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On the robustness of optimal solutions for combinatorial optimization problems*

by

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Abstract: We consider the so-called *generic combinatorial optimization problem*, where the set of feasible solutions is some family of subsets of a finite ground set with specified positive initial weights of elements, and the objective function represents the total weight of elements of a feasible solution. We assume that the weights of all elements may be perturbed simultaneously and independently up to a given percentage of their initial values. A feasible solution which minimizes the worst-case relative regret, is called a *robust* solution. The maximum percentage level of perturbations, for which an initially optimal solution remains robust, is called the *robustness radius* of this solution. In this paper we study the robustness aspect of initially optimal solutions and provide lower bounds for their robustness radii.

Keywords: robustness and sensitivity analysis, combinatorial optimization, accuracy function, robustness radius.

1. Introduction

Robust optimization (see, e.g., Kouvelis and Yu, 1997) deals usually with an optimization problem in which the set of feasible solutions is known precisely, but parameters defining the objective function may be uncertain. All possible realizations of these parameters compose the set of so-called scenarios. It is required to find a feasible solution – called a robust solution – which is reasonably close, in terms of the objective function value, to the optimal one for all possible scenarios. There are various measures of such a 'closeness', leading to various robust optimization models. For example, in minmax relative regret optimization (see, e.g., Averbakh, 2005; Kouvelis and Yu, 1997) one seeks a feasible solution which minimizes the worst-case relative regret, taken as the maximum percentage deviation from the optimality of the considered solution over the set of all scenarios.

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This paper deals with the minmax relative regret optimization model, but instead of a single set of scenarios we are faced with a family of such sets depending on a parameter $\delta \in [0,1)$. Namely, we consider the generic combinatorial optimization problem, sometimes called the subset-type problem, defined for a finite ground set with given positive initial weights of elements. The set of feasible solutions is some fixed family of subsets of the ground set and the objective function, which we want to minimize, represents the total weight of the elements of a feasible solution. To define the set of scenarios for a fixed value of δ, we assume that the weights of the elements may be simultaneously and independently perturbed (increased or decreased) by at most δ percent of their initial values. In this case, the so-called accuracy function of a feasible solution, considered in the sensitivity analysis context in Libura (1999), provides the worst-case relative regret for this solution for any δ in the interval $[0,1)$. Thus, a feasible solution is robust for a particular value of δ, if the corresponding value of the accuracy function at this point is minimum among all the feasible solutions.

In contrast to standard robust optimization approach, our focus in this paper is not a problem of finding a robust solution for a given set of scenarios (corresponding to some δ), but rather a question of the robustness of a solution being optimal for the initial weights. In particular, we are interested in the largest value of δ, for which this solution remains robust. Such a value of δ is called the robustness radius of the considered solution. Main results of this paper concern some lower bounds for this radius.

The paper is organized as follows: In Section 2, we formally describe the considered robustness model and provide the definition of the accuracy function. In Section 3, we define the regret function as a point-wise minimum of the accuracy functions of all feasible solutions, and then we introduce the robustness radius of an optimal solution. In Section 4, we provide lower bounds for the robustness radius in two essentially different cases: If there is a single optimal solution, then we present a lower bound for its robustness radius using derived properties of the accuracy function. In case of multiple optimal solutions, first we characterize these optimal solutions, which may be robust in the neighborhood of $\delta = 0$, and then we provide analogous bounds for their robustness radii. Section 5 contains some concluding remarks.

2. The accuracy function

Let $E = \{e_1, \ldots, e_n\}$ be a finite *ground set* and let $c(e) > 0$ denote the *weight* of element $e \in E$. Consider a family $\mathcal{F} \subseteq 2^E \setminus \{\emptyset\}$ of nonempty subsets of E, called the *feasible solutions*, and for $X \subseteq E$ and $c = (c(e_1), \ldots, c(e_n))$ let

$$w(c, X) = \sum_{e \in X} c(e)$$

denote the weight of subset X.

The *generic combinatorial optimization problem*

$$v(c) = \min\{w(c, X) : \ X \in \mathcal{F}\} \tag{1}$$

consists in finding a feasible solution of minimum weight.

Various discrete optimization problems, like the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0-1 programming problem, can be stated in this general form. In the following we assume that the set of feasible solutions $\mathcal{F}$ is fixed, but the vector of weights c may be perturbed or is given with errors. Namely, we assume that $c \in C(c^o, \delta)$, where for $c^o = (c^o(e_1), \ldots, c^o(e_n)) \in \mathbb{R}^n$, $c^o > 0$, and $\delta \in [0, 1)$,

$$C(c^o, \delta) = \{(d_1, \ldots, d_n) \in \mathbb{R}^n : \ |c^o(e_i) - d_i| \le c^o(e_i) \cdot \delta, \quad i = 1, \ldots, n\}.$$

Thus, there is some initial vector of weights $c^o > 0$, and for a given value of the parameter $\delta \in [0, 1)$ the maximum perturbation of any weight does not exceed δ percent of its initial value.

