**Control and Cybernetics**

vol. **38** (2009) No. 4B

# Shape optimization for stationary Navier-Stokes equations*

by

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**Abstract:** This work discusses geometric optimization problems governed by stationary Navier-Stokes equations. Optimal domains are proved to exist under the assumption that the family of admissible domains is bounded and satisfies the Lipschitz condition with a uniform constant, and in the absence of the uniqueness property for the state system. Through the parametrization of the admissible shapes by continuous functions defined on a larger universal domain, the optimization parameter becomes a control, i.e. an element of that family of continuous functions. The approximating extension technique via the penalization of the Navier-Stokes equation enables the approximation of the associated shape optimization problem by an optimal control problem. Results on existence and uniqueness are proved for the approximating problem and a gradient-type algorithm is indicated.

**Keywords:** optimal design, optimal control method, fluid mechanics, penalization, gradient method.

## 1. Introduction

Optimal design and optimal control in fluid mechanics constitute a difficult and important subject, with many applications. Among the numerous publications devoted to this research direction, we quote the monograph of Mohammadi and Pironneau (2001) and the articles of Borrvall and Peterson (2003), Gao and Ma (2007), Posta and Roubícek (2007), Roubícek and Tröltzsch (2003).

In this work, domain optimization problems associated to stationary Navier-Stokes equations and with general cost functionals will be considered. The unknown is the domain where the state equation is defined, assumed to satisfy

*Submitted: February 2009; Accepted: September 2009.

certain regularity and boundedness constraints. Our methods are different from the above mentioned references and are close to the optimal control approach proposed in Neittaanmäki et al. (2009) and to singular control problems since the uniqueness property is not imposed for the state system (see the existence results in Section 2). We use a direct penalization of the Navier-Stokes equation that approximates its solutions by functions defined on a larger bounded given domain. This allows for the introduction of new domain variations, of functional type, obtained via the variations of the functional parametrization of the shapes.

A general background in shape optimization and optimal control may be found in Delfour and Zolesio (2001), Lions (1971), Neittaanmäki and Tiba (1994) and Pironneau (1984). For the theory of singular control and for geometrical controllability properties the monographs by Neittaanmäki, Sprekels and Tiba (2006) and Lions (1983) are indicated.

The paper is organized as follows: in Section 2 a general shape optimization problem for stationary Navier-Stokes equations is introduced and existence questions are discussed. Optimal pairs are proved to exist under the assumption that the family of admissible domains is bounded and satisfies the Lipschitz condition with a uniform constant. Under supplementary hypotheses, in Section 3 a regularization/approximation procedure is developed. Through parametrization of admissible domains by functions in a certain space of continuous functions on a larger domain, the control parameter becomes an element of that space. The approximating extension technique allows for the approximation of the shape optimization problem by an optimal control problem. The directional derivative of the cost functional is studied in Section 4, results on existence and uniqueness are proved and a gradient type algorithm for shape optimization is given. We underline that some of the results in this paper are also valid in the case when the uniqueness condition is not satisfied for the Navier-Stokes system, i.e. for small viscosity and/or for big forces (see (2.7)). In particular, our Theorem 1 is a partial extension of the existence result of Wang and Yang (2008).

## 2. Formulation of the problem and existence

Let $j : R^d \times R^d \times R^{d\times d} \to R$, $d \geq 2$, be nonnegative and measurable with $j(x,\cdot,\cdot)$ continuous and $j(x,y,\cdot)$ convex. Define the following general minimization problem

$$\operatorname*{Min}_{\Omega}\{J(\Omega) = \int_{\Lambda} j(x, y(x), \nabla y(x))dx\} \tag{2.1}$$

where $\Omega \subset R^d$ is some unknown domain and $y = y_\Omega \in H_0^1(\Omega)^d$ is one of the weak solutions of the stationary Navier-Stokes equation (2.7). The set $\Lambda$ will be made precise below.

The domain $\Omega$ is lipschitzian and the following constraint is imposed on it

$$E \subset \Omega \subset D \subset R^d \tag{2.2}$$

where $E$ and $D$ are given bounded lipschitzian domains. The set $\Lambda$ in (2.1) is either $\Omega$ or $E$. We denote by $\mathcal{O}$ the family of all admissible domains, defined by (2.2) and by some uniform Lipschitz condition on the boundary $\partial\Omega$, for any $\Omega \in \mathcal{O}$.

The generality of the cost functional (2.1) covers examples of velocity tracking type: if $y_0 \in H_0^1(D)^d$ is given,

$$J(\Omega) = \int_E ||y - y_0||_e^2 dx + \int_E ||\nabla y - \nabla y_0||_e^2 dx \tag{2.3}$$

or vorticity minimization ($d = 3$)

$$J(\Omega) = \int_E ||\nabla \times y||_e^2 dx \tag{2.4}$$

and other examples, as in Borrvall and Peterson (2003) and Posta and Roubícek (2007). Here $|| \, ||_e$ denotes the euclidean norm, $\nabla \times y$ is the rotor of $y$ and $u \cdot v$ is the scalar product in $\mathbf{R}^d$.

