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On the theorem of Filippov–Pliś and some applications*

by

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Abstract: In the paper some known and new extensions of the famous theorem of Filippov (1967) and a theorem of Pliś (1965) for differential inclusions are presented. We replace the Lipschitz condition on the set-valued map in the right-hand side by a weaker one-sided Lipschitz (OSL), one-sided Kamke (OSK) or a continuity-like condition (CLC). We prove new Filippov-type theorems for singularly perturbed and evolution inclusions with OSL right-hand sides. In the CLC case we obtain two extended theorems, one of which implies directly the relaxation theorem. We obtain also a theorem in Banach spaces for OSK multifunctions. Some applications to exponential formulae are surveyed.

Keywords: differential inclusions, one-sided, Kamke, Lipschitz, Filippov, Pliś.

1. Introduction

This paper is devoted to some known and new extensions of the celebrated Filippov theorem (1967) (sometimes called also Gronwall-Filippov-Wazewski theorem), and the less known, earlier and not less important theorem of Pliś (1965) for differential inclusions.

These theorems provide approximation and stability properties of the solutions and reachable sets of a differential inclusion, with respect to perturbations in the right-hand side and in the initial state of the inclusion. They play a crucial role in the qualitative and numerical analysis of differential inclusions.

The central assumption of the Filippov theorem is the Lipschitz condition of the right-hand side with respect to the state variable. The classical proof of Filippov uses the method of successive approximations and provides exponential stability estimates not only for the state variables but also for the velocities.

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We prove here some extensions of these theorems for differential inclusions with possibly discontinuous right-hand sides, replacing the Lipschitz condition on the right-hand side with some weaker conditions: one-sided Lipschitz (OSL), one-sided Kamke (OSK) or a continuity-type condition (CLC). Let us note that the class of multifunctions satisfying the one-sided Kamke condition is so large that it is generic, while the subclasses of OSL and the smallest set of Lipschitz functions are of first Baire category (see Donchev, 2005).

Our method of proving the existence does not exploit successive approximations, since they may not converge, as it is shown by an example in Lakshmikantham and Leela (1981) p. 37. We use existence theorems for solutions of differential inclusions with upper (lower) semi-continuous right-hand sides (see e.g. Deimling, 1992, Tolstonogov, 2000) and do not obtain estimates for the velocities, but only for the states. The stability of the velocities may be achieved by supposing continuity of the right-hand side of the differential inclusion as it is done in Donchev, Farkhi and Mordukhovich (2007). A strengthened version of the one-sided Lipschitz condition with better approximation order may be found also in Lempio and Veliov (1998).

The most important applications of this type of theorems lie, for us, in the relaxation theorems, in analysis of convergence and stability of discrete approximations and in semigroup property of the reachable set. Notice that a refined version of the theorem of Pliś, proved here (Theorem 8), implies in a straightforward way the relaxation theorem important in control theory. We remark that the application of relaxation stability to optimization problems is harder than the relaxation in differential inclusions only. One study in this direction and a proof of a Bogolyubov-type theorem may be found in Donchev, Farkhi and Mordukhovich (2007).

We then present some applications of these extensions to the theory and numerics of differential inclusions, like the exponential formula in finite and infinite time interval for the one-sided Lipschitz case. Some of these results are published in Donchev and Farkhi (2000), Donchev, Farkhi and Reich (2003, 2007), Donchev, Farkhi and Wolenski (2003). Other applications of the one-sided Lipschitz conditions may be found in Donchev and Dontchev (2003), Sokolovskaya (2003, 2004), Sokolovskaya and Filatov (2005).

The paper is organized as follows. After we recall some basic notations and theorems of Filippov and Pliś in the next section, we present some extensions for the one-sided Lipschitz maps in Section 3. We cite there our known results on ordinary differential inclusions in $\mathbb{R}^n$, and provide two new theorems in the cases of singularly perturbed and evolution inclusions. The case of continuous-like and one-sided Kamke multifunctions is discussed in Section 4, where some new theorems are proved. Some known and new results, and open problems are presented as applications and conclusions in Section 5.

2. Preliminaries

In the paper we work in the finite dimensional Banach space $E = \mathbb{R}^n$ if it is not pointed out otherwise.

First we give some definitions and notations.

We denote by $\mathbb{B}$ the closed unit ball in E. The set of continuous functions from $I = [0, T]$ to the space E is denoted by $C(I, E)$.

We denote by $\overline{\text{co}}\ A$ the closed convex hull of the set A and by $\overline{\text{ext}}A$ the closed hull of the set of extreme points of A. The sum of sets is defined in the sense of Minkowski: $A + B = \{a + b : \ a \in A, \ b \in B\}$. The support function of a set $A \subset E$ is $\sigma(x, A) = \max\limits_{a \in A} \langle x, a\rangle$. The distance from a point to a set is $\text{dist}(x, A) = \inf\limits_{a \in A}\{|x - a|\}$, and $\text{Proj}(x, A) = \{a \in A : \ |x - a| = d(x, A)\}$ is the projection (or the set of projections) of the point x onto the set A. For the bounded set A we denote $|A| = \sup\{|x| : \ x \in A\}$. The Hausdorff distance between the sets A, B is denoted by $D_H(A, B)$. The continuity of set-valued maps is in the sense of the Hausdorff metric.

The multivalued map F is said to be upper semicontinuous (USC) on A if for every $x \in A$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x + \delta\mathbb{B}) \subset F(x) + \varepsilon\mathbb{B}$; it is called lower semicontinuous (LSC) at x when for every $f_x \in F(x)$ and every $x_n \to x$ there exist $f_n \in F(x_n)$ such that $f_n \to f_x$. $F(\cdot)$ is called *upper hemicontinuous (UHC)* when the support function $\sigma(l, F(\cdot))$ is upper semicontinuous as a real valued function.

The set-valued map $F : I \times E \rightrightarrows E$ is said to be *almost USC (LSC, UHC, continuous)* if for any $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with $\text{meas}(I_\varepsilon) > \text{meas}(I) - \varepsilon$ such that F is USC (LSC, UHC, continuous) on $I_\varepsilon \times E$. Note that in the case of separable E and compact valued functions, the almost continuity is equivalent to the *Caratheodory property*, i.e. the measurability of $F(\cdot, x)$ for every $x \in E$ and continuity of $F(t, \cdot)$ for a.e. $t \in I$ (see Tolstonogov, 2000, Ch. I, sect. 1).

The function $L : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *integrally bounded* on the bounded sets, if for any bounded set $U \subset \mathbb{R}^+$ there is an L_1 function $m(\cdot)$ such that $|L(t, r)| \leq m(t)$ for any $r \in U$. The non-negative almost continuous function $L : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *Kamke function* if it is integrally bounded on the bounded sets, and moreover, $L(t, 0) \equiv 0$, and the differential equation

$$\dot{s} = L(t, s), \ \ s(0) = 0$$

admits only the solution $s(t) \equiv 0$.