Consider a feasible solution $X \in \mathcal{F}$. The quality of this solution for a given $c \in C(c^o, \delta)$ can be measured by its *relative error* (relative regret) $\varepsilon(c, X)$, where

$$\varepsilon(c, X) = \frac{w(c, X) - v(c)}{v(c)}. \tag{2}$$

Observe that for any $c \in C(c^o, \delta)$ and for arbitrary $X \in \mathcal{F}$, $\varepsilon(c, X) \ge 0$. Moreover, $\varepsilon(c, X) = 0$ if and only if X is an optimal solution to problem (1).

For a given feasible solution $X \in \mathcal{F}$ and $\delta \in [0, 1)$ the *accuracy function* $a(X, \delta)$, considered in Libura (1999), gives the maximum value of the relative error $\varepsilon(c, X)$ over the set $C(c^o, \delta)$, i.e.,

$$a(X, \delta) = \max\{\varepsilon(c, X) : \ c \in C(c^o, \delta)\}. \tag{3}$$

It is shown in Libura (1999) that for an arbitrary feasible solution X, $a(X, \delta)$ is a nondecreasing and convex function of δ. Also general formulae for calculating its value for $\delta \in [0, 1)$ are given in Libura (1999, 2000). In Libura and Nikulin (2004, 2006) some extensions and properties of the accuracy function for multicriteria combinatorial optimization problems are studied.

The accuracy function has a finite number of breakpoints in the interval $[0, 1)$. If X^o is an optimal solution to problem (1) for $c = c^o$, then obviously $a(X^o, 0) = 0$, but when δ grows, then $a(X^o, \delta)$ may become positive, which means that X^o is no longer an optimal solution of (1) for some $c \in C(c^o, \delta)$. From the practical point of view it is of special interest to know the first breakpoint of the accuracy function, corresponding to the largest value of δ, for which $a(X^o, \delta) = 0$. This value is called the *accuracy radius* of the solution X^o and is formally defined as follows:

$$r^a(X^o) = \sup\{\delta \in [0, 1) : \ a(X^o, \delta) = 0\}. \tag{4}$$

The accuracy radius of X^o gives thus the maximum percentage perturbation of any weight which does not destroy the optimality of X^o. In Libura (2000) a general formula for calculating the exact value of the accuracy radius is given, and an approach to determine some lower bounds for this value is described.

Example 1. Consider an undirected graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$. Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (14, 11, 14, 15, 13, 18, 17)$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is just the minimum spanning tree problem in the weighted graph G. The subset of edges $X^o = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{4, 5\}\}$ is the unique optimal solution for this problem. The graph G and the minimum spanning tree X^o are shown in Fig. 1.

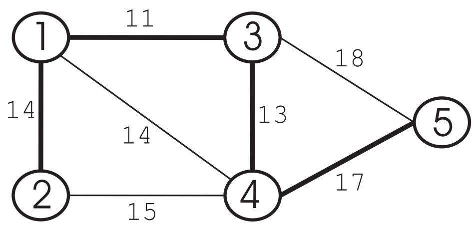

Figure 1. Graph G and its minimum spanning tree indicated with bold lines.

In Fig. 2, the accuracy function $a(X^o, \delta)$ of the solution X^o is shown for $\delta \in [0, 0.5]$. From this picture one can read that the solution X^o remains optimal if the maximum percentage perturbation of any weight does not exceed approximately 2.8% of its initial value. This level of perturbations corresponds to the accuracy radius of X^o, which is equal to $1/35$. For larger values of δ the solution X^o may become suboptimal and – for example – for $\delta = 0.3$, i.e., when the maximum perturbations of weights are equal 30% of their initial values, the maximum relative error of X^o reaches 60%.

3. The regret function and the robustness radius

In the framework of *robust optimization* the set $C(c^o, \delta)$ for a given fixed value of δ is interpreted as a set of possible *scenarios*. Then the accuracy function $a(X, \delta)$ provides the value of so-called *worst-case relative regret* of the solution X over the set of all possible scenarios. In minmax relative regret optimization (see, e.g., Averbakh, 2005; Kouvelis and Yu, 1997) one wants to find such a feasible solution that the worst-case relative regret for it is minimum among the feasible solutions of problem (1). Therefore, we will consider the following