Recall from Temam (1979) the definition of the following spaces

$$\mathcal{V}(\Omega) = \{y \in \mathcal{D}(\Omega)^d; \ \operatorname{div} y = 0\} \tag{2.5}$$

$$V(\Omega) = \text{closure of } \mathcal{V}(\Omega) \text{ in } H_0^1(\Omega)^d. \tag{2.6}$$

Remark 1 *$V(\Omega) = \{y \in H_0^1(\Omega)^d;\ div\, y = 0\}$ as $\partial\Omega$ is assumed lipschitzian, for any $\Omega \in \mathcal{O}$. For $y \in V(\Omega)$, denote by $\tilde{y}$ its extension by 0 to $D$. It follows that $\tilde{y} \in V(D)$. Conversely, if $\tilde{z} \in V(D)$ and $\tilde{z} = 0$ a.e. in $D \setminus \Omega$, then $z = \tilde{z}|_\Omega \in V(\Omega)$. If $\partial\Omega$ is not lipschitzian (for instance if $\partial\Omega$ has just the segment property), the above properties may be not true as Lions' lemma (Proposition 1.2 ii) in Temam, 1979, Ch. I) fails according to the counterexample in Geymonat and Gilardi (1998). Notice that for $d = 2, 3$; by using Hodge theory, Wang and Yang (2008) have, however, extended the characterization of $V(\Omega)$ to domains with the segment property.*

For the sake of simplicity, we shall assume in the sequel that $d \leq 4$ (otherwise slightly more complicated spaces than $V(\Omega)$ have to be used).

The weak formulation of the stationary Navier-Stokes equation with Dirichlet (no-slip) boundary conditions is

$$\int_\Omega \left( \nu \sum_{i,j=1}^d \frac{\partial y_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} + \sum_{i,j=1}^d y_i \frac{\partial y_j}{\partial x_i} v_j \right) dx = \int_\Omega \sum_{j=1}^d f_j v_j dx, \ \forall v \in V(\Omega), \tag{2.7}$$

where $f = (f_1, \ldots, f_d) \in H^{-1}(D)^d$, $\nu > 0$ ($\nu$ is the viscosity). We shall use the following standard notations:

$$((u,v))_\Omega = \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}, \quad ||u||^2_\Omega = ((u,u))_\Omega \tag{2.8}$$

$$b_\Omega(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \tag{2.9}$$

for any $u, v, w \in H^1_0(\Omega)^d$. By Theorem 1.2 in Temam (1979), the equation (2.7) has at least one solution $y \in V(\Omega)$, not necessarily unique.

The following properties of $b_\Omega$ are proved in Temam (1979), Ch. II, §1, Lemma 1.3

$$b(u,v,v) = 0 \quad \forall u \in V, \quad v \in H^1_0(\Omega)^d \tag{2.10}$$

$$b(u,v,w) = -b(u,w,v) \quad \forall u \in V, \quad v, w \in H^1_0(\Omega)^d. \tag{2.11}$$

Remark 2 *If $d > 4$, the supplementary condition $y \in L^d(\Omega)^d$ included in the definition of $V(\Omega)$ makes the existence result for (2.7) remain valid.*

The shape optimization problem to be studied in the sequel, denoted by (P), consists in the minimization of the cost functional (2.1), subject to (2.2), (2.7) and to $\Omega \in \mathcal{O}$. The solution of (2.7) may be not unique. If the Lipschitz assumption is valid for $\mathcal{O}$ with a uniform constant, then this family of sets is compact with respect to the Hausdorff-Pompeiu complementary metric (see Theorem A3.9 in Neittaanmäki, Sprekels and Tiba, 2006, p. 466, for a general compactness result).

Theorem 1 *If $\mathcal{O}$ is compact and there is $\hat{\Omega} \in \mathcal{O}$ such that together with some corresponding solution of (2.7), denoted $\hat{y} \in V(\hat{\Omega})$, it satisfies:*

$$\int_{\hat{\Lambda}} j(x, \hat{y}(x), \nabla \hat{y}(x)) dx < +\infty, \tag{2.12}$$

*then problem (P) has at least one optimal pair $[\Omega^*, y^*] \in \mathcal{O} \times V(\Omega^*)$.*

Here $\hat{\Lambda}$ is either $E$ or $\hat{\Omega}$.

*Proof.* Since the family of admissible domains $\mathcal{O}$ is nonvoid and (2.7) has at least one solution $y = y_\Omega \in V(\Omega)$ for any $\Omega \in \mathcal{O}$, by (2.12) (P) has a minimizing sequence denoted by $[\Omega_n, y_n] \in \mathcal{O} \times V(\Omega_n)$, $n \geq 1$, that is:

$$\lim_{n\to\infty} J(\Omega_n) = \inf(P) < +\infty.$$

By the boundedness and uniform Lipschitz assumption on $\mathcal{O}$, we may assume that $\Omega_n \to \Omega^* \in \mathcal{O}$, on a subsequence, in the Hausdorff-Pompeiu complementary metric and $\chi_{\Omega_n} \to \chi_{\Omega^*}$ a.e. in $D$ (convergence of the associated characteristic functions - see Theorem A3.9 in Neittaanmäki, Sprekels and Tiba, 2006). By Lemma 1.3, Ch. II in Temam (1979), we have

$$\begin{aligned} &\int_{\Omega_n} \sum_{i,j=1}^{d} (y_n)_i \frac{\partial (y_n)_j}{\partial x_i} (y_n)_j dx = \frac{1}{2} \int_{\Omega_n} \sum_{i,j=1}^{d} (y_n)_i \frac{\partial}{\partial x_i} (y_n)_j^2 dx = \\ &= -\frac{1}{2} \sum_{i,j=1}^{d} \int_{\Omega_n} \frac{\partial (y_n)_i}{\partial x_i} (y_n)_j^2 dx = -\frac{1}{2} \sum_{j=1}^{d} \int_{\Omega_n} (y_n)_j^2 \operatorname{div} y_n dx = 0. \end{aligned} \tag{2.13}$$