The following conditions on a set-valued map, weaker than the Lipschitz continuity, will be used throughout the paper:

Definition 1 *$F(t, x) : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be:*

1) one-sided Lipschitz *(OSL) if there is an integrable function $L : I \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}^n$, $t \in I$ and $v \in F(t, x)$ there exists $w \in F(t, y)$ with*

$$\langle x - y, v - w\rangle \leq L(t)|x - y|^2; \tag{1}$$

2) one-sided Kamke (OSK) if there exists Kamke function $L(\cdot,\cdot)$ such that for every $x, y \in \mathbb{R}^n$, $t \in I$ and $v \in F(t,x)$ there exists $w \in F(t,y)$ with

$$\langle x-y, v-w\rangle \le L(t,|x-y|)|x-y|; \tag{2}$$

3) continuous-like (CLC) when (2) holds with a Caratheodory (not necessarily Kamke) non-negative integrally bounded on the bounded sets, and $L(t,0)\equiv 0$.

Clearly, among the just defined classes of mappings, the smallest one is OSL and the largest is CLC.

Consider the system:

$$\frac{dx}{dt} \in F(t,x),\ x(t_0) = x_0, \tag{3}$$

Denote by $A(t,x_0)$ the *attainable (reachable) set* of the system (3) at time t, i.e. the set of all reachable points for this time.

We now recall the theorem of Filippov in a slightly more general formulation, for a multifunction F measurable w.r. to t (see, e.g., Aubin and Cellina, 1984, Aubin and Frankowska, 1990, Deimling, 1992).

Let $y(\cdot) : [0,T] \to \mathbb{R}^n$ be absolutely continuous (AC). Assume that $F : [0,T]\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has nonempty compact values and is such that:

(i) for all $x \in \mathbb{R}^n$ $F(\cdot,x)$ is measurable;

(ii) there exist $\beta > 0$ and Lebesgue integrable $k(\cdot)$ such that for every $x, z \in y(t) + \beta\mathbb{B}$ and a.a. $t \in [0,T]$ $D_H(F(t,x), F(t,z)) \le k(t)|x-z|$;

(iii) there exists an integrable $f(\cdot)$ such that $\operatorname{dist}(\dot y(t), F(t,y(t))) \le f(t)$.

Theorem 1 *(Filippov) If (i),(ii) and (iii) hold and $\chi(T) \le \beta$, then there exists a solution $x(\cdot)$ of (3) such that:*

$$|x(t)-y(t)| \le \chi(t)\ ,\ t\in[0,T],\ \text{and}\ |\dot{x}(t)-\dot{y}(t)| \le k(t)\chi(t)+f(t),$$
a.e.t $\in [0,T]$
where $\chi(t) = e^{\int_0^t k(s)ds}\left(|x_0 - y(0)| + \int_0^t f(s)ds\right)$.

Note that Theorem 1 estimates not only the distance between the functions $x(\cdot)$ and $y(\cdot)$, but also the distance between their derivatives. The original proof of this theorem (see Filippov, 1967) uses the method of successive approximations. As we see here, this method is applicable also in some cases when $F(\cdot,\cdot)$ is not necessarily compact valued.

For simplicity, we now assume that every solution of

$$\dot y(t) \in F(t,y(t)),\ y(0) = y_0 \tag{4}$$

is extendable on $I = [0,T]$. Here $F : I\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

The following stronger version of the theorem of Pliś is formulated in Blagodatskikh and Filippov (1986) where the original compactness assumption of Pliś is replaced by closedness of the values of F, and the estimate of the velocities difference (missing in the original paper of Pliś), is added. The main contribution of Pliś, in our opinion, is replacing the Lipschitz continuity of the set-valued map by a special uniform continuity condition, proving existence (without uniqueness) by the convexity and the compactness, and also estimating the solutions difference by the maximal solution of a proper differential equation. An estimate of this type for differential equations may be found in the earlier paper of Filippov (1960).

Theorem 2 (Pliś) *Let* $D_H(F(t,x),F(t,y)) \leq L(t,r)$ *for* $|x-y| \leq r$, *with* $L(\cdot,\cdot)$ *an integrally bounded Caratheodory function from* $\mathbb{R}^2$ *to* $\mathbb{R}^+$, *such that* $L(\cdot,0)=0$ *and let* $F(\cdot,x)$ *be measurable, with closed convex values. If* $x(\cdot)$ *is* AC *and satisfies* $\operatorname{dist}(\dot{x}(t),F(t,x(t))) \leq f(t)$, *where* $f(\cdot)$ *is a nonnegative* L_1 *function, then there exists a solution* $y(\cdot)$ *of (4) such that*

$$|x(t)-y(t)| \leq r(t), \quad t \in [0,T], \quad |\dot{x}(t)-\dot{y}(t)| \leq L(t,r(t))+f(t), \text{ a.e. } t \in [0,\mathrm{T}].$$

Here $r(\cdot)$ *is the maximal solution of the differential equation*

$$\dot{r}(t) = L(t,r(t)) + f(t), \; r(0) = |x(0)-y_0|. \tag{5}$$

There is no proof of this theorem in Blagodatskikh and Filippov (1986) but one can adopt the method of proof of Theorem 7 to prove it (see Remark 3), when the values of F are compact, as Pliś has supposed originally.

Note that in Theorem 1 F is Lipschitz and not necessarily convex-valued, while in Theorem 2 F is convex-valued and not necessarily Lipschitz.

In the Lipschitz case the distance between the derivatives is easily estimated, although in general (if $L(t,\cdot)$ is non-linear), it is harder to obtain explicit estimates of the distance between the functions x,y and between their derivatives from (5).

To prove Theorem 2 in general Banach spaces, one needs additional compactness assumptions on F or $L(\cdot,\cdot)$ has to be a Kamke function.

3. Extensions to one-sided Lipschitz mappings

In this section we will present some extensions of the Filippov theorem. We call such extensions Filippov-type theorems.

We relax the Lipschitz continuity of F to the weaker one-sided Lipschitz property. In this case, however, we are not able to estimate the difference between the derivatives of $x(\cdot)$ and $y(\cdot)$ in the Filippov-type theorem. The following one-sided Lipschitz condition formulated in Donchev and Farkhi (1998), combined with the compactness and the upper-semicontinuity of F implies existence and the approximation stability of the solutions (but not of the velocities), even for some inclusions with discontinuous in the state variable right-hand sides.

In terms of the support function σ, the OSL condition is equivalent to (see Donchev and Farkhi, 1998)

$$\sigma(x-y, F(t,x)) - \sigma(x-y, F(t,y)) \leq L(t)|x-y|^2.$$

Examples of OSL multifunctions are given, e.g., in Donchev and Farkhi (1998). Although OSL maps may be discontinuous, we succeed to prove Lipschitz continuity of the solution set with respect to the initial point and the "outer" perturbation of the right-hand side $f(t)$, and Hölder continuity of the solution set with respect to "inner" perturbations of the state variable in the right-hand side of the inclusion (see Theorem 4).