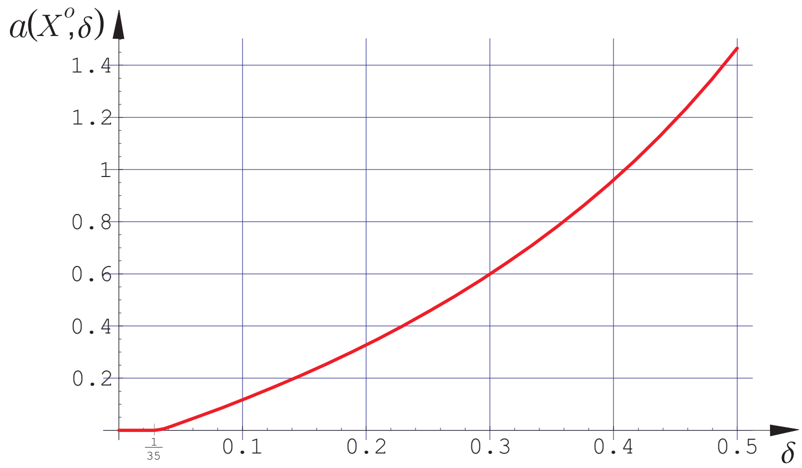

Figure 2. The accuracy function of the optimal spanning tree X^o.

function of $\delta \in [0, 1)$:

$$z(\delta) = \min_{X \in \mathcal{F}} a(X, \delta). \tag{5}$$

We will call this function the *minimum relative regret function* or – for short – the *regret function* for problem (1). A feasible solution X will be called a *robust solution* for a given $\delta \in [0, 1)$ if and only if $a(X, \delta) = z(\delta)$.

It is obvious that if $a(X, \delta) = 0$ for some $\delta \in [0, 1)$, then the solution X is a robust solution for this value of δ. Thus, if X^o is an optimal solution for $\delta = 0$, then it remains robust for any $\delta \leq r^a(X^o)$. But it may be robust also for larger values of δ (see an example below). On the other hand, a feasible solution which is non-optimal for $\delta = 0$ may become a robust solution for larger values of perturbations.

If X^o is an optimal solution to problem (1) for $c = c^o$, then the maximum value of δ, for which X^o remains robust, is called the *robustness radius* of X^o and is denoted by $r^r(X^o)$. Formally:

$$r^r(X^o) = \sup\{\delta \in [0, 1) : \ a(X^o, \delta) = z(\delta)\}. \tag{6}$$

Thus, $r^r(X^o)$ determines maximum percentage perturbations of the initial weights, for which the solution X^o still guarantees minimum value of the worst-case relative regret among all the feasible solutions. The robustness radius of an initially optimal solution may be therefore regarded as some measure of quality of this solution from the robustness point of view. If, in particular, there are multiple optimal solutions of problem (1) for $c = c^o$, then a solution with the largest robustness radius may be considered as a preferable one.

EXAMPLE 2. In Fig. 4, the regret function for the minimum spanning tree problem in graph G from Fig. 1 is shown. According to (5), this function is a point-wise minimum of the accuracy functions for all spanning trees in graph G. Although there are 21 different spanning trees in G, in this case the regret function is determined by the following three spanning trees: $X^o = \{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$, $X' = \{\{1,2\},\{1,3\},\{2,4\},\{4,5\}\}$ and $X'' = \{\{1,2\},\{2,4\},\{3,5\},\{4,5\}\}$; all other feasible solutions may be neglected in (5). The corresponding accuracy functions for the feasible solutions X^o, X' and X'' are shown in Fig. 3.

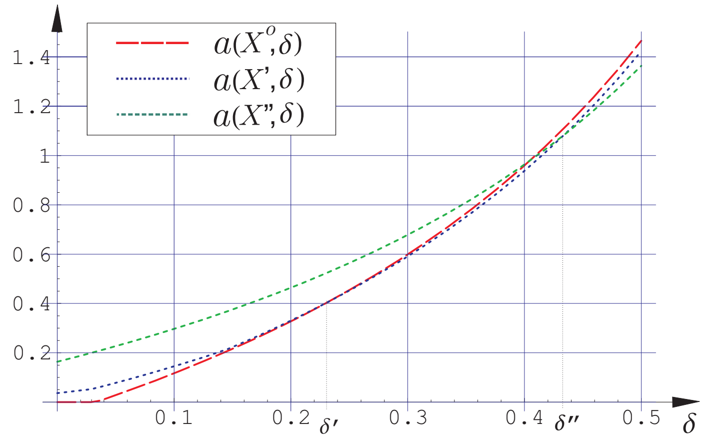

Figure 3. Accuracy functions for spanning trees X^o, X' and X''.

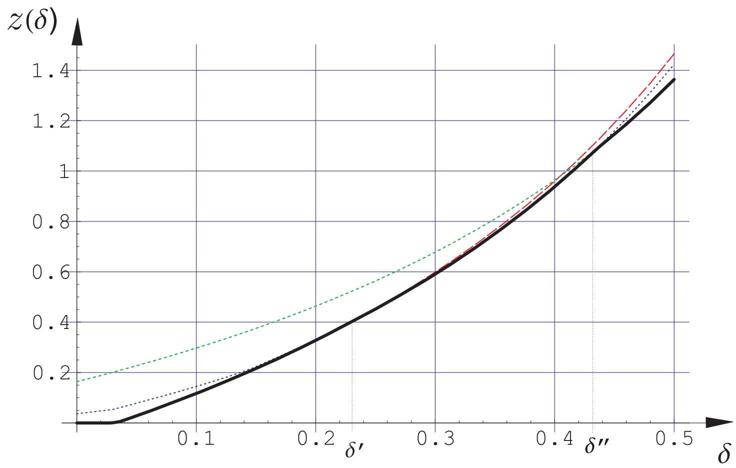

Figure 4. The regret function for the minimum spanning tree problem.