By (2.7) and (2.13), we obtain

$$\begin{aligned} &\nu ||y_n||_{\Omega_n}^2 = \int_{\Omega} \sum_{j=1}^{d} f_j \cdot (y_n)_j dx \le ||f||_{H^{-1}(\Omega_n)^d} ||y_n||_{\Omega_n} \le \\ &\le ||f||_{H^{-1}(D)^d} ||y_n||_{\Omega_n} \end{aligned} \tag{2.14}$$

where $f \in H^{-1}(D)^d$ is given. Relation (2.14) shows that $\{||y_n||_{\Omega_n}\}$ is bounded for $n \ge 1$. Let $\tilde{y}_n$ be the extension of $y_n$ by 0 to $D$. Then $\{\tilde{y}_n\}$ is bounded in $H_0^1(D)^d$ and, passing to a subsequence, one may assume $\tilde{y}_n \to \tilde{y}$ weakly in $H_0^1(D)^d$. Take any open set $Q$ such that $\bar{Q} \subset D \setminus \bar{\Omega}^*$. By Proposition A3.8 in Neittaanmäki, Sprekels and Tiba (2006), there exists $n_Q$ such that $\bar{Q} \subset D \setminus \bar{\Omega}_n$ for $n \ge n_Q$. Therefore, $\tilde{y}_n|_Q = 0$ for $n \ge n_Q$, that is: $\tilde{y}|_Q = 0$ a.e. It follows that $\tilde{y} = 0$ a.e. in $D \setminus \Omega^*$ as $Q$ was arbitrary. The trace theorem ($\Omega^*$ is lipschitzian) shows that $\tilde{y}|_{\Omega^*} \in H_0^1(\Omega^*)^d$. Passing to the limit in distributions it also follows that $\operatorname{div} \tilde{y} = 0$ in $D$, thus in $\Omega^*$. Then, $\tilde{y}|_{\Omega^*} \in V(\Omega^*)$ by the Remark 1, after (2.6).

Consider any $\varphi \in \mathcal{V}(\Omega^*)$ and extend it by 0 to $D$ under the notation $\tilde{\varphi}$. Then $\tilde{\varphi}|_{\Omega_n} \in \mathcal{V}(\Omega_n)$ as $\operatorname{supp} \varphi \subset \Omega_n$ for $n \ge n_\varphi$ by Proposition A3.8 in Neittaanmäki, Sprekels and Tiba (2006). We can use $\tilde{\varphi}|_{\Omega_n}$ as the test function in (2.7) for $n \ge n_\varphi$

$$\begin{aligned} &\nu((y_n, \tilde{\varphi}))_{\Omega_n} + b_{\Omega_n}(y_n, y_n, \tilde{\varphi}|_{\Omega_n}) = \\ &= \nu((y_n, \varphi))_{\operatorname{supp} \varphi} + b_{\operatorname{supp} \varphi}(y_n, y_n, \varphi) = \int_{\operatorname{supp} \varphi} f \cdot \varphi dx. \end{aligned} \tag{2.15}$$

One can pass to the limit in the last equality in (2.15) and obtain

$$\nu((\tilde{y}, \varphi))_{\Omega^*} + b_{\Omega^*}(\tilde{y}, \tilde{y}, \varphi) = \int_{\Omega^*} f \cdot \varphi dx. \tag{2.16}$$

Since $\varphi$ is arbitrary in $\mathcal{V}(\Omega^*)$, a density argument applied in (2.16) for the test functions shows that $\tilde{y} \in V(\Omega^*)$ is a solution of (2.7).

The convergence properties of the minimizing sequence $\{\Omega_n, y_n\}$ and the semicontinuity theorem A3.15, in Neittaanmäki, Sprekels and Tiba (2006), p. 472, shows that

$$0 \leq J(\Omega^*) \leq \liminf_{n\to\infty} J(\Omega_n) = \inf(P). \tag{2.17}$$

Relation (2.16) gives that the pair $[\Omega^*, \tilde{y}] \in \mathcal{O} \times V(\Omega^*)$ is admissible for (P) and indeed an optimal pair by (2.17), which we redenote by $[\Omega^*, y^*]$. ■

Remark 3 *One may remove the condition $E \subset \Omega$ in (2.2) and/or choose $\Lambda = \Omega$ in (2.1). Moreover, state constraints (on y) may be added and the above result remains true if the family of admissible pairs $[\hat{\Omega}, \hat{y}]$ is nonvoid.*

Remark 4 *Theorem 1 should be understood in the sense of singular control/design problems (Neittaanmäki, Sprekels and Tiba, 2006, §3.1.3.1, or Lions, 1983): although the state system (2.7) is ill-posed (nonuniqueness), the optimization problem (2.1), (2.2), (2.7) is well defined. Its solution is, generally, nonunique due to the nonconvex character of optimal design problems.*

