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On a degenerate Riccati equation*

by

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Abstract: In this paper, we study the existence of solutions to a degenerate algebraic Riccati equation associated to an optimal control problem with infinite time horizon. Under some assumptions on the control system, we can select a solution to this Riccati equation providing a feedback control law able to stabilize the system.

Keywords: algebraic Riccati equation, optimal control, parabolic systems.

1. Introduction

Algebraic Riccati equations occur naturally when solving linear quadratic regulation problems in Hilbert spaces.

Let U, Y and Z be real Hilbert spaces. Let $A : D(A) \subset Z \to Z$ be the infinitesimal generator of a c_0-semigroup. Let B belong to $\mathcal{L}(U, Z)$ and C belong to $\mathcal{L}(Z, Y)$ (as usual, if X and Y are Hilbert spaces, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y, and if $X = Y$, we set $\mathcal{L}(X, X) = \mathcal{L}(X)$). Given $z_0 \in Z$ and $u \in L^2(0, \infty; U)$, let $z \in L^2(0, \infty; Z)$ be the solution of the initial value problem

$$\left.\begin{array}{rcl} z'(t) &=& Az(t) + Bu(t),\ t > 0, \\ z(0) &=& z_0. \end{array}\right\} \tag{1.1}$$

Define

$$J(z, u) \;=\; \frac{1}{2}\int_0^\infty \|Cz(t)\|_Y^2\ dt + \frac{1}{2}\int_0^\infty \|u(t)\|_U^2\ dt.$$

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The linear regulation problem for equation (1.1) and the cost function J is

$$(\mathcal{P}_{z_0}) \qquad \inf\Big\{J(z,u) \mid u \in L^2(0,\infty;U) \text{ and } (z,u) \text{ satisfies } (1.1)\Big\}.$$

If the semigroup $(e^{tA})_{t\geq 0}$ is exponentially stable, then $J(z_{z_0,u},u)$ is finite for all $z_0 \in Z$ and $u \in L^2(0,\infty;U)$, where $z_{z_0,u}$ is the solution to equation (1.1). Otherwise, that is, if $(e^{tA})_{t\geq 0}$ is not exponentially stable, the functional $J(z_{z_0,u},u)$ may not be necessarily finite valued. In the general case, we need to make the following assumption for the problem $(\mathcal{P}_{z_0})$ to be well posed.

Finite Cost Condition (FCC): for every $z_0 \in Z$, there exists a control $u \in L^2(0,\infty;U)$ such that $J(z_{z_0,u},u) < \infty$.

If we assume that FCC is valid, then it is easy to see that problem $(\mathcal{P}_{z_0})$ admits a unique solution. In this case, it can also be shown (see Zabczyk, 2008; Bensoussan et al., 1993; Lasiecka and Triggiani, 2000) that

$$\inf(\mathcal{P}_{z_0}) = \frac{1}{2}(\tilde{P}z_0, z_0)_Z \quad \text{for all} \;\; z_0 \in Z,$$

where $\tilde{P}$ is the so-called minimal solution to the algebraic Riccati equation

$$\begin{array}{l} P \in \mathcal{L}(Z), \quad P = P^* \geq \mathbf{0}, \\ A^*P + PA - PBB^*P + C^*C \;=\; \mathbf{0}. \end{array} \tag{1.2}$$

The optimal control $u \in L^2(0,\infty;U)$ of problem $(\mathcal{P}_{z_0})$ is given in the feedback form

$$u(t) = -B^*\tilde{P}\, z_{z_0,u}(t) \quad \text{for all } t > 0.$$

Thus, we can obtain the optimal state $\tilde{z}$ as the solution of the initial value problem:

$$\left.\begin{array}{rcl} z'(t) &=& (A - BB^*\tilde{P})z(t), \; t > 0, \\ z(0) &=& z_0. \end{array}\right\} \tag{1.3}$$

At this point, let us clarify some notation. In equation (1.2), $\mathbf{0}$ denotes the null operator of $\mathcal{L}(Z)$, P^*, B^* and C^* denote the adjoint of P, B and C, and we assume that Z and U are identified with their duals. Two self-adjoint operators S and T in Z are related by the partial order relation $S \geq T$ if, for every vector $z \in Z$, we have $((S-T)z,z)_Z \geq 0$, where $(\cdot,\cdot)_Z$ denotes the inner-product.

Let us recall that the pair (A,C) is *exponentially detectable* if there exists $L \in \mathcal{L}(Y,Z)$ such that $A + LC$, with domain $D(A)$, is the generator of an exponentially stable semigroup on Z.

If, in addition to the FCC, the pair (A,C) is exponentially detectable, then the solution to equation (1.2) is unique.

Let us assume that equation (1.2) admits a unique solution. When Z is of infinite dimension, for example, if equation (1.1) is a partial differential equation, the solution to equation (1.2) can be only approximated by a numerical algorithm. An approximation A_h, B_h and C_h of the operators A, B and C is obtained by discretizing equation (1.1) by some numerical scheme (a finite element method, or a finite difference method, or a finite volume method...) and equation (1.2) is replaced by an equation of the form

$$\begin{aligned} &P_h \in \mathcal{L}(\mathbf{R}^N), \quad P_h = P_h^* \geq \mathbf{0}, \\ &A_h^* P_h + P_h A_h - P_h B_h B_h^* P_h + C_h^* C_h \; = \; \mathbf{0}. \end{aligned} \tag{1.4}$$

For control of fluid flows or control of thermal processes the dimension N can be very large and in that case classical algorithms are not efficient in solving equation (1.4). This is why new algorithms are still developed nowadays to solve equation (1.4) (see Benner and Baur, 2008). The Kleinman-Newton algorithm (see Kleinman, 1968) is still efficient in solving equation (1.4) for large scale equation (see Benner and Baur, 2008; Burns et al., 2008). The inconvenience of that method is that, in order to guarantee the convergence of the Newton method, the initial guess P_0 must be chosen so that $A_h - B_h B_h^* P_0$ is exponentially stable. Finding such an initialization can be nearly as complicated as solving equation (1.4) itself.