From Fig. 4 one can see that the solution X^o remains robust behind its accuracy radius. Indeed, the robustness radius of this solution is determined by the value of $\delta = \delta' = r^r(X^o)$, for which the accuracy functions of X^o and X' coincide. In our example δ' is equal approximately 0.23. This means that the solution X^o remains robust if the maximum percentage perturbation of any edge weight does not exceed approximately 23% of its initial value.
For $\delta > \delta'$ the solution X' becomes a robust solution and it remains robust till $\delta = \delta'' \approx 0.43$. For larger levels of perturbations, again, we have a new robust solution: this time X''.

Computational complexity results in robust optimization (see, e.g., Averbakh, 2005; Kasperski, 2008; Kouvelis and Yu, 1997) suggest that calculating the exact value of the robustness radius may be a difficult task. Therefore in the next section we give some simple bounds for the accuracy function of an arbitrary feasible solution and derive corresponding bounds for the regret function and for the robustness radius of an optimal solution.

4. Bounds for the regret function and for the robustness radius

In Libura (1999) it is shown that for $X \in \mathcal{F}$ and $\delta \in [0, 1)$ the accuracy function of X is expressed by the following formula:

$$a(X, \delta) = \max_{Y \in \mathcal{F}} \frac{w(c^o, X) - w(c^o, Y) + \delta\, w(c^o, X \otimes Y)}{(1 - \delta)\, w(c^o, Y)}, \tag{7}$$

where $X \otimes Y = (X \cup Y) \setminus (X \cap Y)$. It will be convenient to rewrite (7) in the following equivalent form:

$$a(X, \delta) = \max_{Y \in \mathcal{F}} \frac{(1 + \delta)w(c^o, X) - (1 - \delta)w(c^o, Y) - 2\,\delta\, w(c^o, X \cap Y)}{(1 - \delta)\, w(c^o, Y)}. \tag{8}$$

Lemma 1 gives an upper bound for the accuracy function of an arbitrary feasible solution in problem (1).

LEMMA 1. *For $X \in \mathcal{F}$ and $\delta \in [0, 1)$,*

$$a(X, \delta) \leq \frac{2\delta}{1 - \delta} + \frac{1 + \delta}{1 - \delta} \cdot a(X, 0). \tag{9}$$

Proof. For arbitrary $X, Y \in \mathcal{F}$ we have $w(c^o, X \cap Y) \geq 0$ and

$$w(c^o, X) \leq w(c^o, Y) + w(c^o, X) - v(c^o).$$

Thus, after replacing in (8) $w(c^o, X)$ with $w(c^o, Y) + w(c^o, X) - v(c^o)$ and removing a nonnegative component $2\delta\, w(c^o, X \cap Y)$, we obtain:

$$\begin{aligned} a(X,\delta) &\leq \max_{Y\in\mathcal{F}} \left\{ \frac{(1+\delta)w(c^o,Y) - (1-\delta)w(c^o,Y)}{(1-\delta)w(c^o,Y)} \right. \\ &\qquad + \left. \frac{1+\delta}{1-\delta} \cdot \frac{w(c^o,X) - v(c^o)}{w(c^o,Y)} \right\} \\ &= \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot \max_{Y\in\mathcal{F}} \frac{w(c^o,X) - v(c^o)}{w(c^o,Y)} \\ &= \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot a(X,0). \end{aligned}$$ ■

If X^o is an optimal solution in (1) for $c = c^o$, then $a(X^o, 0) = 0$, and from (9) we have immediately:

COROLLARY 1. *If X^o is an optimal solution in* (1) *for $c = c^o$, then for any $\delta \in [0, 1)$,*

$$a(X^o,\delta) \leq \frac{2\delta}{1-\delta}. \tag{10}$$

Corollary 1 provides an upper bound for the maximum relative error of an arbitrary optimal solution of problem (1) under the assumption that percentage perturbations of weights do not exceed $\delta{\cdot}100\%$. The same bound has been obtained earlier in Oguz (2000) in the framework of the so-called tolerance approach (see Wendell, 2004) for linear programs.

Observe also that now, directly from the definition of the regret function and from the inequality (10), we have the following fact:

COROLLARY 2. *For $\delta \in [0, 1)$,*

$$z(\delta) \leq \frac{2\delta}{1-\delta}. \tag{11}$$

It is easy to see that the bound (11) is tight for any $\delta \in [0, 1)$. Indeed, it is enough to consider problem (1) where $E = \{e_1, e_2\}$, $\mathcal{F} = \{\{e_1\}, \{e_2\}\}$ and $c = c^o = (1, 1)$. Then $z(\delta) = \min\{a(\{e_1\}, \delta), a(\{e_2\}, \delta)\}$. Moreover, $w(c^o, \{e_1\}) = w(c^o, \{e_2\}) = 1$ and $w(c^o, \{e_1\} \otimes \{e_2\}) = 2$. Thus, from (5) and (7) it follows that in this case $z(\delta) = \frac{2\delta}{1-\delta}$ for any $\delta \in [0, 1)$.