## 3. Approximating extensions

The results of the previous section will be used in what follows for a specific family of admissible domains. These will be indexed by a subspace of the space of continuous functions on some fixed domain. From now on, $D$ will be a bounded open lipschitzian domain in $\mathbf{R}^3 (d = 3)$, but all the arguments below extend with small modifications for all $d$ if the definition of $V(\Omega)$ is appropriately adapted, as mentioned in Section 2. Let $X(D) \subset C(\bar{D})$ be a functional space on $D$. For $g \in X(D)$ define

$$\Omega = \Omega_g = \text{int}\{x \in D | g(x) \geq 0\}, \tag{3.1}$$

$g$ is called a parametrization of $\Omega_g$ and $\Omega = \Omega_g$ is an admissible domain. When (2.2) is to be satisfied, then we require

$$g(x) \geq 0 \quad \forall x \in E. \tag{3.2}$$

Some simple examples of spaces $X(D)$ are the finite element spaces. Since $X(D) \subset C(\bar{D})$, then $\partial\Omega = \partial\bar{\Omega}$ and $\bar{\Omega} = \{x \in D | g(x) \geq 0\}$ for every $\Omega$ defined by (3.1). As in the previous section, we shall also assume that the family $\mathcal{O}$ of all the admissible domains $\Omega = \Omega_g$ is uniformly Lipschitz.

Let $H$ denote the Heaviside function, $H : \mathbf{R} \to \{0, 1\}$,

$$H(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r < 0. \end{cases} \tag{3.3}$$

It is obvious that $H \circ g = \chi_{\bar{\Omega}_g}$, the characteristic function of $\bar{\Omega}_g$. Some solution of (2.7) in $\Omega_g$ will be denoted by $y_g$.

As in Neittaanmäki, Pennanen and Tiba (2009) define, for $\varepsilon > 0$, a smoothing of the Yosida approximation $H_\varepsilon$ of the maximal monotone extension of (3.3), to be denoted $H^\varepsilon$. For instance

$$H^\varepsilon(r) = \begin{cases} 1, & r \geq 0 \\ \dfrac{(\varepsilon - 2r)(r+\varepsilon)^2}{\varepsilon^3} & -\varepsilon < r < 0 \\ 0, & r \leq -\varepsilon. \end{cases} \tag{3.4}$$

Remark that $H^\varepsilon \in C^1(\mathbf{R})$ and is lipschitzian. This type of domain $\Omega_g$ regularization $H^\varepsilon(g)$ via characteristic functions was introduced in Mäkinen, Neittaanmäki and Tiba (1992).

Consider the approximating extension of the boundary value problem (2.7) from $\Omega = \Omega_g$ to $D$, given by

$$\begin{aligned} \nu((y_\varepsilon, v))_D + b_D(y_\varepsilon, y_\varepsilon, v) + \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)] y_\varepsilon \cdot v dx = \\ = \int_D f \cdot v, \quad y_\varepsilon, v \in V(D). \end{aligned} \tag{3.5}$$

In the sequel, we denote $V = V(D)$ and $V^*$ is the dual of $V$ with norms $||.||, ||.||_*$, respectively. They are generated as in (2.8). The following estimation will be essential when uniqueness properties are investigated:

PROPOSITION 1 *There exists $c_1 > 0$ such that*

$$|b_D(u, v, w)| \leq c_1 ||u|| \, ||v|| \, ||w|| \quad \forall u, v, w \in H_0^1(D)^3. \tag{3.6}$$

*Proof.* Denote $b_D$ by $b$. From Hölder inequality it follows that

$$\left| \int_D u_i \frac{\partial v_j}{\partial x_i} w_j \right| \leq ||u_i||_{L^6(D)} \left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^2(D)} ||w_j||_{L^3(D)}. \tag{3.7}$$

From Sobolev inequalities (Galdi, 1998, p. 31, relation (2.5)) for $i \in \{1, 2, 3\}$, $n = 3$

$$||u_i||_{L^6(D)} \leq \tfrac{2}{3^{1/2}} ||u_i||_{H_0^1(D)}. \tag{3.8}$$

In the same vein, if $m(D)$ is the Lebesgue measure of $D$, Hölder inequality gives

$$||w_j||^3_{L^3(D)} = \int_D |w_j|^3 dx \leq \left( \int_D |w_j|^6 \right)^{1/2} m(D)^{1/2}.$$

Then, by (3.8),

$$||w_j||_{L^3(D)} \leq ||w_j||_{L^6(D)} m(D)^{1/6} \leq \frac{2}{3^{1/2}} m(D)^{1/6} ||w_j||_{H_0^1(D)}. \tag{3.9}$$

Relations (3.7), (3.8) and (3.9) imply

$$\sum_{i,j=1}^{3}\left|\int_D u_i\frac{\partial v_j}{\partial x_i}w_j\right|\le\frac{4}{3}\sum_{i,j=1}^{3}m(D)^{1/6}||u_i||_{H_0^1(D)}\left\|\frac{\partial v_j}{\partial x_i}\right\|_{L^2(D)}||w_j||_{H_0^1(D)}.$$

Then, from (2.9), it follows that (3.6) is proved with

$$c_1=9m(D)^{1/6}. \tag{3.10}$$

The constant $c_1$ in (3.10) is related to outstanding properties of the solutions of (3.5).