Very recently, Benner et al. (2008) proposed to choose P_0 as the solution to the degenerate Riccati equation:

$$\begin{aligned} &P \in \mathcal{L}(Z), \quad P = P^* \geq \mathbf{0}, \\ &A^* P + PA - PBB^* P \; = \; \mathbf{0}, \\ &A - BB^* P \text{ generates an exponentially stable semigroup.} \end{aligned} \tag{1.5}$$

(In Benner and Baur, 2008, the authors consider the case where $\dim(Z) < \infty$.) Indeed, there are specific algorithms to solve equation (1.5) even when the dimension of Z is relatively high (Amodei and Buchot, 2008; Benner and Baur, 2008). (Equation (1.5) corresponds to the algebraic Riccati equation (1.2) when $C = \mathbf{0}$.) It is well known that there exists a unique solution to equation (1.2) such that $A - BB^* P$ generates an exponentially stable semigroup (see Bensoussan et al., 1993, Part III, Chapter 1, Corollary 4.1). The same proof can be applied to equation (1.5). Thus, the condition $A - BB^* P$ *generates an exponentially stable semigroup* that guarantees the uniqueness of solution to problem (1.5).

In this paper, we are interested in solutions to equation (1.5) when Z is of finite or of infinite dimension.

The paper is organized as follows. In Section 2, we recall a comparison principle for solutions of the algebraic Riccati equation (1.2). This will be helpful in Section 5 to characterize the solution to problem (1.5). An immediate consequence of this comparison principle is that, if A is itself exponentially stable, then the only solution of (1.5) is the trivial one. In Section 3, we will consider the other extreme, i.e. when $-A$ is exponentially stable. In particular, we will show that this, together with the condition that $(-A, B)$ is exactly controllable, is sufficient for the degenerate equation (1.5) to have a nontrivial solution P which is, on the one hand, invertible and, on the other hand, such that $A - BB^*P$ is exponentially stable. Actually, P^{-1} will turn out to be the solution of an associated Lyapunov equation and so we can explicitly write down the solution.

In Section 4, we will generalize this to cover certain cases which includes the finite dimensional case where A has no eigenvalues on the imaginary axis and the parabolic case (see Bensoussan et al., 2007). In particular, we will show that, when A has no eigenvalues on the imaginary axis, the eigenvalues of $A - BB^*P$ are related to the eigenvalues of A in the following manner. Let the spectrum of A, denoted $\sigma(A)$, be the disjoint union of $\sigma_s(A)$ of eigenvalues with strictly negative real part and $\sigma_u(A)$ of eigenvalues with strictly positive real part. If $\lambda = \mu + i\nu \in \mathbf{C}$, denote its reflection on the imaginary axis by $\widetilde{\lambda}$, i.e. $\widetilde{\lambda} = -\mu + i\nu$. Then

$$\sigma(A - BB^*P) \ = \ \sigma_s(A) \cup \{\widetilde{\lambda} \mid \lambda \in \sigma_u(A)\}.$$

To see why we can expect such a result, it is illuminating to consider the (albeit trivial) one-dimensional case. If a and $b \in \mathbf{R}$, we seek $p \geq 0$ such that $2ap - b^2p^2 = 0$. This has two solutions, namely $p = 0$ and $p = 2a/b^2$ when $b \neq 0$. Now consider the perturbed equation

$$2ap_\varepsilon - b^2p_\varepsilon^2 + \varepsilon^2 \ = \ 0.$$

The positive solution to this quadratic equation is given by

$$p_\varepsilon \ = \ \frac{a + \sqrt{a^2 + \varepsilon^2 b^2}}{b^2}.$$

When $a < 0$, $p_\varepsilon \to p = 0$; when $a > 0$, $p_\varepsilon \to p = \frac{2a}{b^2}$. Thus

$$a - b^2p \ = \ \begin{cases} a & \text{when } a < 0 \\ -a & \text{when } a > 0. \end{cases}$$

In Section 5, we will show that the solution P is the limit of the family of operators P_ε, as $\varepsilon \to 0$, where $P_\varepsilon = P_\varepsilon^* \geq \mathbf{0}$ and

$$A^*P_\varepsilon + P_\varepsilon A - P_\varepsilon BB^*P_\varepsilon + \varepsilon^2 I \ = \ \mathbf{0},$$

where I is the identity operator on Z. Our analysis in Section 5 relies on the notion of maximal solutions to algebraic Riccati equations introduced by A. Bensoussan (1987) (see also Bensoussan et al., 1993, Part III, or Bensoussan et al., 2007, Part V, pp. 497–500). As in Bensoussan et al. (1993) Part III, we will give a variational characterization for the control given by the usual feedback law $u = -B^*Pz$, where z is the solution of (1.3) and P is the (maximal) solution to (1.5). Some of the results in Bensoussan et al. (1993), Part III, may be applied to the degenerate case (that is, when $C = \mathbf{0}$) and not some other ones. Thus, for clarity we shall write all the proofs.

Let us finally mention that the results established in Sections 4 and 5 are well known for finite dimensional systems (see Amin, 1985; Ibbini and Amin, 1993; Zhou et al., 2008).

In Section 6, we end the paper by giving a concrete example where results of Section 4 apply.

2. A comparison principle

In this section, we recall a comparison principle for solutions of the algebraic Riccati equation.

LEMMA 1 *Let U, Y and Z be real Hilbert spaces and let $A : D(A) \subset Z \to Z$ be the infinitesimal generator of a c_0-semigroup. Let $B \in \mathcal{L}(U, Z)$ and let $C_i \in \mathcal{L}(Z, Y)$, for $i = 1, 2$. Assume that $C_1^*C_1 \geq C_2^*C_2$. Let $P_i \in \mathcal{L}(Z)$, where $P_i = P_i^* \geq \mathbf{0}$, be a solution to the algebraic Riccati equation:*

$$P_iA + A^*P_i - P_iBB^*P_i + C_i^*C_i \;=\; \mathbf{0}$$

for $i = 1, 2$. Finally, assume that $A - BB^*P_1$ is exponentially stable. Then $P_1 \geq P_2$.