We like to stress also that the OSL constant $L(t)$ may be negative, which is impossible in the classical Lipschitz condition. The negative OSL constant implies weak asymptotic stability (see Donchev and Farkhi, 2000).

To get existence of solutions of the differential inclusion, it is sufficient to require that $F(t,\cdot)$ be only USC instead of continuous.

In Donchev and Farkhi (1998), a generalization of the Filippov theorem was proved under the conditions:

A1. F is defined on $I \times \mathbb{R}^n$, with nonempty compact and convex values, $F(\cdot, x)$ measurable for each x and $F(t,\cdot)$ USC for all t and F is OSL.

A2. (*Linear growth*) There is an integrable function $\lambda(t) > 0$ such that $|F(t,x)| \leq \lambda(t)(1+|x|)$ for all $x \in X$ and almost all $t \in I$.

REMARK 1 *It is not hard to show (see Donchev, Farkhi, 2000) that under A1, A2 the system*

$$\dot{x}(t) \in \overline{\text{co}}\ F(t, x+\mathbb{B}),\ \ x(0) = x_0$$

admits a nonempty compact solution set and there exist constants M and K such that $|x(t)| \leq M$ and $|F(t,x)| \leq K$ for every such a solution.

THEOREM 3 *(Donchev and Farkhi, 1998) Suppose A1 and A2 hold, $I = [0,T]$ is a finite interval, and $y: I \to \mathbb{R}^n$ is an absolutely continuous function satisfying $dist(\dot{y}(t), F(t,y(t))) \leq g(t)$ for a.e. $t \in I$, where g is integrable. Then, for each $x_0 \in R^n$, there exists a solution $x(\cdot)$ of (3) on I such that*

$$|x(t) - y(t)| \leq u(t), \quad t \in I, \tag{6}$$

where $u(\cdot)$ is the solution of the initial value problem

$$\frac{du}{dt} = L(t)u + g(t),\ \ u(t_0) = |x_0 - y_0|. \tag{7}$$

We have in the case $L(t) \leq L = const$ the following estimate of the distance between the attainable sets:

$$D_H(A(t,x_0), A(t,y_0)) \leq e^{Lt}|x_0 - y_0|. \tag{8}$$

If $L < 0$, this leads to the conclusion that the attainable map is contractive for a fixed positive time and the system is weakly asymptotically stable. For non-positive L the system is weakly stable (see Donchev and Farkhi, 2000).

Remark 2 *If F is OSL with a constant L, it has been proved in Donchev, Farkhi (2000) that all solutions of the inclusion*

$$\dot{x}(t) \in F([0,T], x(t) + \mathbb{B}), \quad x(0) \in X_0,$$

and their velocities are globally bounded on $[0,T]$, provided X_0 is bounded. Namely, there are constants M, K such that $|x(t)| \leq M$, $|F(t, x(t))| \leq K$ for any solution $x(t)$ and any $t \in [0,T]$. Thus, the assumption that F is bounded on bounded sets and OSL with a constant L may replace the growth condition A2 in the corresponding existence theorem (see Deimling, 1992, Theorem 5.2).

In the following modification of the Filippov - Pliś theorem (see Donchev and Farkhi, 2000), the given quasitrajectory solves the inclusion where also the state argument in the right-hand side is perturbed. In the Lipschitz case, the estimate also with these perturbation follows from the classical Filippov theorem, while in the OSL case we arrive at a square root in the distance estimate.

Theorem 4 *(Donchev and Farkhi, 2000) Let A1 and A2 hold with a constant L, and let $g(t), h(t)$ be measurable functions such that $|h(t)| \leq 1$, $0 \leq g(t) \leq 1$ for a.e. $t \in I$. If $y(\cdot)$ is a solution of*

$$\dot{y}(t) \in F(t, y(t) + h(t)) + g(t)\mathbb{B}, \qquad y(0) = y_0$$

then there exists a solution $x(\cdot)$ of (4) such that

$$\begin{aligned} |x(t) - y(t)| \quad \leq \quad & e^{Lt}\Big\{|x_0 - y_0| + \int_0^t e^{-Ls} g(s)\, ds \\ & + \Big[\int_0^t e^{-2Ls}|h(s)|(4K + 8|L|M + 2L_+)\, ds\Big]^{\frac{1}{2}}\Big\}, \end{aligned}$$

where $L_+ = \max\{0, L\}$.

Proof. We provide here a sketch of the proof. First we show that there exists a solution $z(\cdot)$ of the inclusion

$$\dot{z}(t) \in F(t, z(t) + h(t)), \quad z(0) = x_0 \tag{9}$$

such that $|z(t) - y(t)| \leq e^{Lt}\big\{|x_0 - y_0| + \int_0^t e^{-Ls} g(s)\, ds\big\}$.

Consider the mapping

$$H(t,u) = \{v \in F(t, u + h(t)) : \langle y(t) - u, \dot{y}(t) - v\rangle \\ \leq |y(t) - u|(L|y(t) - u| + g(t))\}.$$

$H(t,u)$ is nonempty for every $t \in I, u \in X$ and convex-valued, since F is convex-valued. $H(t,\cdot)$ is closed valued USC. $H(\cdot,u)$ is measurable since F is almost continuous. Hence, there exists a solution $z(\cdot)$ of the differential inclusion:

$$\dot{z}(t) \in H(t, z(t)), \quad z(0) = x_0. \tag{10}$$

It follows from the definition of $H(\cdot,\cdot)$ that

$$\langle y(t) - z(t), \dot{y}(t) - \dot{z}(t)\rangle \leq L|y(t) - z(t)|^2 + g(t)|y(t) - z(t)|.$$

Denoting $m(t) = |y(t) - z(t)|$, we obtain for a.e. $t \in I$

$$\dot{m}(t) \leq Lm(t) + g(t), \quad m(0) = |y_0 - x_0|.$$

Hence

$$|z(t) - y(t)| \leq e^{Lt}\big\{|x_0 - y_0| + \int_0^t e^{-Ls} g(s)\, ds\big\}. \tag{11}$$

Defining the mapping

$$G(t,u) = \big\{v \in F(t,u) : \langle z(t) + h(t) - u, \dot{z}(t) - v\rangle \leq L|z(t) + h(t) - u|^2\big\},$$

we obtain from the OSL condition that $G(t,u) \neq \emptyset$. As above, we establish that it is closed and convex valued, $G(\cdot,u)$ is measurable, $G(t,\cdot)$ is USC. Let

$$\dot{x}(t) \in G(t, x(t)), \quad x(0) = x_0. \tag{12}$$

Therefore, one derives by the OSL condition and Remark 2,

$$\langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\rangle \\ \leq L|x(t) - z(t)|^2 + 2|L||h(t)||x(t) - z(t)| + 2K|h(t)| + L|h(t)|^2.$$

Denote $p(t) = |x(t) - z(t)|^2$. The last inequality gives

$$\dot{p}(t) \leq 2Lp(t) + 2(2K + 4M|L|)|h(t)| + 2L|h(t)|^2.$$

This, together with (11), implies the desired estimate. ∎

3.1. Singularly perturbed systems

The singularly perturbed systems with Lipschitz and non-Lipschitz right-hand sides are widely investigated, see e.g. Donchev and Dontchev (2003), Sokolovskaya (2003), Donchev and Slavov (1995), Veliov (1994, 1997). We extend here to the OSL case the well-known results for the Lipschitz case (Veliov, 1994), and also generalize a result of Donchev and Dontchev (2003) for OSL control systems.