The following lemma provides a simple lower bound for the accuracy function of any feasible solution X.

LEMMA 2. *For $X \in \mathcal{F}$ and $\delta \in [0, 1)$,*

$$a(X,\delta) \geq \frac{1+\delta}{1-\delta} \cdot a(X,0). \tag{12}$$

Proof. For a given feasible solution X and arbitrary $Y \in \mathcal{F}$ we have from (8) the following inequality:

$$a(X,\delta) \geq \frac{(1+\delta)w(c^o,X) - (1-\delta)w(c^o,Y) - 2\delta w(c^o, X \cap Y)}{(1-\delta)w(c^o,Y)}.$$

Taking $Y = X^o$, where X^o is an optimal solution in (1) for $c = c^o$, we have:

$$a(X,\delta) \geq \frac{(1+\delta)w(c^o,X) - (1-\delta)v(c^o) - 2\delta w(c^o, X \cap X^o)}{(1-\delta)v(c^o)}.$$

Replacing $w(c^o, X \cap X^o)$ with $v(c^o) = w(c^o, X^o) \geq w(c^o, X \cap X^o)$, we obtain:

$$\begin{aligned} a(X,\delta) &\geq \frac{(1+\delta)w(c^o,X) - (1-\delta)v(c^o) - 2\delta v(c^o)}{(1-\delta)v(c^o)} \\ &= \frac{1+\delta}{1-\delta} \cdot \frac{w(c^o,X) - v(c^o)}{v(c^o)} \\ &= \frac{1+\delta}{1-\delta} \cdot a(X,0). \end{aligned}$$ ■

Let Ω denote the set of all optimal solutions of problem (1) for $c = c^o$, and let a be minimum non-zero value of the relative error for a feasible solution in (1), i.e.,

$$a = \min_{X \in \mathcal{F} \setminus \Omega} \frac{w(c^o,X) - v(c^o)}{v(c^o)}. \tag{13}$$

Observe that according to a standard convention, $a = \infty$ when $\mathcal{F} \setminus \Omega = \emptyset$.

If we know the exact value of a (or some positive lower bound for a), then the bounds for the accuracy function provided by Lemma 1 and Lemma 2 allow for calculation of the lower bound for the robustness radius of an arbitrary optimal solution of (1). In the following, we will distinguish two cases: single optimal solution for problem (1), and multiple optimal solutions for problem (1).

Assume first that X^o is a single optimal solution of problem (1) for $c = c^o$. The following fact holds:

THEOREM 1. *If X^o is a single optimal solution of problem* (1) *for $c = c^o$, then*

$$r^r(X^o) \geq \begin{cases} \frac{a}{2-a} & if \quad a < 1, \\ 1 & otherwise. \end{cases} \tag{14}$$

Proof. Consider the following two convex functions of δ on the interval $[0,1)$: $f'(\delta) = \frac{2\delta}{1-\delta}$, which – according to Lemma 1 – is an upper bound on $a(X^o,\delta)$ and $f''(\delta) = \frac{1+\delta}{1-\delta} \cdot a$, which – according to Lemma 2 – is a lower bound for the accuracy function $a(Y,\delta)$ of any feasible solution $Y \in \mathcal{F} \setminus \{X^o\}$.

The solution X^o is obviously robust for $\delta = 0$ and it remains robust for all such $\delta \in [0, 1)$ that $f'(\delta) \leq f''(\delta)$. If $a \geq 1$, then this inequality holds for any $\delta \in [0, 1)$ which means that $r^r(X^o) = 1$. For $a < 1$ the inequality $f'(\delta) \leq f''(\delta)$ is valid for $\delta \leq \frac{a}{2-a}$ and this value provides a lower bound on the robustness radius of X^o. ■

Consider now the case of multiple optimal solutions for problem (1), i.e., assume that $|\Omega| = p,\ p > 1$. Although all the solutions belonging to Ω give the same optimal objective value for $\delta = 0$, they may differ from the robustness point of view. It is obvious that any solution in Ω is robust for $\delta = 0$, but an interesting question arises, how to select an optimal solution which remains robust in some neighborhood of $\delta = 0$.

From the formula (7) on the accuracy function it follows directly that for any $X \in \Omega$, $|\Omega| > 1$, we have $a(X, \delta) = 0$ for $\delta = 0$, and $a(X, \delta) > 0$ for $\delta > 0$. Moreover, the following lemma states that for some neighborhood of $\delta = 0$ the accuracy function of any solution belonging to Ω depends only on the solutions from this set, and does not depend on any feasible solution belonging to the set $\mathcal{F} \setminus \Omega$.