Proof. Substracting the equation for P_2 from that for P_1, we get

$$\begin{aligned}&(P_1 - P_2)(A - BB^*P_1) + (A - BB^*P_1)^*(P_1 - P_2)+\\&\quad +(P_1 - P_2)BB^*(P_1 - P_2) = C_2^*C_2 - C_1^*C_1.\end{aligned}$$

Thus, if $\zeta \in D(A)$,

$$\begin{aligned}&\frac{d}{dt}\Big((P_1 - P_2)e^{t(A-BB^*P_1)}\zeta\ ,\ e^{t(A-BB^*P_1)}\zeta\Big) =\\&\quad = -\left\|B^*(P_1 - P_2)e^{t(A-BB^*P_1)}\zeta\right\|_U^2 +\\&\quad + \Big((C_2^*C_2 - C_1^*C_1)e^{t(A-BB^*P_1}\zeta, e^{t(A-BB^*P_1)}\zeta\Big)_Z.\end{aligned}$$

Integrating this from 0 to T and using the fact that $C_1^*C_1 \geq C_2^*C_2$, we get

$$\Big((P_1 - P_2)e^{T(A-BB^*P_1)}\zeta, e^{T(A-BB^*P_1)}\zeta\Big)_Z - ((P_1 - P_2)\zeta, \zeta)_Z \;\leq\; 0.$$

Since $A - BB^*P_1$ is assumed to be exponentially stable, the above relation implies, on letting T tend to infinity, that $((P_1 - P_2)\zeta, \zeta)_Z \geq 0$. By density of $D(A)$ in Z, this is true for all $\zeta \in Z$, which completes the proof. ■

As an immediate consequence of this lemma, we deduce the following results.

COROLLARY 1 *The algebraic Riccati equation (1.2) and the degenerate algebraic Riccati equation (1.5) each admit at most one solution P such that $A - BB^*P$ is exponentially stable. In particular, if A is itself exponentially stable, then the degenerate algebraic Riccati equation (1.5) has no non-trivial solution P such that $A - BB^*P$ is exponentially stable.*

Recall that (see Zabczyk, 2008) if the pair (A, C) is exponentially detectable, then any solution P of the algebraic Riccati equation (1.2) is such that $A-BB^*P$ is exponentially stable. Thus, if (A, C) is exponentially detectable, then (1.2) admits a unique solution.

3. A special case

In the previous section, we saw that if A was itself exponentially stable, then the trivial solution was the only one possible for the degenerate Riccati equation. In this section, we consider the other extreme, when $-A$ is exponentially stable. Thus, we are in the case when A is the infinitesimal generator of a group. (Some generalization corresponding to the case when A is the infinitesimal generator of a semigroup is studied in the next section.) We will see that we can expect not only a nontrivial solution, but one that is invertible as well. More precisely, we prove the following result.

THEOREM 1 *Let U and Z be Hilbert spaces. Let $A : D(A) \subset Z \to Z$ be the infinitesimal generator of a c_0-group. Let $B \in \mathcal{L}(U, Z)$. Then, the following are equivalent:*
(i) *$-A$ is exponentially stable and there exists $\alpha > 0$ such that for all $z \in Z$,*

$$\int_0^\infty \left\| B^* e^{-tA^*} z \right\|_U^2 \, dt \;\geq\; \alpha \|z\|_Z^2. \tag{3.1}$$

(ii) *The degenerate algebraic Riccati equation (1.5) admits a solution $P \in \mathcal{L}(Z)$ and P is invertible.*

Proof. Step 1. Assume that $-A$ is exponentially stable and that (3.1) holds. Then, the operator

$$Q \;=\; \int_0^\infty e^{-tA} BB^* e^{-tA^*} \, dt$$

is well-defined and, clearly, $Q = Q^*$. Let $z \in Z$. Then

$$(Qz, z)_Z \;=\; \int_0^\infty \left\| B^* e^{-tA^*} z \right\|_U^2 \, dt \;\geq\; \alpha \|z\|_Z^2 \;\geq\; 0$$

by (3.1). It then follows, from the Lax-Milgram lemma, that Q is invertible.

Step 2. Set $Q(t) = e^{-tA}BB^*e^{-tA^*}$. Then,

$$\frac{d}{dt}Q(t) \;=\; (-A)e^{-tA}BB^*e^{-tA^*} + e^{-tA}BB^*e^{-tA^*}(-A^*).$$

Thus,

$$-\int_0^\infty \frac{d}{dt}Q(t)\;dt \;=\; Q(0) \;=\; BB^*.$$

Thus, we deduce that

$$AQ + QA^* \;=\; BB^* \quad \text{with} \quad Q = \int_0^\infty e^{-tA}BB^*e^{-tA^*}\;dt. \tag{3.2}$$

Setting $P = Q^{-1}$ and multiplying the above equation on both sides by P, we deduce that P satisfies the degenerate algebraic Riccati equation (1.5). Further, it is clear that $P = P^* \geq \mathbf{0}$.

Step 3. Since P is invertible and satisfies (1.5), we see immediately that

$$P(A - BB^*P)P^{-1} \;=\; -A^*. \tag{3.3}$$

Thus, $A - BB^*P$ is similar to $-A^*$ which is exponentially stable, by hypothesis, and so the exponential stability of $A - BB^*P$ is established. Thus, P satisfies all the conditions laid down in statement (ii) of the theorem.