Consider the following singularly perturbed multivalued Cauchy problem:

$$\begin{pmatrix} \dot{x} \\ \varepsilon\dot{y} \end{pmatrix} \in F(t, x(t), y(t)), \quad \begin{array}{l} x(0) = x_0 \\ y(0) = y_0. \end{array} \tag{13}$$

Let $F : I \times \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$. Here we denote the closed unit balls in $\mathbb{R}^n$ and in $\mathbb{R}^m$ respectively by $\mathbb{B}_1$ and $\mathbb{B}_2$. We suppose that

S1. $F(\cdot, \cdot, \cdot)$ is almost USC, with nonempty convex compact values, bounded on the bounded sets.

Notice that for every fixed $\varepsilon > 0$ the system (13) is an ordinary differential inclusion. Singular perturbations theory studies the behavior of the solution set $Z(\varepsilon)$ of (13) as $\varepsilon \to 0^+$. For a bibliography we refer to Donchev and Dontchev (2003), Veliov (1997) and the references therein.

In this section will call $F(t, \cdot, \cdot)$ OSL if there exist positive constants A, B, C, μ such that for every $x_1,\ x_2 \in E_1$, every $y_1,\ y_2 \in E_2$ and every $(f_1, g_1) \in F(t, x_1, y_1)$ there exists $(f_2, g_2) \in F(t, x_2, y_2)$ such that

$$\begin{aligned} \langle x_1 - x_2, f_1 - f_2 \rangle &\le A|x_1 - x_2|^2 + B|y_1 - y_2|^2, \\ \langle y_1 - y_2, g_1 - g_2 \rangle &\le C|x_1 - x_2|^2 - \mu|y_1 - y_2|^2. \end{aligned}$$

It is not difficult to see (as, e.g., in Donchev and Slavov, 1995) that if F is OSL and satisfies **S1**, there exist constants M and N such that $|x(t)| + |y(t)| \le M$ and $\|F(t, x(t) + \mathbb{B}_1, y(t) + \mathbb{B}_2)\| \le N$ for every $\varepsilon > 0$ and for every solution $(x(\cdot), y(\cdot))$ of

$$\begin{pmatrix} \dot{x} \\ \varepsilon\dot{y} \end{pmatrix} \in \overline{\text{co}}\ F(t, x(t) + \mathbb{B}_1, y(t) + \mathbb{B}_2), \quad \begin{array}{l} x(0) = x' \in x^0 + \mathbb{B}_1 \\ y(0) = y' \in y^0 + \mathbb{B}_2. \end{array} \tag{14}$$

The following Filippov-type theorem is very applicable in singular perturbation theory (see Veliov, 1994, and Donchev and Dontchev, 2003, where special cases are considered).

THEOREM 5 *Let F be OSL satisfying **S1** and let (u, v) be a solution of the system:*

$$\begin{pmatrix} \dot{u} \\ \varepsilon\dot{v} \end{pmatrix} \in F(t, u(t) + f(t), v(t) + g(t)), \quad \begin{array}{l} u(0) = u^0 \\ v(0) = v^0, \end{array}$$

where $f(\cdot)$, $g(\cdot)$ are bounded and measurable functions.

Then there exist constants K_x, K_y and a solution (x,y) of (13) such that

$$\dot r(t) \le 2\{Ar + Bs + K_x(|f(t)| + |g(t)|)\}, \quad r(0) = |x^0 - u^0|^2$$
$$\varepsilon\dot s(t) \le 2\{Cr - \mu s + K_y(|f(t)| + |g(t)|)\}, \quad s(0) = |y^0 - v^0|^2,$$

where $r(t) = |x(t) - u(t)|^2$ *and* $s(t) = |y(t) - v(t)|^2$,

Proof. Consider the map

$$\begin{aligned}(t,x,y) \to \tilde F(t,x,y) = \mathrm{cl}\{(\alpha,\beta) \in F(t,x,y) :\ &\langle u(t) - x + f(t), \dot u(t) - \alpha\rangle \\ &\le A|u(t) - x + f(t)|^2 + B|v(t) - y + g(t)|^2, \\ \varepsilon\langle v(t) - y + g(t), \dot v(t) - \beta\rangle \\ &\le C|u(t) - x + f(t)|^2 - \mu|v(t) - y + g(t)|^2\}.\end{aligned}$$

The so defined $\tilde F$ is almost USC with nonempty convex compact values. Furthermore, the system

$$\begin{pmatrix}\dot x \\ \varepsilon\dot y\end{pmatrix} \in \tilde F(t,x,y), \quad \begin{matrix} x(0) = x' \\ y(0) = y'.\end{matrix}$$

has a solution $(x(t), y(t))$ such that

$$\langle u(t) - x(t) + f(t), \dot u(t) - \dot x(t)\rangle \le A|u(t) - x(t) + f(t)|^2 + B|v(t) - y(t) + g(t)|^2$$

and

$$\varepsilon\langle v(t) - y(t) + g(t), \dot v(t) - \dot y(t)\rangle \le C|u(t) - x(t) + f(t)|^2 - \mu|v(t) - y(t) + g(t)|^2.$$

Denoting $\mathcal{P} := A|u(t) - x(t) + f(t)|^2 + B|v(t) - y(t) + g(t)|^2$, we get

$$\langle x(t) - u(t), \dot x(t) - \dot u(t)\rangle \le \mathcal{P} + |\dot x(t) - \dot u(t)||f(t))| \le \mathcal{P} + 2N|f(t)|.$$

On the other hand

$$\begin{aligned}&\Big|\ |x(t) - u(t) + f(t)|^2 - |x(t) - u(t)|^2\ \Big| \\ = &\Big|\ |x(t) - u(t) + f(t)| - |x(t) - u(t)|\ \Big| \cdot \Big||x(t) - u(t) + f(t)| + |x(t) - u(t)|\ \Big| \\ &\le |f(t)|(2|x(t)| + 2|u(t)| + |f(t)|) \le |f(t)|(4M + 1).\end{aligned}$$

Hence

$$\langle x - u, \dot x - \dot u\rangle \le A|x-u|^2 + B|y-v|^2 + 2N|f(t)| + (4M+1)A|f(t)| + (4M+1)B|g(t)|.$$

Analogously

$$\varepsilon\langle y - v, \dot y - \dot v\rangle \le C|x-u|^2 - \mu|y-v|^2 + 2N\varepsilon|g(t)| + (4M+1)C|f(t)| + (4M+1)\mu|g(t)|.$$

Since $\langle x - u, \dot x - \dot u\rangle = \frac{1}{2}\frac{d}{dt}x - u|^2$, $\langle y - v, \dot y - \dot v\rangle = \frac{1}{2}\frac{d}{dt}|y - v|^2$, we get the conclusion of the theorem ∎

3.2. Evolution inclusions

In this section we will prove a Filippov-type theorem for evolution inclusions in a Banach space E with uniformly convex dual E^*. Although our paper is devoted mainly to differential inclusions in $\mathbb{R}^n$, we believe that this result is important, since parabolic partial differential inclusions are often represented as evolution inclusions with unbounded operators.