Lemma 3. *If $X \in \Omega$ and $\delta \leq \frac{a}{2+a}$, then*

$$a(X, \delta) = \frac{2\delta}{(1-\delta)} \left(1 - \min_{Y \in \Omega} \frac{w(c^o, X \cap Y)}{v(c^o)} \right). \tag{15}$$

Proof. For arbitrary $X \in \mathcal{F}$ and $\delta \in [0, 1)$ the formula (8) can be stated as follows:

$$a(X, \delta) = \max\{a'(X, \delta), a''(X, \delta)\}, \tag{16}$$

where

$$a'(X, \delta) = \max_{Y \in \Omega} \frac{(1+\delta)w(c^o, X) - (1-\delta)w(c^o, Y) - 2\,\delta\, w(c^o, X \cap Y)}{(1-\delta)\ w(c^o, Y)}$$

and

$$a''(X, \delta) = \max_{Y \in \mathcal{F} \setminus \Omega} \frac{(1+\delta)w(c^o, X) - (1-\delta)w(c^o, Y) - 2\,\delta\, w(c^o, X \cap Y)}{(1-\delta)\ w(c^o, Y)}.$$

If $X \in \Omega$, then $w(c^o, X) = v(c^o)$, and for $\delta \in [0, 1)$ we have:

$$a'(X, \delta) = \frac{2\delta}{1-\delta} \left(1 - \min_{Y \in \Omega} \frac{w(c^o, X \cap Y)}{v(c^o)} \right), \tag{17}$$

and

$$a''(X, \delta) \leq \frac{2\delta + a\delta - a}{(1-\delta)(1+a)}. \tag{18}$$

Indeed, substituting $w(c^o, Y) = v(c^o)$ for $Y \in \Omega$, and observing that for any $Y \in \mathcal{F} \setminus \Omega,\ \ w(c^o, Y) \geq (1+a)v(c^o)$, we obtain:

$$\begin{aligned} a'(X,\delta) &= \max_{Y\in\Omega} \frac{(1+\delta)v(c^o) - (1-\delta)v(c^o) - 2\,\delta\, w(c^o, X\cap Y)}{(1-\delta)\ v(c^o)} \\ &= \frac{2\delta}{(1-\delta)}\left(1 - \min_{Y\in\Omega}\frac{w(c^o, X\cap Y)}{v(c^o)}\right) \end{aligned}$$

and

$$\begin{aligned} a''(X,\delta) &= \max_{Y\in\mathcal{F}\setminus\Omega} \frac{(1+\delta)v(c^o) - (1-\delta)w(c^o,Y) - 2\,\delta\, w(c^o, X\cap Y)}{(1-\delta)\ w(c^o,Y)} \\ &= \max_{Y\in\mathcal{F}\setminus\Omega} \frac{(1+\delta)v(c^o) - 2\,\delta\, w(c^o, X\cap Y)}{(1-\delta)\ w(c^o,Y)} - 1 \\ &\leq \max_{Y\in\mathcal{F}\setminus\Omega} \frac{(1+\delta)v(c^o)}{(1-\delta)\ w(c^o,Y)} - 1 \leq \frac{(1+\delta)v(c^o)}{(1-\delta)(1+a)v(c^o)} - 1 \\ &= \frac{(1+\delta)-(1-\delta)(1+a)}{(1-\delta)(1+a)} = \frac{2\delta + a\delta - a}{(1-\delta)(1+a)}. \end{aligned}$$

But $a'(X,\delta) \geq 0$ for any $\delta \in [0,1)$ whereas $a''(X,\delta) \leq 0$ when $\delta \leq \frac{a}{2+a}$. This implies that for $X \in \Omega$ and $\delta \leq \frac{a}{2+a}$ we have $a(X,\delta) = a'(X,\delta)$, which proves (15). ■

Lemma 3 allows for formulating a necessary condition for a solution from the set Ω to be robust in the neighborhood of $\delta = 0$. Directly from the definition of the regret function and from (15) we have the following corollary:

Corollary 3. *If an optimal solution $X^o \in \Omega$ remains robust in some neighborhood of $\delta = 0$, then the following condition must hold:*

$$\min_{Y\in\Omega} w(c^o, X^o\cap Y) = \max_{X\in\Omega}\min_{Y\in\Omega} w(c^o, X\cap Y). \tag{19}$$

Proof. A solution $X^o \in \mathcal{F}$ is robust for a given $\delta \in [0,1)$ if and only if $a(X^o,\delta) = z(\delta) = \min_{X\in\mathcal{F}} a(X,\delta)$. When for $X^o \in \Omega$, $|\Omega| > 1$, the condition (19) does not hold, i.e., $\min_{Y\in\Omega} w(c^o, X^o\cap Y) < \max_{X\in\Omega}\min_{Y\in\Omega} w(c^o, X\cap Y)$, then for $\delta \in (0, \frac{a}{2+a}]$ it follows from (15) that $a(X^o,\delta) > \min_{X\in\Omega} a(X,\delta) \geq \min_{X\in\mathcal{F}} a(X,\delta)$ and therefore X^o is not a robust solution. ■

Let

$$\Omega_r = \left\{ X \in \Omega :\ \min_{Y\in\Omega} \frac{w(c^o, X\cap Y)}{v(c^o)} = b \right\}, \tag{20}$$

where

$$b = \max_{X\in\Omega}\min_{Y\in\Omega} \frac{w(c^o, X\cap Y)}{v(c^o)}. \tag{21}$$

It is easy to see that $b < 1$ if $|\Omega| > 1$.