Step 4. Conversely, let us now assume the validity of statement (ii) of the theorem. Once again, since P is an invertible operator, which solves (1.5), we deduce that (3.3) is valid. Thus, if $A - BB^*P$ is exponentially stable, the same is true for $-A^*$ and so for $-A$ as well. Also we deduce from (1.5) and the invertibility of P, that

$$AP^{-1} + P^{-1}A^* \;=\; BB^* \tag{3.4}$$

which is just (3.2) with P^{-1} replacing Q. Thus we easily see that

$$\frac{d}{dt}\left(e^{-tA}P^{-1}e^{-tA^*}\right) \;=\; -e^{-tA}BB^*e^{-tA^*},$$

whence we deduce that

$$P^{-1} \;=\; \int_0^\infty e^{-tA}BB^*e^{-tA^*}\;dt.$$

Thus, if $z \in Z$, then

$$\int_0^\infty \left\|B^*e^{-tA^*}z\right\|_U^2\;dt \;=\; \int_0^\infty \left(e^{-tA}BB^*e^{-tA^*}z, z\right)_Z\;dt \;=\; (P^{-1}z, z)_Z.$$

Since P^{-1} is self-adjoint and non-negative, it admits a square root, i.e. there exists $S \in \mathcal{L}(Z)$ such that $P^{-1} = S^2$. Clearly, S is also invertible and, further, $\|Sz\|_Z \geq \beta\|z\|_Z$, where

$$\beta \; = \; \frac{1}{\|S^{-1}\|_{\mathcal{L}(Z)}}.$$

It then follows that

$$\int_0^\infty \left\| B^* e^{-tA^*} z \right\|_U^2 \, dt \; \geq \beta^2 \|z\|_Z^2,$$

which establishes (3.1) with $\alpha = \beta^2$. This completes the proof. ■

REMARK 3.1 Notice that the operator Q, defined in Step 1 of the proof, is nothing else than the Gramian corresponding to $(-A, B)$ and that (3.2) is just the Lyapunov equation associated to the pair (A, B).

REMARK 3.2 If the pair $(-A, B)$ is exactly controllable in some time $T > 0$, then there exists $\alpha > 0$ such that, for every $z \in Z$,

$$\int_0^T \left\| B^* e^{-tA^*} z \right\|_U^2 \, dt \; \geq \; \alpha \|z\|_Z^2.$$

Thus, in this case, (3.1) holds and the above result is applicable.

REMARK 3.3 The idea of using the dual Riccati equation (3.4) when P is invertible is classical in control theory. It has been used in connection with exact controllability in Flandoli et al. (1988), Theorem 2.6. See also Lasiecka and Triggiani (2000b).

4. Generalizations

We assume, henceforth, that the Hilbert space Z and the operator $A : D(A) \subset Z \to Z$, which is assumed to be the infinitesimal generator of a c_0-semigroup, satisfy the following hypothesis:

(H) There exist closed subspaces Z_s and Z_u of Z such that:
(i) $Z = Z_s \oplus Z_u$.
(ii) $Z_u \cap D(A)$ and $Z_s \cap D(A)$ are invariant under A.
(iii) The operator $A|_{Z_s}$, the restriction of A to Z_s, is exponentially stable.
(iv) The operator $-A|_{Z_u}$ is exponentially stable.

EXAMPLE 4.1 If Z is of finite dimension and if $A : Z \to Z$ is linear and has no eigenvalues on the imaginary axis, then using the Jordan form of the matrix, we can find Z_s and Z_u invariant under A such that all the eigenvalues of the restriction of A to Z_s have negative real part and those of the restriction of A to Z_u have positive real part. Further, $Z = Z_s \oplus Z_u$.

EXAMPLE 4.2 If $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on Z and if the resolvent of A is a compact operator in Z, then the spectrum of A is discrete (see Kato, 1995). Let us assume that A has no eigenvalue on the imaginary axis. From Kato (1995), pp. 178–182, it follows that the space Z can be decomposed in the form $Z = Z_s \oplus Z_u$, where $Z_u \cap D(A) = Z_u$ and $Z_s \cap D(A)$ are invariant under A, Z_s is the stable space of A and Z_u is the unstable subspace. Moreover, Z_u is of finite dimension. Thus, hypothesis (H) is satisfied. Further, let us notice that, since Z is a real Hilbert space and the operator A takes values in Z, then $\lambda \in \mathbf{C}$ and its conjugate $\overline{\lambda} \in \mathbf{C}$ either both are or both are not eigenvalues of A. The same observation holds true for A^*, and A and A^* have obviously the same eigenvalues.

Let $\pi_s : Z \to Z_s$ and $\pi_u : Z \to Z_u$ be the canonical projections with respect to this decomposition of Z. Notice that

$$\pi_u A \;=\; A\pi_u \;=\; \pi_u A \pi_u. \tag{4.1}$$

A similar relation holds with π_s as well. The restriction of A to Z_u (respectively Z_s) is, in fact, equal to $\pi_u A$ (respectively $\pi_s A$). Observe that $\pi_u B$ maps U into Z_u.

Assume that (3.1) holds for the pair $(\pi_u A, \pi_u B)$ in place of (A, B), i.e. there exists $\alpha > 0$ such that

$$\int_0^\infty \left\| (\pi_u B)^* e^{-t(\pi_u A)^*} z \right\|_U^2 \, dt \;\geq\; \alpha \|z\|_Z^2 \tag{4.2}$$

for every $z \in Z_u$. Then, by Theorem 3.1, there exists a self-adjoint, non-negative and invertible operator $P_u \in \mathcal{L}(Z_u)$ satisfying the relation

$$P_u(\pi_u A) + (\pi_u A)^* P_u - P_u(\pi_u B)(\pi_u B)^* P_u \;=\; \mathbf{0}. \tag{4.3}$$

Further, $\pi_u A - (\pi_u B)(\pi_u B)^* P_u$ will be exponentially stable.

THEOREM 2 *Assume that the hypothesis (H) and condition (4.2) hold. Let $P_u \in \mathcal{L}(Z_u)$ be as detailed above. Define*

$$P \;=\; \pi_u^* P_u \pi_u. \tag{4.4}$$

Then $P \in \mathcal{L}(Z)$ is such that $P = P^* \geq \mathbf{0}$ and solves the degenerate algebraic Riccati equation (1.5). Further, $A - BB^*P$ is exponentially stable.