Denote by $J(\cdot)$ the duality mapping. It is well known that in this case $J(\cdot)$ is single-valued and uniformly continuous on the bounded sets (see Deimling, 1992).

The OSL condition here is $\sigma(J(x-y),F(x))-\sigma(J(x-y),F(y)) \le L|x-y|^2$ for every $x,y\in E$.

DEFINITION 2 *The operator $A : D(A) \rightrightarrows E$ is said to be* m-dissipative *if $\langle J(x-y), v_x - v_y\rangle \le 0$ for every $x,y\in D(A)\subset E$ and every $v_x\in Ax$, $v_y\in Ay$, and for every $\lambda>0$, $Range(Id-\lambda A)=E$, where $D(A)$ is the domain of A and Id is the identity operator.*

We refer to Chapter 3 of Lakshmikantham and Leela (1981) for the theory of m-dissipative operators.

Consider the inclusion:

$$\dot{x}(t) \in Ax + F(t,x), \quad x(0) = x_0 \in \overline{D(A)}, \quad t \in I. \tag{15}$$

Solutions of (15) are understood in the following integral sense: $x(\cdot)$ is said to be a solution of (15) if there exists $f(t)\in F(t,x(t))$ such that $x(\cdot)$ is a solution of

$$\dot{x}(t) \in Ax(t) + f(t), \quad x(0) = x_0 \in \overline{D(A)}. \tag{16}$$

The continuous function $x(\cdot) : I \to \overline{D(A)}$ is said to be a solution of (16) if $x(0)=x_0$ and for every $u\in D(A)$, $v\in Au$, $0\le s\le t\le 1$ the following inequality holds true

$$|x(t)-u|^2 \le |x(s)-u|^2 + 2\int_s^t \langle J(x(\tau)-u), f(\tau)-v\rangle d\tau.$$

We refer to Vrabie (1987) for more information related to equation (16).

Suppose:

AA1. A is m-dissipative operator, which is an infinitesimal generator of a compact semigroup $T(\cdot)$.

AF1. $F(\cdot,\cdot)$ is almost UHC with nonempty convex weakly compact values. Further, there exists an integrable $\lambda(\cdot)$ such that $|F(t,x)|\le \lambda(t)(1+|x|)$.

Notice that under the above assumptions for every $K>0,\eta>0$ there exists $M>0$ such that the differential inclusion

$$\dot{y}(t) \in Ay + F(t, y+\eta\mathbb{B}) + K\mathbb{B},\ y(0)=x_0,\ t\in I \tag{17}$$

admits a nonempty $C(I,E)$ compact set of solutions (hence equicontinuous), satisfying $|x(t)| \leq M$ (see, e.g., Bressan and Staicu, 1994).

Let $\Omega_J : [0, 2M+1] \to \mathbb{R}^+$ be the modulus of continuity of the duality map J on $M\mathbb{B}$.

A special case of the following theorem has been proved in Donchev (2007) for a linear operator A.

THEOREM 6 *Assume AA1 and AF1 hold. Let $\eta \in (0,1]$ be given. If F is OSL (with a constant L_F), then there exists a constant C such that if $x(\cdot)$ is a solution of*

$$\dot{x}(t) \in Ax(t) + F(t, x(t) + \eta\mathbb{B}), \ x(0) = x_0,$$

then there exists a solution $y(\cdot)$ of (15) such that $|x(t) - y(t)| \leq C\sqrt{\eta + \Omega_J(\eta)}$ on I.

Proof. It is easy to see that there exists a strongly measurable function $|\tilde{f}(t)| < \eta$ such that $\dot{x}(t) \in Ax(t) + F(t, x(t) + \tilde{f}(t))$, $x(0) = x_0$. Moreover, there exists a strongly measurable selection $f(t) \in F(t, x(t) + \tilde{f}(t))$. Define the multivalued map

$$F_1(t,z) := \{v \in F(t,z) : \langle J(x(t) + \tilde{f}(t) - z), f(t) - v\rangle \leq L_F|x(t) + \tilde{f}(t) - z|^2\}.$$

It is easy to see that $F_1(\cdot,\cdot)$ is almost UHC with nonempty convex weakly compact values. Therefore, there exists a solution $y(\cdot)$ of

$$\dot{y}(t) \in Ay(t) + F_1(t, y(t)), \ y(0) = x_0.$$

Now, there exists a strongly measurable selection $f_1(t) \in F_1(t, y(t))$ such that

$$\dot{y}(t) = Ay(t) + f_1(t), \ y(0) = x_0.$$

Consequently, $\langle J(x(t) - y(t) + \tilde{f}(t)), f(t) - f_1(t)\rangle \leq L_F|x(t) + \tilde{f}(t) - y(t)|^2$.

Therefore

$$\begin{aligned}
&\langle J(x(t) - y(t)), f(t) - f_1(t)\rangle \leq L_F|x(t) - y(t)|^2 \\
&+|J(x(t) - y(t)) - J(x(t) - y(t) + \tilde{f}(t))|\,(|f(t)| + |f_1(t)|) \\
&+ \left|L_F|x(t) - y(t)|^2 - L_F|x(t) - y(t) + \tilde{f}(t)|^2\right| \\
&\leq L_F|x(t) - y(t)|^2 + 2\Omega_J(\eta)(M + 1 + \eta)\lambda(t) + |L_F|\eta(4M + \eta).
\end{aligned}$$

Since $|x(t) - y(t)|^2 - |x(0) - y(0)|^2 \leq 2\displaystyle\int_0^t \langle J(x(s) - y(s), f(s) - f_1(s)\rangle ds$, and $\eta \in (0,1]$, one has that there exists a constant C such that $|x(t) - y(t)| \leq C\sqrt{\eta + \Omega_J(\eta)}$ on I. ∎

4. Further extensions of the Filippov-Pliś theorem

In this section we present extensions of Theorem 2 in the case of CLC and OSK right-hand sides.

In terms of the support function σ, the CLC is equivalent to

$$\sigma(x-y, F(t,x)) - \sigma(x-y, F(t,y)) \le L(t, |x-y|)|x-y|,$$

for every $x, y \in X$ and a.e. $t \in I$. The OSK conditions in terms of support function are similar.

The next theorem may be considered as an extension of Theorem 2 for CLC multifunctions. The method of proof avoids successive approximations, but requires convexity and compactness in order to ensure existence of solutions. Also, there is no estimate of the velocities difference.