One of the properties of the solution of (3.5) involving $c_1$ is uniqueness. ∎

Theorem 2 *If*

$$\nu^2>c_1||f||_{V^*} \tag{3.11}$$

*with $c_1$ given in (3.6), (3.10), the solution of equation (3.5) is unique and depends continuously on $g$ from $(C(D),||\ ||_\infty)$ to $L^2(D)^3$.*

*Proof.* Denote $b_D$ by $b$ and suppose that $y^1$ and $y^2$ are two solutions of (3.5). Define $y=y^1-y^2$. Subtracting equations (3.5) for $y^1$ and $y^2$ one gets, with notation $((\ ,\ ))=((\ ,\ ))_D$,

$$\begin{aligned}&\nu((y,v))+b(y^1,y,v)+b(y,y^2,v)+\\&+\frac{1}{\varepsilon}\int_D[1-H^\varepsilon(g)]y\cdot v dx=0\quad\forall v\in V.\end{aligned} \tag{3.12}$$

With $v=y$, using (2.11), (2.12) and (3.6), (3.12) becomes

$$\nu||y||^2+\frac{1}{\varepsilon}\int_D[1-H^\varepsilon(g)]||y||_e^2dx=b(y,y^2,y)=b(y,y,y^2)\le c_1||y||^2||y^2||.$$

Using (2.14) for $y^2$ it follows that $||y||^2\left(\nu-\frac{c_1}{\nu}||f||_{V^*}\right)\le 0$, thus hypothesis (3.11) implies $y=0$ so $y^1=y^2$ and uniqueness is proved.

Suppose now that $g_n\to g$ uniformly on $D$. Denote by $y_n=y(g_n)$ the unique solution of (3.5). From (3.11) and (2.14) it follows that $||y_n||\le M$ $\forall n$, so, passing to a subsequence, one can suppose $y_n\to y$ weakly in $H_0^1(D)^3$, strongly in $L^2(D)^3$. By passing to the limit in (3.5) it follows that $y$ is a solution of (3.5) relative to $g$, and from uniqueness, $y=y(g)$. Since every subsequence has the same limit, we conclude that $y(g_n)\to y(g)$ in $L^2(D)^3$. ∎

Remark 5 *Under hypothesis (3.11), we get immediately*

$$\nu^2\ge 9m(\Omega_g)^{1/6}||f||_{V(\Omega_g)^*}$$

*for any $\Omega_g\in\mathcal{O}$. Then, Theorem 1.3 in Temam (1979), p. 167 also gives the uniqueness of the solution $y_g\in V(\Omega_g)$ of (2.7).*

We denote by $y_\varepsilon \in V$ the unique solution of (3.5).

THEOREM 3 *Suppose that (3.11) holds for $f$ in (2.7). If $\Omega = \Omega_g$ is as in (3.1), then there exists a sequence $\varepsilon_n \to 0$ such that $y_{\varepsilon_n}|_{\Omega_g} \to y_g$ weakly in $H^1(\Omega_g)^3$ and strongly in $L^2(\Omega_g)^3$.*

*Proof.* As before, we consider $v = y_\varepsilon$ in (3.5). Then

$$\nu||y_\varepsilon||^2 + \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)]||y_\varepsilon||_e^2 dx = \int_D f \cdot y_\varepsilon dx. \tag{3.13}$$

Relation (3.13) implies that $\{y_\varepsilon | \varepsilon \in (0, \varepsilon_0]\}$ is bounded in $H_0^1(D)^3$. From the Sobolev theorem (the inclusion $H_0^1 \subset L^2$ is compact) it follows that there exists a sequence $\varepsilon_n \to 0$ such that $y_{\varepsilon_n} \to \bar{y}$ in $L^2(D)^3$. Moreover, (3.13) yields

$$\lim_{n\to\infty} \int_D [1 - H^{\varepsilon_n}(g)]||y_{\varepsilon_n}||_e^2 dx = 0. \tag{3.14}$$

Since $g(x) < 0$ on $D - \bar{\Omega}_g$ and $g$ is continuous, for every compact set $K \subset D - \bar{\Omega}_g$ there exists a constant $c_K > 0$ such that $g(x) \le -c_K$, $\forall x \in K$. If $\varepsilon > 0$ is small enough, $H^\varepsilon(g) \equiv 0$ on $K$, thus, by (3.14), $y_{\varepsilon_n} \to 0$ in $L^2(K)^3$, so $\bar{y} = 0$ a.e. in $D - \bar{\Omega}_g$. From the trace theorem for Lipschitz domains, $\bar{y}|_{\Omega_g} \in H_0^1(\Omega_g)^3$. By passing to the limit in distributions we also get that the restriction to $\Omega_g$ of $\bar{y}_g$ is in $V(\Omega_g)$. Since $H^\varepsilon(g) = 1$ in $\Omega_g$, relation (3.5) with $v \in \mathcal{D}(\Omega_g)^3$, gives

$$\nu((y_{\varepsilon_n}, v)) + b(y_{\varepsilon_n}, y_{\varepsilon_n}, v) = \int_{\Omega_g} f \cdot v dx. \tag{3.15}$$

As $n \to \infty$ in (3.15) it follows that $\bar{y}|_{\Omega_g} \in V(\Omega_g)$ satisfies (2.7) and since (3.11) implies also the uniqueness of the solution of (2.7), $\bar{y}|_{\Omega_g} = y_g$, and the theorem is proved. ∎

Theorem 3 allows for the approximation of the shape optimization problem (2.1), (2.7) by the optimal control problem (2.1), (3.5). The function $g \in X(D)$ is the control parameter. If (2.2) is imposed to the unknown domain $\Omega_g$, (3.2) should be added to (2.1), (3.5).

## 4. The directional derivative

In order to develop a gradient type algorithm for solving the optimal control problem associated to (2.1), (3.5), an important step is to compute the directional derivative of the mapping $g \mapsto J[y_\varepsilon(g)]$ in the direction $w \in X(D)$. We start by proving a result on the directional derivative of the mapping $g \mapsto y_\varepsilon(g)$ in the direction $w \in X(D)$.