From Corollary 3 it follows that the robustness radius of any optimal solution from the set $\Omega \setminus \Omega_r$ is equal to zero. In the following we provide a lower bound for the robustness radius of an arbitrary optimal solution belonging to Ω_r. We will need the following fact:

LEMMA 4. *If $X \in \Omega_r$ and*

$$a\delta - 2ab\delta - 2b\delta + a \geq 0, \tag{22}$$

then

$$a(X, \delta) = \frac{2\delta}{1-\delta}\,(1-b). \tag{23}$$

Proof. If $X \in \Omega_r$, then, according to (17) and (18), for $\delta \in [0, 1)$,

$$a'(X, \delta) = \frac{2\delta}{1-\delta}\,(1-b)$$

and

$$a''(X, \delta) \leq \frac{2\delta + a\delta - a}{(1-\delta)(1+a)}.$$

But if the inequality (22) holds, then

$$\frac{2\delta}{1-\delta}\,(1-b) \geq \frac{2\delta + a\delta - a}{(1-\delta)(1+a)},$$

which implies that $a'(X, \delta) \geq a''(X, \delta)$ and consequently

$$a(X, \delta) = \max\{a'(X, \delta), a''(X, \delta)\} = a'(X, \delta) = \frac{2\delta}{1-\delta}\,(1-b). \qquad \blacksquare$$

Lemma 4 allows for finding such a neighborhood of $\delta = 0$, depending on a and b, in which the exact value of the accuracy function of any solution $X \in \Omega_r$ is given by (23). The following theorem uses this fact to provide a lower bound for the robustness radius of arbitrary $X \in \Omega_r$.

THEOREM 2. *If $X \in \Omega_r$ and $a \geq \frac{b}{1-b}$, then*

$$r^r(X) \geq \begin{cases} \frac{a}{2(1-b)-a} & \text{if} \quad a < 1-b, \\ 1 & \text{otherwise.} \end{cases} \tag{24}$$

If $X \in \Omega_r$ and $a < \frac{b}{1-b}$, then

$$r^r(X) \geq \begin{cases} \min\left\{\frac{a}{2(1-b)-a}, \frac{a}{2b+2ab-a}\right\} & \text{if} \quad a < 1-b, \\ \frac{a}{2b+2ab-a} & \text{otherwise.} \end{cases} \tag{25}$$

Proof. Consider first the case, when $a \geq \frac{b}{1-b}$. Then the inequality (22) holds for any $\delta \in [0,1)$. Indeed, we have $a\delta - 2ab\delta - 2b\delta + a \geq a\delta - 2ab\delta - 2b\delta + a\delta = 2\delta(a - ab - b) \geq 0$.

The solution $X \in \Omega_r$ remains robust for a given $\delta \in [0,1)$ if the value of its accuracy function $a(X,\delta) = \frac{2\delta}{1-\delta}(1-b)$ does not exceed the lower bound of the accuracy function for any solution $Y \in \mathcal{F} \setminus \Omega$, which, according to Lemma 2, is equal to $\frac{1+\delta}{1-\delta}\,a$. But this holds for arbitrary $\delta \in [0,1)$ if $a \geq 1-b$, and for $\delta \leq \frac{a}{2(1-b)-a}$ if $a < 1-b$, which proves (24).

If $a < \frac{b}{1-b}$, then the inequality (22) holds for any $\delta \leq \frac{a}{2b+2ab-a} < 1$. Using the same arguments as before we obtain that $\min\left\{\frac{a}{2(1-b)-a}, \frac{a}{2b+2ab-a}\right\}$ and $\min\left\{1, \frac{a}{2b+2ab-a}\right\} = \frac{a}{2b+2ab-a}$ provide lower bounds for the robustness radius $r^r(X)$ for $a < 1-b$ and $a \geq 1-b$, respectively. ■