THEOREM 7 *Let $F(\cdot,\cdot)$ be almost continuous, convex and compact valued and CLC with linear growth. If $x(\cdot)$ is AC, satisfying* $\operatorname{dist}(\dot{x}(t), F(t, x(t))) \le f(t)$, *where $f(\cdot)$ is a nonnegative L_1 function, then there exists a solution $y(\cdot)$ of (4) such that*

$$|x(t) - y(t)| \le r(t).$$

Here $r(\cdot)$ is the maximal solution of the differential equation

$$\dot{r}(t) = L(t, r(t)) + f(t), \ r(0) = |x(0) - y_0|. \tag{18}$$

Proof. Define the multifunction:

$$G(t,y) := \{v \in F(t,y) : \langle x(t)-y, \dot{x}(t)-v\rangle \le [L(t, |x(t)-y|) + f(t)]\,|x(t)-y|\}.$$

It is easy to see that the map $G(\cdot,\cdot)$ is almost USC with nonempty convex compact values. Indeed, let $\dot{x}(\cdot)$, $f(\cdot)$, $L(\cdot,\cdot)$ and $F(\cdot,\cdot)$ be continuous on $t_i \to t_\infty$ and $x_i \to x_\infty$. Let $f_n \in F(t_n, x_n)$ be such that $f_n \to f_\infty$. Since $\langle\cdot,\cdot\rangle$ is continuous, one has that $f_\infty \in G(t_\infty, x_\infty)$ and hence $G(\cdot,\cdot)$ is with almost closed graph. Furthermore, $G(t,x) \subset F(t,x)$, i.e. $G(\cdot,\cdot)$ is almost USC. $G(t,x) \ne \emptyset$ due to its definition. Moreover, it has convex values, because $\langle x(t)-y, \cdot\rangle$ is linear. Let $y(\cdot)$ be a solution of $\dot{y}(t) \in G(t, y(t))$. Therefore $\frac{d}{dt}|x(t)-y(t)|^2 \le 2|x(t) - y(t)|\,(L(t, |x(t)-y|) + f(t) + \varepsilon)$. Since the function $t \to |x(t)-y(t)|$ is AC one has that it is a.e. differentiable. Hence, $\frac{d}{dt}|x(t)-y(t)| \le L(t, |x(t)-y|) + f(t)$ a.e. on I. The proof is therefore complete. ■

REMARK 3 *The proof of Theorem 7 may be easily modified to prove also Theorem 2. Then one can define*

$$G(t,y) := \{v \in F(t,y) : \ |\dot{x}(t) - v| \le (L(t, |x(t)-y|) + f(t))\}.$$

In this case, thanks to the continuity of $F(t,\cdot)$, also the distance between the derivatives $|\dot{x}(t) - \dot{y}(t)|$ can be estimated.

We will extend Theorem 2 in a form, which is very convenient to prove a relaxation theorem.

Denote

$$H(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\, F(t, x+\varepsilon\mathbb{B}). \tag{19}$$

It is clear from the above definitions that F is CLC if and only if $\overline{\text{co}}\, F$ is CLC. To show this also for $H(t,x)$, we prove the following useful property:

Lemma 1 *Let $\mathcal{A} \subset \mathbb{R}^n$ be open. If F is bounded on bounded sets and for every $x, y \in \mathcal{A}$*

$$\sigma(x-y, F(t,x)) - \sigma(x-y, F(t,y)) \le w(t, |x-y|),$$

where $w(\cdot\,,\cdot\,)$ is a Caratheodory function with $w(t,0) \equiv 0$ then

$$\sigma(x-y, H(t,x)) - \sigma(x-y, F(t,y)) \le w(t, |x-y|).$$

Proof. It follows from the definition of $H(\cdot,\cdot)$ that for $x, y \in \mathcal{A}$ and a given $\varepsilon > 0$ there exists $l_\varepsilon \in \varepsilon\mathbb{B}$ such that

$$\sigma(x-y, H(t,x)) \le \sigma(x-y, F(t, x+l_\varepsilon)). \tag{20}$$

Besides, by the Lipschitz continuity of the support function it is well-known that for any compact F:

$$|\sigma(x-y+l_\varepsilon, F) - \sigma(x-y, F) \le |l_\varepsilon||F|.$$

Applying twice the last inequality, we get

$$\begin{aligned}&\sigma(x-y, F(t, x+l_\varepsilon)) - \sigma(x-y, F(t,y))\\ &\le \sigma(x-y+l_\varepsilon, F(t, x+l_\varepsilon)) - \sigma(x-y+l_\varepsilon, F(t,y))\\ &\qquad +|l_\varepsilon|\{|F(t, x+l_\varepsilon)| + |F(t,y)|\}.\end{aligned}$$

Now, applying (20) and the given inequality

$$\sigma(x-y+l_\varepsilon, F(t, x+l_\varepsilon)) - \sigma(x-y+l_\varepsilon, F(t,y)) \le w(t, |x-y+l_\varepsilon|),$$

we get

$$\begin{aligned}&\sigma(x-y, H(t,x)) - \sigma(x-y, F(t,y))\\ &\le w(t, |x-y+l_\varepsilon|) + |l_\varepsilon|\{|F(t, x+l_\varepsilon)| + |F(t,y)|\}.\end{aligned}$$

Finally, since $|l_\epsilon| \le \varepsilon$, letting $\varepsilon \to 0$, we obtain the claim of the lemma. ∎

It is clear from Lemma 1 that $F(\cdot,\cdot)$ is CLC iff its regularization $H(\cdot,\cdot)$ is CLC.

Now we prove a Filippov-type relaxation theorem.

Theorem 8 *Let $F(\cdot,\cdot)$ be almost continuous with convex compact values, CLC and bounded on bounded sets. If $x(\cdot)$ is absolutely continuous with*

$$\operatorname{dist}(\dot{x}(t), F(t,x(t))) \le f(t),$$

where $f(\cdot)$ is nonnegative L_1 function, then for every $\varepsilon > 0$ there exists a solution $y(\cdot)$ of

$$\dot{z}(t) \in \overline{\operatorname{ext}}\ F(t,z(t)),\ z(0) = y_0 \tag{21}$$

such that $|x(t) - y(t)| \le \max\{\varepsilon, r(t)\}$. Here $r(\cdot)$ is the maximal solution of the differential equation

$$\dot{r}(t) = L(t, r(t)) + f(t) + \varepsilon,\ r(0) = |x(0) - y_0|.$$

Proof. Here we follow the idea of Donchev (2004), presented there in the case of OSL right-hand side. We consider the multivalued map:

$$G_\varepsilon(t,z) := \overline{\left\{v \in \overline{\operatorname{ext}}\ F(t,z) :\ \langle x(t) - z, \dot{x}(t) - v\rangle < (L(t,|x(t)-z|) + f(t))\,|x(t)-z| + \varepsilon^2\right\}}$$

when $|x(t) - z| > \varepsilon$ and $G_\varepsilon(t,z) := \operatorname{Proj}(\dot{x}(t), \overline{\operatorname{ext}}\ F(t,z))$ for $|x(t) - z| \le \varepsilon$.