PROPOSITION 2 *The mapping $g \mapsto y_\varepsilon(g) y_\varepsilon$, the unique solution of (3.5) under condition (3.11), is Gâteaux differentiable between $X(D)$ and $V(D)$ and the* *derivative in direction $w \in X(D)$, denoted by $z = (z_1, z_2, z_3) \in V(D)$, satisfies the equation in variations*

$$\begin{aligned} &\int_D \left( \nu \sum_{i,j=1}^{3} \frac{\partial z_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} + \sum_{i,j=1}^{3} y_{\varepsilon_i} \frac{\partial z_j}{\partial x_i} v_j + \sum_{i,j=1}^{3} z_i \frac{\partial y_{\varepsilon_j}}{\partial x_i} v_j \right) dx + \\ &\quad + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g)] z \cdot v dx = \frac{1}{\varepsilon} \int_D ((H^\varepsilon)'(g) w) y_\varepsilon \cdot v dx \end{aligned} \tag{4.1}$$

*with $v \in V(D)$ arbitrary.*

*Proof.* Denote by $y_\varepsilon^\lambda = y_\varepsilon(g + \lambda w)$ the unique solution of (3.5) corresponding to $g + \lambda w$. So

$$\nu((y_\varepsilon^\lambda, v)) + b(y_\varepsilon^\lambda, y_\varepsilon^\lambda, v) + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g + \lambda w)] y_\varepsilon^\lambda \cdot v dx = \int_D f \cdot v dx$$

and

$$\nu((y_\varepsilon, v)) + b(y_\varepsilon, y_\varepsilon, v) + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g)] y_\varepsilon \cdot v dx = \int_D f \cdot v dx.$$

Subtracting the two equations and dividing by $\lambda \neq 0$ lead to

$$\begin{aligned} &\nu \left( \left( \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, v \right) \right) + b \left( \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, y_\varepsilon^\lambda, v \right) + b \left( y_\varepsilon, \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, v \right) + \\ &\quad + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g + \lambda w)] \left( \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda} \right) \cdot v dx = \\ &\quad = \frac{1}{\varepsilon} \int_D \frac{H^\varepsilon(g + \lambda w) - H^\varepsilon(g)}{\lambda} y_\varepsilon \cdot v dx. \end{aligned} \tag{4.2}$$

$H^\varepsilon \in C^1(\mathbf{R})$ implies

$$\lim_{\lambda \to 0} \frac{H^\varepsilon(g + \lambda w) - H^\varepsilon(g)}{\lambda} = (H^\varepsilon)'(g) w \tag{4.3}$$

uniformly in $\bar{D}$ since $H^\varepsilon(\cdot)$ is lipschitzian and $w \in C(\bar{D})$. Moreover, $\left\{ \frac{H^\varepsilon(g + \lambda w) - H^\varepsilon(g)}{\lambda} | \lambda > 0 \right\}$ is bounded in $L^\infty(D)$ by a constant independent of $\lambda > 0$. With $v = \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}$ in (4.2) it follows, using also (2.10) and (2.11), that

$$\begin{aligned} &\nu \left\| \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda} \right\|^2 + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g + \lambda w)] \left\| \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda} \right\|_e^2 dx = \\ &= \frac{1}{\varepsilon} \int_D \frac{H^\varepsilon(g + \lambda w) - H^\varepsilon(g)}{\lambda} \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda} + b \left( \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, y_\varepsilon^\lambda \right). \end{aligned} \tag{4.4}$$

Since, by (3.6), $b\left(\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}, y_\varepsilon^\lambda\right) \leq c_1 \left\|\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}\right\|^2 ||y_\varepsilon^\lambda||$, (4.4) implies that

$$\begin{aligned}&(\nu - c_1||y_\varepsilon^\lambda||)\left\|\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}\right\|^2 + \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g+\lambda w)]\left\|\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}\right\|_e^2 \leq \\ &\leq \frac{1}{\varepsilon}\int_D \frac{H^\varepsilon(g+\lambda w) - H^\varepsilon(g)}{\lambda}\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}.\end{aligned} \tag{4.5}$$

Since, from (2.14) and (3.11), $\nu > c_1||y_\varepsilon^\lambda||$ (see (4.7) below) it follows from (4.5),(4.3) that $\left\{\frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda}\right\}_{\lambda\in(0,\lambda_0]}$ is bounded in $H_0^1(D)$.

Let $z \in V(D)$ be a weak limit point, for $\lambda_k \to 0$,

$$z = w - \lim_{k\to 0}\frac{y_\varepsilon^{\lambda_k} - y_\varepsilon}{\lambda_k}. \tag{4.6}$$

From (4.2) and (4.3) it follows that $z$ verifies (4.1).

Use now the following Lemma that will be proved in the sequel:

LEMMA 1 *Under condition (3.11) equation (4.1) has a unique solution.*

Uniqueness in (4.1) implies that the convergence in (4.6) is valid without taking subsequences and the solution $z$ depends linearly on $w$ and by (4.2) $z$ is indeed $\frac{\partial y_\varepsilon}{\partial w}(g)$, the derivative of $y_\varepsilon$ in $g$ in the direction $w$.