Example 3. Consider again the minimum spanning tree problem in graph $G = (V,E)$ shown in Fig. 1, where $V = \{1,2,3,4,5\}$ and $E = \{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\},\{3,5\},\{4,5\}\}$, and assume that $c^o = (2,2,2,2,1,2,2)$ is a vector of the initial weights of edges. The set Ω of optimal solutions contains now 10 elements, corresponding to the following subsets of edges:

$$\begin{aligned}
&\{\{1,2\},\{1,4\},\{3,4\},\{3,5\}\},\\
&\{\{1,2\},\{1,4\},\{3,4\},\{4,5\}\},\\
&\{\{1,2\},\{1,3\},\{3,4\},\{3,5\}\},\\
&\{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\},\\
&\{\{1,2\},\{2,4\},\{3,4\},\{3,5\}\},\\
&\{\{1,2\},\{2,4\},\{3,4\},\{4,5\}\},\\
&\{\{1,3\},\{2,4\},\{3,4\},\{3,5\}\},\\
&\{\{1,3\},\{2,4\},\{3,4\},\{4,5\}\},\\
&\{\{1,4\},\{2,4\},\{3,4\},\{3,5\}\},\\
&\{\{1,4\},\{2,4\},\{3,4\},\{4,5\}\}.
\end{aligned}$$

But according to Corollary 3, the set Ω_r of optimal solutions which are robust in the neighborhood of $\delta = 0$ contains only the following two solutions:
$\{\{1,2\},\{2,4\},\{3,4\},\{3,5\}\}$ and $\{\{1,2\},\{2,4\},\{3,4\},\{4,5\}\}$.
Indeed, it is easy to see that for any $X \in \Omega_r$ we have

$$\min_{Y \in \Omega} w(c^o, X \cap Y) = 3,$$

whereas for any $X \in \Omega \setminus \Omega_r$,

$$\min_{Y \in \Omega} w(c^o, X \cap Y) = 1.$$

We can now calculate a lower bound for the robustness radius of any solution belonging to Ω_r. For any $X \in \Omega$ we have $w(c^o, X) = v(c^o) = 7$ and for any non-optimal feasible solution Y, $w(c^o, Y) = 8$, which gives $a = \frac{1}{7}$. Calculating b from (21) we obtain $b = \frac{3}{7}$. Finally, from Theorem 2 it follows, that for any $X \in \Omega_r$,

$$r^r(X) \geq \frac{1}{7}.$$

5. Conclusions

In this paper we consider the generic combinatorial optimization problem with inexact data. It is assumed that any coefficient in the objective function may differ from its initial value by at most δ percent of this value. Thus, in the framework of the so-called robust optimization with interval data, the parameter $\delta \in [0, 1)$ determines a particular set of scenarios.

We exploit our previous results concerning the accuracy function to derive lower bounds for perturbations of the objective function coefficients, for which an optimal solution, obtained for nominal values of these coefficients, remains robust.

To use directly these results one has to know the set of optimal solutions of the problem and at least some nontrivial lower bound for the relative error of any non-optimal solution. A straightforward approach to get such data consists in generating a sequence of so-called k-best solutions until the first one non-optimal solution is obtained. Algorithms of this type are developed for various combinatorial optimization problems (see, e.g., Hamacher and Queyranne, 1985; Katoh et al., 1981; van der Poort et al., 1999).

References

Averbakh, I. (2005) Computing and minimizing the relative regret in combinatorial optimization with interval data. *Discrete Optimization* **2**, 273-287.

Hamacher, H.W. and Queyranne, M. (1985) K best solutions to combinatorial optimization problems. *Annals of Operations Research* **4**, 123-143.

Kasperski, A. (2008) *Discrete Optimization with Interval Data. Minmax Regret and Fuzzy Approach.* Springer-Verlag, Berlin, Heidelberg.

Katoh, N., Ibaraki, T, and Mine, H. (1981) An algorithm for finding k minimum spanning trees. *SIAM Journal on Computing*, **10**, 245-255.

Kouvelis, P. and Yu, G. (1997) *Robust Discrete Optimization and Its Applications.* Kluwer Academic Publishers, Boston.

Libura, M. (1999) On accuracy of solutions for combinatorial optimization problems with perturbed coefficients of the objective function. *Annals of Operations Research* **86**, 53-62.

LIBURA, M. (2000) Quality of solutions for perturbed combinatorial optimization problems. *Control and Cybernetics* **29**, 199-219.

LIBURA, M. and NIKULIN, Y. (2004) Stability and accuracy functions in multicriteria combinatorial optimization problems with Σ-MINMAX and Σ-MINMIN partial criteria. *Control and Cybernetics* **33**, 511-524.

LIBURA, M. and NIKULIN, Y (2006) Stability and accuracy functions in multicriteria linear combinatorial optimization problems. *Annals of Operations Research* **147**, 255-267.

OGUZ, O. (2000) Bounds on the opportunity costs of neglecting reoptimization in mathematical programming. *Management Science* **46**, 1009-1012.

VAN DER POORT, E., LIBURA, M., SIERKSMA, G. and VAN DER VEEN, J.A.A. (1999) Solving the k-best traveling salesman problem. *Computers & Operations Research* **26**, 409-425.

WENDELL, R.E. (2004) Tolerances sensitivity and optimality bounds in linear programming. *Management Science* **50**, 797-803.