Since $F(\cdot,\cdot)$ is CLC iff $\overline{\operatorname{ext}}\ F(\cdot,\cdot)$ is CLC, $G_\varepsilon(\cdot,\cdot)$ is nonempty closed valued. From Lemma 2.3.7 of Tolstonogov (2000) we know that $\overline{\operatorname{ext}}\ F(\cdot,\cdot)$ is almost LSC. It is straightforward to prove that $G_\varepsilon(\cdot,\cdot)$ is itself almost LSC. Consequently, there exists a solution $y(\cdot)$ of

$$\dot{z}(t) \in G_\varepsilon(t, z(t)),\ z(0) = y_0.$$

It is easy to see that $\langle x(t)-y(t), \dot{x}(t)-\dot{y}(t)\rangle \le (L(t,|x(t)-y(t)|) + f(t) + \varepsilon)\,|x(t) -y(t)|$ when $|x(t) - y(t)| > \varepsilon$. Since $t \to |x(t) - y(t)|$ is absolutely continuous, one has that it is a.e. differentiable. Thus

$$|x(t) - y(t)|\frac{d}{dt}|x(t) - y(t)| \le (L(t,|x(t)-y(t)|) + f(t) + \varepsilon)\,|x(t) - y(t)|,$$

when $|x(t) - y(t)| \ge \varepsilon$. The proof can be completed by a standard argument. ∎

Corollary 1 *Relaxation Theorem. Let $L(\cdot,\cdot)$ be a Kamke function; then solution set of (21) is dense (with respect to $C(I,\mathbb{R}^n)$) in the solution set of (4).*

Proof. Apply Theorem 8 with $f(x) \equiv 0$. ∎

Now we give an analog of Theorem 8 for the regularized map $H(t,x)$ defined in (19). It may be regarded as a Filippov-Pliś-type theorem without the convexity and the Lipschitz assumptions, but with a small ε appearing in the right-hand side of the differential inequality.

THEOREM 9 *Assume that $F(\cdot,\cdot)$ is almost LSC, CLC with linear growth and nonempty compact values. Let $x(\cdot)$ be AC with $\dot{x}(t) \in H(t, x(t)) + f(t)\mathbb{B}$. Then, for every $\varepsilon > 0$ there exists a solution $y(\cdot)$ of (4) such that*

$$\frac{d}{dt}|x(t) - y(t)| \leq L(t, |x(t) - y(t)|) + f(t) + \varepsilon, \ |x(0) - y(0)| = |x(0) - y_0|.$$

Proof. Fix $\varepsilon > 0$ and define the following multifunction

$$G_\varepsilon(t, y) = \overline{\{v \in F(t, y) : \ \langle x(t) - y, \dot{x}(t) - v\rangle < (L(t, |x(t) - y(t)|) + f(t) + \varepsilon)|x(t) - y(t)|\}},$$

when $y \neq x(t)$. If $y = x(t)$ we let $G_\varepsilon(t, y) = \overline{F(t, y) \bigcap (\dot{x}(t) + (f(t) + \varepsilon)\mathbb{B})}$.

Due to the CLC condition and to Lemma 1, $G_\varepsilon(\cdot,\cdot)$ has nonempty compact values. Furthermore, it is easy to see that it is almost LSC. Therefore there exists a solution $y(\cdot)$ of

$$\dot{y}(t) \in G_\varepsilon(t, y(t)), \ y(0) = y_0.$$

This solution satisfies the conclusion of the theorem. ■

4.1. Filippov-Pliś-Type theorems in Banach spaces

In this section we study the problem (3) in an arbitrary Banach space E.

Let $x, y \in E$. Define $[x, y]_+ := \lim_{h\to 0^+} h^{-1}\{|x + hy| - |x|\}$, the right derivative of $|x|$ in direction y. The map $[\cdot,\cdot]_+$ is upper semicontinuous as a function from $E \times E$ into $\mathbb{R}$ and moreover, $|[x, y]_+ - [x, z]_+| \leq |y - z|$. If also for the function $x(\cdot)$ $\dot{x}(t)$ exists, then the upper Dini derivative of $|x(t)|$ is $[x(t), \dot{x}(t)]_+$. In addition, if the duality map $J(\cdot)$ is single valued, then $\langle J(y), x\rangle = |y|[x, y]_+$. We refer to Lakshmikantham and Leela (1981), Chapter 1, for more details.

Since we work in infinite-dimensional spaces, we need some compactness type assumptions. The Hausdorff measure of noncompactness of the bounded set $A \subset E$ is defined as:

$$\chi(A) = \inf\{R > 0 : A \text{ can be covered by finitely many balls of radius } \leq R\}$$

To prove a version of Pliś theorem for almost LSC maps (see Donchev, 1997), we use the following assumptions:

As1. There exists a Kamke function $u : I \times \mathbb{R}^+ \to \mathbb{R}$ such that: for every $x, y \in E$ and every $f_x \in F(t, x)$ there exists $f_y \in F(t, y)$ with $[x - y, f_x - f_y]_+ \leq u(t, |x - y|)$.

As2. There exists a Kamke function $w : I \times R^+ \to R$ such that

$$\lim_{h\to 0} \chi(F([t, t + h] \times A)) \leq w(t, \chi(A))$$

for every bounded $A \subset E$.

Theorem 10 *Suppose $F(\cdot,\cdot)$ is almost LSC with linear growth satisfying As1, As2. Let $\varepsilon > 0$ and $x(\cdot)$ be AC function such that $\operatorname{dist}(\dot{x}(t), F(t, x(t))) \leq f(t)$ for a.e. $t \in I$, with $f(\cdot)$ an L_1-function. Then there exists a solution $y(\cdot)$ of (4) such that $|x(t)-y(t)| \leq r(t)$, where $r(0) = |x_0 - y_0|$ and $\dot{r}(t) = u(t, r) + f(t) + \varepsilon$.*

Proof. We claim that the following map is almost LSC nonempty compact valued

$$\Gamma(t, w) := cl\{v \in F(t, w) : [x(t) - w, \dot{x}(t) - v]_+ < u(t, |x(t) - w|) + f(t) + \varepsilon\}.$$

