I. (1980) A unified theory for abstract parabolic boundary problems. *Applied Mathematics and Optimization* **6**, 287–333.

LASIECKA, I. (2002) *Mathematical Control Theory of Coupled PDE's*, NSF-CMBS Lecture Notes, SIAM.

LASIECKA, I. (2004) *Optimal Control Problems and Riccati Equations for Systems with Unbounded Controls and Partially Analytic Generators: Applications to Boundary and Point Control Problems.* **LNM 1855**. *Functional Analytic Methods for Evolution Equations.* Springer Verlag.

LASIECKA, I. and TRIGGIANI, R. (1983) Regularity of hyperbolic equations under $L_2(0,T;L_2(\Gamma))$-Dirichlet boundary terms. *Appl. Math. Optimiz.* **10**, 275–286.

LASIECKA, I., LIONS, J.-L. and TRIGGIANI, R. (1986) Non-Homogeneous Boundary Value Problems for Second Order Hyperbolic Operators. *Journal de Mathèmatiquée Pure et Appliquée* **65**, 149–192.

LASIECKA, I. and TRIGGIANI, R. (1986) Riccati equations for hyperbolic partial differential equations with $L_2(0,T;L_2(\Gamma))$ -Dirichlet boundary data. *SIAM Journal on Control* **24**, 884–925

LASIECKA, I. and TRIGGIANI, R. (1991) *Differential and Algebraic Riccati Equations with Applications to Point-Boundary Control Problems.* **LNCIS 164**, Springer Verlag.

LASIECKA, I. and TRIGGIANI, R. (2000) *Control Theory for Partial Differential Equations and Approximation Theories, Vol. I: Abstract Parabolic Systems.* Encyclopedia of Mathematics and its Applications **74**, Cambridge University Press.

LASIECKA, I. and TRIGGIANI, R. (2000) *Control Theory for Partial Differential Equations Continuous and Approximation Theories, Vol. 2: Abstract Hyperbolic-like Systems over a Finite Time Horizon.* Encyclopedia of Mathematics and its Applications **75**, Cambridge University Press.

LASIECKA, I. and TRIGGIANI, R. (2004) Optimal control and Differential Riccati Equations under Singular estimates for $e^{At}B$ in the absence of analyticity. *Advances in Dynamics and Control*, Special Volume dedicated to A. V. Balakrishnan, Chapman and Hall/CRC Press, 271–309.

LASIECKA, I. and TUFFAHA, A. (2008) Riccati Equations for the Bolza Problem Arising in Boundary/Point Control Problems Governed by $C_0$ Semigroups Satisfying a Singular Estimate. *J. Optim. Theory Appl.* **136**, 229-246.

LASIECKA, I. and TUFFAHA, A. (2009) Riccati theory and singular estimates for a Bolza control problem arising in linearized fluid-structure interactions. *Systems Control Letters*.

LIONS, J.-L. (1969) *Quelques methodes de resolution des problèmes aux limites nonlinéaries*. Dunod. Paris.

MOUBACHIR, M. and ZOLESIO, J. (2006) *Moving Shape Analysis and Control: applications to Fluid-Structure Interactions*. Chapman, Hall/CRC.

PAZY, A. (1983) *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer.

RUSSELL, D. (1978) Controllability and stabilizability theory for linear partial differential equations, recent progress and open questions. *SIAM Review* **20**, 639–739 .

SALAMON, D.-A. (1989) Realization theory in Hilbert space. *Math. Systems Theory*, **21**, 147–164.

STAFFANS, O.-J. (2005) *Well-Posed Linear Systems.* Cambridge University Press, Cambridge.

TEMAM, R. (1977)*Navier-Stokes Equations. Studies in Mathematics and its Applications*, North-Holland.

TEMAM, R. (1995) *Navier-Stokes Equations and Nonlinear Functional Analysis*. CBMS-NSF **66**.

WEISS, G. (1989) Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, **27**, 527–545.

WEISS, G. and ZWART, H. (1998) An example in lq optimal control. *Systems Control Lett.* **8**, 339–349.