The Proposition is proved. ∎

*Proof of Lemma 1.* Hypothesis (3.11) and (2.14) yield

$$||y_\varepsilon|| < ||f||_{H^{-1}(D)^d}\nu^{-1} < \frac{\nu^2}{c_1}\nu^{-1} = \frac{\nu}{c_1}. \tag{4.7}$$

Suppose $z^1$ and $z^2$ are solutions of (4.1) and take $z = z^1 - z^2$. Then, for every $v \in V$,

$$\nu((z,v)) + b(y_\varepsilon, z, v) + b(z, y_\varepsilon, v) + \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)]z\cdot v dx = 0. \tag{4.8}$$

With $v = z$ in (4.8) it follows from (2.10), (2.11) and (3.6) that

$$\nu||z||^2 + \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)]||z||_e^2 dx = b(z, z, y_\varepsilon) \leq c_1||z||^2||y_\varepsilon||$$

so $||z||^2(\nu - c_1||y_\varepsilon||) \leq 0$ and then (4.7) implies $z = 0$, thus $z^1 = z^2$. ∎

Consider now the cost functional

$$J(y) = \frac{1}{2} \int_E ||y - y_0||_e^2 dx \tag{4.9}$$

with $y_0 \in L^2(E)^3$, $E \subset \Omega \subset D$ for any $\Omega \in \mathcal{O}$. Define the adjoint (or co-state) equation of (3.5), (4.9) (see Dede, 2007; Gunzburger, 2000; Posta and Roubícek, 2007) through

$$\begin{aligned} &\int_D \left( \nu \sum_{i,j=1}^{3} \frac{\partial p_{\varepsilon j}}{\partial x_i} \frac{\partial v_j}{\partial x_i} - \sum_{i,j=1}^{3} y_{\varepsilon i} \frac{\partial p_{\varepsilon j}}{\partial x_i} v_j + \sum_{i,j=1}^{3} \frac{\partial y_{\varepsilon j}}{\partial x_i} p_{\varepsilon j} v_i \right) dx + \\ &+ \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g)] p_\varepsilon \cdot v dx = \int_D (y_\varepsilon - y_0) \cdot v dx \end{aligned} \tag{4.10}$$

for any $v \in V$. The classical formulation in vectorial form of (4.10) is

$$-\nu \Delta p_\varepsilon - \sum_{i=1}^{3} y_{\varepsilon_i} \frac{\partial p_\varepsilon}{\partial x_i} + \sum_{i=1}^{3} p_{\varepsilon_i} \nabla y_{\varepsilon_i} + \frac{1}{\varepsilon}(1 - H^\varepsilon(g)) p_\varepsilon + \nabla q = y_\varepsilon - y_0$$

with some $q \in L^1_{loc}(D)$ and with $div p_\varepsilon = 0, p_\varepsilon = 0$ on $\partial D$.

THEOREM 4 *Let condition (3.11) be satisfied. Then (4.10) has a unique solution* $p_\varepsilon \in V$.

*Proof.* Rewrite (4.10) as

$$\begin{aligned} &a_{y_\varepsilon}(p, v) := \nu((p, v)) - b(y_\varepsilon, p, v) + b(p, y_\varepsilon, v) + \\ &+ \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g)] p_\varepsilon \cdot v dx = \int_D (y_\varepsilon - y_0) \cdot v dx; \end{aligned}$$

$a_{y_\varepsilon}$ is bilinear and bounded in $V$ and by (3.6), (2.10), (2.11) (recall $y_\varepsilon \in V$) we have

$$\begin{aligned} &a_{y_\varepsilon}(p, p) = \nu ||p||^2 - b(y_\varepsilon, p, p) - b(p, p, y_\varepsilon) + \frac{1}{\varepsilon} \int_D [1 - H^\varepsilon(g)] ||p_\varepsilon||_e^2 dx \geq \\ &\nu ||p||^2 - b(p, p, y_\varepsilon) \geq (\nu - c_1 ||y_\varepsilon||) ||p||^2 \end{aligned}$$

and (4.7) implies that $a_{y_\varepsilon}$ is coercive in $V$. A standard application of Lax-Milgram theorem gives now existence and uniqueness for the solution of (4.10). ■

THEOREM 5 *The directional derivative in $g \in X(D)$ of $J[y_\varepsilon(g)]$ in the direction $w \in X(D)$ is given by*

$$\frac{\partial J}{\partial w}[y_\varepsilon(g)] w = \frac{1}{\varepsilon} \int_D ((H^\varepsilon)'(g) w) y_\varepsilon \cdot p_\varepsilon dx, \tag{4.11}$$

*where $p_\varepsilon \in V$ satisfies (4.10).*

*Proof.* Consider, as before, $y_\varepsilon^\lambda = y_\varepsilon(g+\lambda w)$, $\lambda > 0$, and compute the limit for $\lambda \to 0$ of

$$\frac{J(y_\varepsilon^\lambda) - J(y_\varepsilon)}{\lambda} = \int_E \frac{y_\varepsilon^\lambda - y_\varepsilon}{\lambda} \cdot \frac{y_\varepsilon^\lambda + y_\varepsilon - 2y_0}{2} dx.$$