If $z \in F(t, x(t))$, $|z - \dot{x}(t)| \leq f(t)$, then there exists $v \in F(t, w)$ with $[x(t)-w, z-v]_+ \leq u(t, |x(t) - w|)$. Therefore, $[x(t) - w, \dot{x}(t) - v]_+ \leq f(t) + u(t, |x(t) - w|)$, i.e. $\Gamma(t, x)$ is nonempty compact valued. From Scorza Dragoni and Lusin's properties we obtain that for every $\varepsilon > 0$ there exists a compact $I_\varepsilon \subset I$ with $\mu(I \backslash I_\varepsilon) \leq 1 - \varepsilon$, such that: $F(\cdot,\cdot)$ is LSC on $I_\varepsilon \times E$; $u(\cdot,\cdot)$ is continuous on $I_\varepsilon \times R^+$ and $\dot{x}(\cdot), f(\cdot)$ are continuous on I_ε. Therefore it remains to show that $\Gamma(\cdot,\cdot)$ is LSC on $I_\varepsilon \times E$. Let $l \in \Gamma(t, w)$, $(t \in I_\varepsilon)$ and let $[x(t) - w, \dot{x}(t) - l]_+ \leq u(t, |x(t) - w|) + f(t) + \varepsilon - \gamma$, where $\gamma > 0$. Since $F(\cdot,\cdot)$ is LSC and $[\cdot,\cdot]_+$ is USC, there exist $\tau > t$ and $\mu > 0$ such that $[x(t') - w', \dot{x}(t') - l']_+ - [x(t) - w, \dot{x}(t) - l]_+ < \gamma$, whenever $|l' - l| < \mu$, $|w' - w| < \mu$ and $t' \in I_\varepsilon \bigcap [t, \tau)$. Thus, there exists $l' \in F(t', u')$ such that $[x(t') - u', \dot{x}(t') - l']_+ < u(t', |x(t') - u'|) + f(t') + \varepsilon$. Therefore, $l' \in \Gamma(t', x(t'))$ and hence $\Gamma(\cdot,\cdot)$ is almost LSC. This, together with As2 implies that there exists a solution $y(\cdot)$ of

$$\dot{y}(t) \in \Gamma(t, y(t)), \ y(0) = y_0.$$

This is the desired solution. ∎

We close the section with a theorem for the special case of uniformly convex E^*.

Theorem 11 *Assume AA1, AF1 hold. Let $f(\cdot)$ be non-negative Lebesgue integrable function. If F is CLC with a function $w(\cdot,\cdot)$ then for every solution $x(\cdot)$ of*

$$\dot{x}(t) \in Ax(t) + F(t, x(t)) + f(t)\mathbb{B}, \ x(0) = x_0,$$

there exists a solution $y(\cdot)$ of

$$\dot{z}(t) \in Az(t) + F(t, z(t)), \ z(0) = y_0$$

such that $|x(t) - y(t)| \leq r(t)$, where $r(\cdot)$ is the maximal solution of $\dot{r}(t) = w(t, r(t))$, $r(0) = |x_0 - y_0|$.

Proof. Let $x(\cdot)$ be a solution of $\dot{x} = Ax + g_x(t) + s(t)$, where $g_x(t) \in F(t, x(t))$ and $s(t) \in f(t)\mathbb{B}$. Define the multifunction:

$$G(t, y) := \{z \in F(t, y) : \langle J(x(t) - y), g_x(t) + s(t) - z\rangle \leq (w(t, |x(t) - y|) + |s(t)|)\, |x(t - y|\}.$$

It is easy to see that $G(\cdot,\cdot)$ is almost UHC with nonempty closed convex values. Therefore the problem

$$\dot{y}(t) \in Ay + G(t, y(t)),\ y(0) = y_0$$

has a solution $y(\cdot)$ with corresponding $g_y(t) \in G(t, y(t))$. Since $x(\cdot)$ and $y(\cdot)$ are continuous, the set $\mathcal{A} = \{x(t) \neq y(t)\}$ is open. On every interval (κ, τ), contained in $\mathcal{A}$, one has $[x(t)-y(t), g_x(t)-g_y(t)]_+ \leq w(t, |x(t)-|y(t)|) + |s(t)|$. It is standard now to establish that $y(\cdot)$ satisfies the requirement of the theorem.■

5. Applications and conclusions

In this section we present an application of the Filippov-type theorems to the Euler discrete approximation of differential inclusions. For simplicity, we restrict our consideration to the autonomous problem in $\mathbb{R}^n$

$$\dot{x}(t) \in F(x(t)),\ \text{for } t \in (0, T),\ x(0) = x_0. \tag{22}$$

We are able to determine the reachable set by the exponential formula of Donchev, Farkhi and Wolenski (2003), which is a direct consequence of the Filippov theorem.

Denote $h = T/N$ and define the Euler set-valued iterates of a given initial set X:

$$(I + hF)\{X\} = \{x + hf :\ x \in X,\ f \in F(x)\}.$$

We denote $(I + hF)^k\{X\} = (I + hF)\{(I + hF)^{k-1}\{X\}\}$ for $k = 2, 3, \ldots$.

The following theorem proved in Donchev, Farkhi (2000) provides convergence and distance estimate for the Euler set-valued approximant to the reachable set.

Corollary 2 *(*The Exponential Formula*) Suppose $F(\cdot)$ is USC with convex compact values, bounded on bounded sets and OSL with a constant L. The reachable set $A(t, x_0)$ of the system (22) satisfies*

$$A(t, x_0) = \lim_{N\to\infty} (I + \frac{t}{N} F)^N \{x_0\}$$

and the error is $\mathrm{D_H}\left(A(t, X_0), (I + \frac{t}{N} F)^N \{X_0\}\right) \leq \frac{C}{\sqrt{N}}$.

If $L < 0$ in Corollary 2, then the following version of the exponential formula holding on an infinite time interval is proved in Donchev, Farkhi and Reich (2003) and in Donchev, Farkhi and Reich (2007) in the infinite-dimensional case:

Theorem 12 *(The Infinite Time Exponential Formula) Let F satisfy the conditions of Corollary 2 with a negative constant L. Then there exists a compact set $A^{\infty} = \lim_{t\to\infty} R(t, X_0)$, and if the sequence of positive numbers $\{h_k\}_{k=1}^{\infty}$ satisfies $\sum_{k\to\infty} h_k = \infty$, $\sum_{k\to\infty} h_k^2 < \infty$, the following exponential formula holds*

$$A^{\infty} = \lim_{N\to\infty} (I + h_N F) \circ (I + h_{N-1}F) \circ \ldots \circ (I + h_1 F)\{X_0\}.$$

We presented here various extensions of the theorems of Filippov and Pliś, with estimates only for the solutions and not for the distance between their derivatives, since the derivatives may belong to discontinuous right-hand sides.

A stronger Filippov-Pliś-type theorem with additional estimates of the distance between the derivatives may be obtained for right-hand sides satisfying the so-called Modified OSL (MOSL) condition. The MOSL condition is weaker than the Lipschitz one and stronger than the OSL (see Donchev, Farkhi, Mordukhovich, 2007). This will be the subject of a future paper.

Open Problems.

1) One of the main open problems related to the Filippov-type theorems, as well as to the discrete approximations, is whether the square root estimate in Theorem 4 is sharp. This implies the order of one half in the error estimate of the Euler discrete approximation (Corollary 2). So far we do not have examples confirming that this rate is sharp.

2) There are many papers devoted to necessary optimality conditions for optimal control systems described by differential inclusions when the right-hand side is locally Lipschitz. It would be very interesting to prove such conditions in case of one-sided Lipschitz (or OSK/CLC) right-hand sides. A first step in this direction has been made in Donchev, Farkhi and Mordukhovich (2007).

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