Proof. Step 1. Clearly $P = P^* \geq \mathbf{0}$, since this follows immediately from the corresponding properties of P_u. Multiplying (4.3) on the left by π_u^* and on the right by π_u, we get

$$\pi_u^* P_u \pi_u A \pi_u + \pi_u^* A^* \pi_u^* P_u \pi_u - \pi_u^* P_u \pi_u B B^* \pi_u^* P_u \pi_u \;=\; \mathbf{0}. \tag{4.5}$$

It follows from (4.1) that $A^*\pi_u^* = \pi_u^* A^* = \pi_u^* A^* \pi_u^*$. Further, since π_u is a projection, we have $\pi_u^2 = \pi_u$ and $(\pi_u^*)^2 = \pi_u^*$. Using these relations in (4.5) and the definition of P as in (4.4), we see immediately that P satisfies the degenerate algebraic Riccati equation (1.5).

Step 2. Since $\pi_u + \pi_s = I$, the identity operator on Z, we have

$$\begin{aligned} A - BB^*P &= A - BB^*\pi_u^* P_u \pi_u \\ &= (\pi_u + \pi_s)A - (\pi_u + \pi_s)BB^*\pi_u^* P_u \pi_u. \end{aligned}$$

If $z \in Z$, we have $z = \pi_u z + \pi_s z$ and, since $\pi_u \pi_s = \pi_s \pi_u = \mathbf{0}$, and since A commutes with π_u and with π_s, we get

$$\begin{aligned} (A - BB^*P)z &= (\pi_u + \pi_s)A(\pi_u z + \pi_s z) - (\pi_u + \pi_s)BB^*\pi_u^* P_u \pi_u z \\ &= \pi_u A \pi_u z + \pi_s A \pi_s z - (\pi_u + \pi_s)BB^*\pi_u^* P_u \pi_u z. \end{aligned}$$

Thus,

$$\pi_u((A - BB^*P)z) = (\pi_u A - \pi_u BB^*\pi_u^* P_u)(\pi_u z)$$

and

$$\pi_s((A - BB^*P)z) = \pi_s A \pi_s z - (\pi_s BB^*\pi_u^* P_u)(\pi_u z).$$

We can combine the above two relations in the following form using matrix notation:

$$\begin{bmatrix} \pi_s((A - BB^*P)z) \\ \pi_u((A - BB^*P)z) \end{bmatrix} = \begin{bmatrix} \pi_s A & -\pi_s BB^*\pi_u^* P_u \\ \mathbf{0} & \pi_u A - \pi_u BB^*\pi_u^* P_u \end{bmatrix} \begin{bmatrix} \pi_s z \\ \pi_u z \end{bmatrix}.$$

We know that

$$\|e^{t(\pi_u A - \pi_u BB^*\pi_u^* P_u)}\pi_u z\|_Z \leq Ce^{-\omega t}\|\pi_u z\|_Z,$$

for some $\omega > 0$, $C > 0$, and that

$$\|e^{t\pi_s A}\pi_s z\|_Z \leq Ce^{-\delta t}\|\pi_s z\|_Z,$$

with $\delta > 0$, $C > 0$. Combining these two stability results, it can be easily shown that $A - BB^*P$ is also exponentially stable (see e.g. Triggiani, 1975). This completes the proof. ∎

Assume now that either Z is finite dimensional and that $A \in \mathcal{L}(Z)$, or that the unbounded operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on Z with a compact resolvent. Then, as already seen in Examples 4.1 and 4.2, the hypothesis (H) is true as long as A has no eigenvalues on the imaginary axis of the complex plane. If the pair $(-\pi_u A, \pi_u B)$ is exactly controllable in Z_u (see Remark 3.2), then the result of the above theorem is valid.

In particular, the eigenvalues of $A-BB^*P$ are those of $\pi_s A$, which are precisely those eigenvalues of A with negative real part, and those of $\pi_u A - \pi_u BB^*\pi_u^* P_u$, which are precisely those of $-(\pi_u A)^*$ (due to (3.3)), and hence the reflections on the imaginary axis of those eigenvalues of A with positive real part.

A sufficient condition for $(-\pi_u A, \pi_u B)$ to be exactly controllable in Z_u in time $T>0$ is that the pair $(-A,B)$ be exactly controllable in time $T>0$. To see this, let $z_0, z_1 \in Z_u$. Then, there exists $u \in L^2(0,T;U)$ and $z \in L^2(0,T;Z)$ such that

$$\begin{aligned} z'(t) &= -Az(t) + Bu(t),\ t>0, \\ z(0) &= z_0,\ z(T) = z_1. \end{aligned}$$

Notice that $\pi_u z_0 = z_0$ and $\pi_u z_1 = z_1$. Since $\pi_u A = \pi_u A \pi_u$, we get that

$$(\pi_u z)'(t) \ = \ -(\pi_u A)(\pi_u z)(t) + (\pi_u B)u(t).$$

Thus, for the control u, the pair $(-\pi_u A, \pi_u B)$ drives the initial state $z_0 \in Z_u$ to the state $z_1 \in Z_u$ in time T via the solution $\pi_u z(t)$. Thus $(-\pi_u A, \pi_u B)$ is controllable in time T.

We can thus summarize these arguments in the following result.

THEOREM 3 *Assume now that either Z is finite dimensional and that $A \in \mathcal{L}(Z)$, or that the unbounded operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on Z with a compact resolvent, and that A has no eigenvalues on the imaginary axis of the complex plane. Let U be a Hilbert space and let $B \in \mathcal{L}(U,Z)$. Assume that the pair $(-\pi_u A, \pi_u B)$ is exactly controllable. Then the degenerate algebraic Riccati equation (1.5) admits a solution P which is self-adjoint, non-negative and such that $A - BB^*P$ is exponentially stable. If $\sigma(.)$ denotes the spectrum of a matrix, then*

$$\sigma(A - BB^*P) \ = \ \sigma_s(A) \cup \{\widetilde{\lambda} \mid \lambda \in \sigma_u(A)\}$$

where

$$\sigma_s(A) \ = \ \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda < 0\},$$

$$\sigma_u(A) \ = \ \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda > 0\},$$

and, if $\lambda = \mu + i\nu$, then $\widetilde{\lambda} = -\mu + i\nu$, is the reflection of λ on the imaginary axis.