By (3.6), for $\lambda_k \to 0$, $\dfrac{y_\varepsilon^{\lambda_k} - y_\varepsilon}{\lambda} \xrightarrow{w} z$. Recalling the continuity of $y_\varepsilon$ with respect to $g$ it follows that, for $v = z$ in (4.10) and $v = p_\varepsilon$ in (4.1)

$$\lim_k \frac{J(y_\varepsilon^{\lambda_k}) - J(y_\varepsilon)}{\lambda_k} = \int_E z(y_\varepsilon - y_0)dx =$$

$$= \int_D \left( \nu \sum_{i,j=1}^3 \frac{\partial p_{\varepsilon j}}{\partial x_i}\frac{\partial z_j}{\partial x_i} - \sum_{i,j=1}^3 y_{\varepsilon i}\frac{\partial p_{\varepsilon j}}{\partial x_i} z_j + \sum_{i,j=1}^3 \frac{\partial y_{\varepsilon j}}{\partial x_i} p_{\varepsilon j} z_i \right) dx +$$

$$+ \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)] p_\varepsilon \cdot z dx \overset{(4.1)}{=} \frac{1}{\varepsilon}\int_D ((H^\varepsilon)'(g)w) y_\varepsilon \cdot p_\varepsilon dx.$$

Remark 6 *It follows from (4.11) that the steepest descent direction is*

$$w_d = -\frac{1}{\varepsilon}(H^\varepsilon)'(g)(y_\varepsilon \cdot p_\varepsilon). \tag{4.12}$$

*Since* $\frac{1}{\varepsilon}(H^\varepsilon)'(g)$ *is positive, another descent direction is*

$$w_d = -(y_\varepsilon \cdot p_\varepsilon).$$

Another general class of cost functionals defined on subdomains of $D$, to which the previous reasonings also apply has the form

$$J(y,\Omega) = \frac{1}{2}\int_\Omega j[x, y(x)]dx \tag{4.13}$$

with $j$ a Caratheodory mapping on $\Omega \times \mathbf{R}^3$, with quadratic growth in $y$ such that $j[x, y_\varepsilon(x)]$ is integrable for $y_\varepsilon$ the solution of (3.5). Approximate (4.13) by

$$J_\varepsilon(g) = \int_D H^\varepsilon[g(x)] j[x, y(x)]dx \tag{4.14}$$

and consider the optimal control problem of minimizing (4.14) subject to (3.5). If $g$ is contained in a compact subset of $X(D)$, existence may be proved for the problem (4.14), (3.5) as in Theorem 1. Then (4.1) continues to be the equation in variations. The adjoint (co-state) equation is now

$$\begin{aligned} &\int_D \left( \nu \sum_{i,j=1}^3 \frac{\partial p_{\varepsilon j}}{\partial x_i}\frac{\partial v_j}{\partial x_i} - \sum_{i,j=1}^3 y_{\varepsilon i}\frac{\partial p_{\varepsilon j}}{\partial x_i} v_j + \sum_{i,j=1}^3 \frac{\partial y_{\varepsilon j}}{\partial x_i} p_{\varepsilon j} v_i \right) dx + \\ &+ \frac{1}{\varepsilon}\int_D [1 - H^\varepsilon(g)] p_\varepsilon \cdot v dx = \int_D H^\varepsilon(g) \nabla_y j[x, y_\varepsilon(x)] \cdot v(x) dx \end{aligned} \tag{4.15}$$

supposing $j$ differentiable with respect to $y$ and with the notation

$$\nabla_y j = \left(\frac{\partial j}{\partial y_1}, \frac{\partial j}{\partial y_2}, \frac{\partial j}{\partial y_3}\right).$$

Existence and uniqueness for (4.15) follow as in Theorem 4. Along the same lines as in Theorem 5 one can prove:

Theorem 6 *The directional derivative of the cost functional $J_\varepsilon[y(g)]$ defined by (4.14), in $g$, in the direction $w \in X(D)$ is given as*

$$\int_D (H^\varepsilon)'[g(x)][j(x, y_\varepsilon(x)) + \frac{1}{\varepsilon} y_\varepsilon(x) \cdot p_\varepsilon(x)]w(x)dx \tag{4.16}$$

*with $p_\varepsilon$ being the solution of (4.15).*

Remark 7 *The steepest descent direction of the cost functional (4.14) is*

$$w_d = -(H^\varepsilon)'(g)\left[j(x, y_\varepsilon) + \frac{1}{\varepsilon} y_\varepsilon \cdot p_\varepsilon\right] \tag{4.17}$$

*and another descent direction is given by*

$$w_d = -\left[j(x, y_\varepsilon) + \frac{1}{\varepsilon} y_\varepsilon \cdot p_\varepsilon\right].$$

An algorithm

Consider the Navier-Stokes stationary homogeneous equations for a viscous incompressible fluid defined by (3.5), subject to (3.11) and consider a cost functional of type (4.9) or, eventually, (4.14). The following gradient type algorithm can be used for shape optimization.

Consider a fixed small regularization parameter $\varepsilon$.

Step 1. Start with $n = 0$ and select $g_n = g_0$

Step 2. Compute $y_n$ solution of (3.5)

Step 3. Compute $p_n$ solution of (4.10) (or, eventually, (4.15))

Step 4. Compute $w_n = -\frac{1}{\varepsilon}(H^\varepsilon)'(g_n)(y_n \cdot p_n)$ in (4.12) (or (4.17))

Step 5. Introduce $\tilde{g}_n = g_n - \lambda_n w_n$ with $\lambda_n$ obtained by line search.

Step 6. $g_{n+1} = P_{U(D)}\tilde{g}_n$ with $U(D)$ a set defined by the supplementary constraints imposed on $g$ (if any)

Step 7. If $|g_{n+1} - g_n|$ or $||w_n||$ are less than a preestablished tolerance parameter - STOP

If not, GO TO Step 2 with $n$ replaced by $n + 1$.

**Acknowledgement.** This work was suported by CNCSIS Romania, Grant ID-PCE 1192/09. The authors are grateful to the referees for useful comments and recommendations.

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