REMARK 4.1 If Z is of finite dimension, the assumption "$(-A,B)$ *is exactly controllable in time* $T>0$" may be easily verified, while when Z is of infinite dimension, there is generally no hope to show that $(-A,B)$ *is exactly controllable in time* $T>0$ (or even stabilizable). But in many applications, if Z_u is of finite dimension, it is easy to check whether $(-\pi_u A, \pi_u B)$ is exactly controllable.

REMARK 4.2 It is proved in Priola and Zabczyk (2003), Theorem 1.3, that if the control system (1.1) is null controllable then the algebraic Riccati equation

$$P \in \mathcal{L}(Z), \quad P = P^* \geq \mathbf{0}, \quad A^*P + PA - PBB^*P \ = \ \mathbf{0},$$

admits the unique solution $P = \mathbf{0}$ if and only if the system (1.1) is null controllable with vanishing energy. Our result stated in Theorem 2 is different and complementary. Indeed it may provide the existence of a solution to the algebraic Riccati equation (1.5) which is non zero.

5. A variational characterization

In this section, we assume that assumption (H) of Section 4 and condition (4.2) are satisfied. According to Theorem 2, $P = \pi_u^* P_u \pi_u$, where $P_u \in \mathcal{L}(Z_u)$ is the solution to (4.3), is the unique solution of (1.5). We will characterize this solution in a different manner, using variational arguments. As mentioned in the introduction, our approach is linked to the notion of maximal solution introduced in Bensoussan et al. (1993), Part III, or Bensoussan et al. (2007), Part V, pp. 497–500.

To begin with, we prove a technical lemma from functional analysis. While we feel that this result should be well known, we nevertheless include a proof of the same for want of a suitable reference (see Remark 5.1).

LEMMA 2 *Let H be a real Hilbert space. Let $\{P_n\}$ be a sequence in $\mathcal{L}(H)$ such that, for every n, we have $P_n = P_n^* \geq \mathbf{0}$. Assume, further, that for every $v \in H$, the sequence $\{(P_n v, v)_H\}$ is decreasing. Then, there exists $P \in \mathcal{L}(H)$ such that $P = P^* \geq \mathbf{0}$ and, for every $v \in H$, $P_n v \to Pv$ in H.*

Proof. Step 1. Since $\{(P_n v, v)_H\}$ is decreasing and is bounded below by zero, we have that $\{(P_n v, v)_H\}$ is convergent for every $v \in H$. Since $P_n = P_n^*$, and since

$$(P_n v, w)_H \ = \ \frac{1}{4}[P_n(v + w), v + w)_H - (P_n(v - w), v - w)_H],$$

we deduce that the sequence $\{(P_n v, w)_H\}$ is convergent for every v and $w \in H$. Thus, by the Banach-Steinhaus theorem, $\{P_n v\}$ is a bounded sequence in H for every $v \in H$ and, again, by the same theorem, it follows that $\{P_n\}$ is bounded in $\mathcal{L}(H)$.

Step 2. Set

$$a(v, w) \ = \ \lim_{n\to\infty} (P_n v, w)_H.$$

Since $\{P_n\}$ is bounded, it follows that $a(\cdot, \cdot)$ is a continuous bilinear form. Consequently, by the Riesz representation theorem, there exists $Pv \in H$ such that $(Pv, w)_H = a(v, w)$ for every $w \in H$. Since $a(\cdot, \cdot)$ is bilinear, continuous and symmetric, it follows that P is linear, continuous and such that $P = P^*$. Finally, since

$$(Pv, v)_H \;=\; \lim_{n\to\infty} (P_n v, v)_H \;\geq\; 0,$$

it follows that $P \geq \mathbf{0}$ as well.

Step 3. We now show that, for every $v \in H$, we have that $P_n v \to Pv$ in H. Since $\{(P_n v, v)_H\}$ is a decreasing sequence with $(Pv, v)_H$ as its limit, we see that

$$((P_n - P)v, v)_H \;\geq\; 0.$$

Thus, $P_n - P \geq \mathbf{0}$ and so it admits a square root, say, $S_n \in \mathcal{L}(H)$, i.e. $S_n^2 = P_n - P$. Since

$$\|S_n v\|_H^2 \;=\; ((P_n - P)v, v)_H \;\to\; 0,$$

a fresh application of the Banach-Steinhaus theorem implies that $\|S_n\| \leq C$. Thus

$$\|P_n v - Pv\|_H \;=\; \|S_n^2 v\|_H \;\leq C\|S_n v\|_H \;\to\; 0$$

and the proof is complete. ∎

Remark 5.1 Steps 1 and 2 of the above proof may be found in Zabczyk (2008), but the arguments in Step 3 are not given there.

Let us now recall that assumption (H) of Section 4 and condition (4.2) are satisfied, and that $P = \pi_u^* P_u \pi_u$ is the unique solution of (1.5) ($P_u \in \mathcal{L}(Z_u)$ is the solution to (4.3). In particular, since $A - BB^*P$ is exponentially stable, it follows that the pair (A, I), where I is the identity operator on Z, is exponentially detectable. Thus, for all $\varepsilon \in \mathbf{R}$, the pair $(A, \varepsilon I)$ is exponentially detectable. Therefore, for every $\varepsilon > 0$, there exists a unique $P_\varepsilon \in \mathcal{L}(Z)$ such that $P_\varepsilon = P_\varepsilon^* \geq \mathbf{0}$ and

$$P_\varepsilon A + A^* P_\varepsilon - P_\varepsilon BB^* P_\varepsilon + \varepsilon^2 I \;=\; \mathbf{0}. \tag{5.1}$$

Further, $A - BB^* P_\varepsilon$ will be exponentially stable. By the comparison principle (see Lemma 1), it follows that $\{(P_\varepsilon v, v)_Z\}$ is decreasing as ε decreases to zero. Hence, by Lemma 2, it follows that there exists $P_0 \in \mathcal{L}(Z)$ such that $P_0 = P_0^* \geq \mathbf{0}$ and such that, for every $z \in Z$, $P_\varepsilon z \to P_0 z$ as $\varepsilon \to 0$. It is now immediate to see that

$$P_0 A + A^* P_0 - P_0 BB^* P_0 \;=\; \mathbf{0}.$$

(The convergence of $P_\varepsilon z$ to $P_0 z$ was needed to pass to the limit in the quadratic term $P_\varepsilon BB^* P_\varepsilon$.)

PROPOSITION 1 *We have that $P_0 = P$, where $P = \pi_u^* P_u \pi_u$.*

Proof. Since $A - BB^*P$ is exponentially stable, the comparison principle (see Lemma 1) implies that $P \geq P_0$. On the other hand, by the same comparison principle, since $A - BB^*P_\varepsilon$ is exponentially stable, we have that $P_\varepsilon \geq P$ for all $\varepsilon > 0$ and so $P_0 \geq P$ as well. ∎

For all $\zeta \in Z$, $\zeta \neq 0$, let us set

$$E_\zeta = \left\{ u \in L^2(0,\infty;U) \mid \int_0^\infty \|z_{\zeta,u}(t)\|_Z^2 \, dt < \infty, \ \lim_{T\to\infty} \|z_{\zeta,u}(T)\|_Z = 0 \right\},$$

where $z_{\zeta,u}(t)$ is the solution of the initial value problem

$$\begin{aligned} z'(t) &= Az(t) + Bu(t), \ t > 0, \\ z(0) &= \zeta. \end{aligned} \tag{5.2}$$

Let us notice that E_ζ is non-empty. Indeed the function $u(t) = -B^*Pe^{t(A-BB^*P)}\zeta$ belongs to E_ζ. Let us consider the problem

$$(\mathcal{Q}_\zeta) \qquad \inf_{u \in E_\zeta} \int_0^\infty \|u(t)\|_U^2 \, dt.$$

PROPOSITION 2 *Assume that assumption (H) of Section 4 and condition (4.2) are satisfied. Then, for all $\zeta \in Z$, $\zeta \neq 0$, problem $(\mathcal{Q}_\zeta)$ admits a unique solution u defined by*

$$u(t) = -B^*Pe^{t(A-BB^*P)}\zeta,$$

where $P = \pi_u^* P_u \pi_u$ is the unique solution of (1.5). Moreover, we have

$$(P\zeta,\zeta)_Z = \min_{u \in E_\zeta} \int_0^\infty \|u(t)\|_U^2 \, dt. \tag{5.3}$$

Proof. Let $\zeta \neq 0$ belong to Z and let u belong to E_ζ. If we multiply equation (5.2) by $Pz_{\zeta,u}$, after integration, we obtain

$$\int_0^\infty \left(z'_{\zeta,u}, Pz_{\zeta,u}\right)_Z dt = \int_0^\infty \left(Az_{\zeta,u}, Pz_{\zeta,u}\right)_Z dt + \int_0^\infty \left(Bu, Pz_{\zeta,u}\right)_Z dt.$$

Using the equation satisfied by P, we can write

$$-\frac{1}{2}\left(\zeta, P\zeta\right)_Z = \frac{1}{2}\int_0^\infty \|B^*Pz_{\zeta,u}\|_U^2 \, dt + \int_0^\infty \left(u, B^*Pz_{\zeta,u}\right)_U dt.$$

Thus we have

$$\int_0^\infty \|u\|_U^2 \, dt = \left(\zeta, P\zeta\right)_Z + \int_0^\infty \|B^*Pz_{\zeta,u} + u\|_U^2 \, dt.$$

Let us notice that if

$$u(t) = -B^*Pe^{t(A-BB^*P)}\zeta,$$

then $B^*Pz_{\zeta,u} + u = 0$. Thus $u(t) = -B^*Pe^{t(A-BB^*P)}\zeta$ is the unique solution to $(\mathcal{Q}_\zeta)$ and the proof is complete. ∎

Remark 5.2 Since $-\pi_u A$ is exponentially stable, $\pi_u A$ is not and so $0 \notin E_\zeta$.

Remark 5.3 It is not known whether E_ζ is weakly closed. If that were the case, then the solution to $(\mathcal{Q}_\zeta)$ could be characterized as the orthogonal projection of 0 onto E_ζ.

Remark 5.4 Let us notice that a characterization of maximal solution to the algebraic Riccati equation

$$P \in \mathcal{L}(Z), \quad P = P^* \geq \mathbf{0}, \quad A^*P + PA - PBB^*P + R \ = \ \mathbf{0},$$

where $R = R^* \geq \mathbf{0}$, is given in Priola and Zabczyk (2003), Theorem 1.4, for null controllable systems. Formula (5.3) is different from the one in Priola and Zabczyk (2003), Theorem 1.4, formula (1.6), even in the case when $R = \mathbf{0}$.

6. An application

Let us give a direct and surprising application to the results stated in Section 4. We are going to show how we can apply these results to a two dimensional viscous Burgers type equation. Let Ω be a two dimensional regular domain with boundary Γ. Let w be a given stationary solution to equation

$$-\nu\Delta w + w\partial_1 w + w\partial_2 w = f, \qquad w = g \quad \text{on } \Gamma. \tag{6.1}$$

The symbols ∂_1 and ∂_2 denote the partial derivatives with respect to x_1 and x_2 respectively, $\nu > 0$ is the viscosity coefficient. Now, let us consider the non-stationary Burgers equation

$$\left.\begin{aligned} &\partial_t z - \nu\Delta z + z\partial_1 z + z\partial_2 z = f + \chi_\omega u \quad \text{in } \Omega \times (0,\infty) = Q_\infty, \\ &z = g \quad \text{on } \Gamma \times (0,\infty) = \Sigma_\infty, \\ &z(0) = w + y_0 \quad \text{in } \Omega. \end{aligned}\right\} \tag{6.2}$$

Here χ_ω is the characteristic function of ω, an open and nonempty subset of Ω. The function u is a control variable. We assume that the solution w to equation (6.1) belongs to $H^3(\Omega)$. If z is the solution to equation (6.2), then $y = z - w$ obeys

$$\left.\begin{aligned} &\partial_t y - \nu\Delta y + y(\partial_1 w + \partial_2 w) + (w + y)(\partial_1 y + \partial_2 y) = \chi_\omega u \quad \text{in } Q_\infty, \\ &y = 0 \quad \text{on } \Sigma_\infty, \\ &y(0) = y_0 \quad \text{in } \Omega. \end{aligned}\right\} \tag{6.3}$$

We denote by $(A, D(A))$ and $(A^*, D(A^*))$ the unbounded operators in $L^2(\Omega)$ defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad D(A^*) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$Ay = \nu\Delta y - y(\partial_1 w + \partial_2 w) - w(\partial_1 y + \partial_2 y), \quad A^* y = \nu\Delta y + w(\partial_1 y + \partial_2 y).$$

Since $w \in H^3(\Omega)$, we can easily verify that there exists $\lambda_0 > 0$ in the resolvent set of A satisfying

$$\begin{aligned} &\big((\lambda_0 I - A)y, y\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{H^1(\Omega)} \qquad \text{for all } y \in D(A),\\ &\text{and}\\ &\big((\lambda_0 I - A^*)y, y\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|y\|^2_{H^1(\Omega)} \qquad \text{for all } y \in D(A^*). \end{aligned} \tag{6.4}$$

Equation (6.3) may be rewritten in the form

$$y' = Ay + Bu + F(y) \quad \text{in } (0, \infty), \qquad y(0) = y_0. \tag{6.5}$$

The nonlinear term $-y(\partial_1 y + \partial_2 y)$, which is equal to $-\partial_1(y^2/2) - \partial_2(y^2/2)$, is rewritten as an element $F(y)$ in $(D(A^*))'$ as follows

$$\langle F(y), \Phi\rangle_{(D(A^*))', D(A^*)} = \frac{1}{2}\int_\Omega y^2(\partial_1\Phi + \partial_2\Phi),$$

for all $\Phi \in D(A^*)$. The operator $B \in \mathcal{L}(L^2(\Omega))$ is defined by $Bu = \chi_\omega u$. Observe that $B = B^*$. The linearized system associated to (6.5) is

$$y' = Ay + Bu \quad \text{in } (0, \infty), \qquad y(0) = y_0, \tag{6.6}$$

Let us recall that $(A, D(A))$ is the infinitesimal generator of an analytic semigroup (see Thevenet et al., 2009) and that the resolvent of $(A, D(A))$ is compact. Thus, the spectrum of A is discrete and the eigenvalues have finite multiplicity. The number of eigenvalues having a real part greater or equal than $-\alpha$ is finite, for any $\alpha > 0$. Moreover, system (6.6) is exponentially stabilizable with any prescribed decay rate, because it is null controllable (see Fursikov and Imanuvilov, 1996). Without loss of generality we can choose $\alpha > 0$ so that the pair $(A + \alpha I, B)$ satisfies the assumptions of Theorem 2. Next, by making the change of variable $\hat{y} = e^{\alpha t}y$, $\hat{u} = e^{\alpha t}u$, for some $\alpha > 0$, we can easily verify that $(\hat{y}, \hat{u})$ obeys the system

$$\hat{y}' = (A + \alpha I)\hat{y} + B\hat{y} \quad \text{in } (0, \infty), \qquad \hat{y}(0) = y_0. \tag{6.7}$$

In that case, the Riccati equation is

$$\begin{aligned} &P \in \mathcal{L}(Z), \quad P = P^* \geq \mathbf{0},\\ &(A^* + \alpha I)P + P(A + \alpha I) - PBB^*P \ = \ \mathbf{0},\\ &A + \alpha I - BB^*P \text{ generates an exponentially stable semigroup.} \end{aligned} \tag{6.8}$$

From Theorem 2, it follows that the algebraic Riccati equation (6.8) has a unique solution P_α. As in Thevenet et al. (2009), it can be shown that the solution P_α inherits from the following regularizing property $P_\alpha \in \mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H^1_0(\Omega))$. Next we apply the linear feedback law to the nonlinear system satisfied by $(\hat{y}, \hat{u})$. Thus, we consider the system

$$\hat{y}' = (A + \alpha I)\hat{y} - BB^* P_\alpha \hat{y} + e^{-\alpha t} F(\hat{y}), \quad \hat{y}(0) = y_0. \tag{6.9}$$

Let us notice that if $(\hat{y}, \hat{u})$ is the solution of system (6.9) then the pair $(y, u) = (e^{-\alpha t}\hat{y}, e^{-\alpha t}\hat{u})$ obeys the system

$$y' = Ay - BB^* P_\alpha y + F(y), \quad y(0) = y_0. \tag{6.10}$$

As in Thevenet et al. (2009), using a fixed point argument, the following local stabilization result can be proved.

THEOREM 4 *There exist $\mu_0 > 0$ and a nondecreasing function η from $\mathbf{R}^+$ into itself, such that if $\mu \in (0, \mu_0)$ and $\|y_0\|_{L^2(\Omega)} \leq \eta(\mu)$, then equation (6.9) admits a unique solution in the set*

$$D_{\alpha,\mu} = \Big\{ y \in L^2(0,\infty; H^1_0(\Omega)) \cap H^{1/2}(0,\infty; L^2(\Omega)) \\ \mid \|e^{\alpha t} y\|_{L^2(0,\infty;H^1_0(\Omega))\cap H^{1/2}(0,\infty;L^2(\Omega))} \leq \mu \Big\}.$$

Thus, the linear feedback law applied to the nonlinear system (6.5) provides an exponential decay rate of the solution (the decay rate α can be chosen arbitrarily large). Following Raymond (2006), similar results can be obtained for the internal stabilization of the 2D Navier-Stokes equations. What is really surprising is that we can find a feedback law stabilizing locally a nonlinear system, with an observation operator which is identically zero.

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