Control and Cybernetics

vol. **38** (2009) No. 3

Minimum variance control of discrete-time and continuous-time LTI MIMO systems – a new unified framework*

by

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Abstract: This paper presents a new uniform framework for solving the problem of minimum variance control of both discrete-time and continuous-time linear time-invariant multi-input multi-output systems described by general input-output models. Rather surprisingly, it is shown that the continuous-time case can be analyzed and synthesized without the necessity of involving the celebrated (and rather complex) theory of output predictor emulation, so that quite similar, simple solution is obtained like for the well-known discrete-time case.

Keywords: minimum variance control, continuous-time minimum variance control, theory of emulation, multivariable systems, multivariable zeros.

1. Introduction

The minimum variance control (MVC) problem has originally been formulated and solved for LTI discrete-time systems, at first SISO (Åström, 1970; Åström, and Wittenmark, 1973) and later square MIMO ones (Borisson, 1979; Keviczky and Hetthessy, 1977; Koivo, 1979). The problem has not since been given extensive research interest, apparently due to the lack of robustness of MVC and its instability for nonminimum phase systems. Taking advantage of the discrete-time control experience, in particular in terms of robust GPC (generalized predictive control) (Bitmead, Gevers and Wertz, 1990; Clarke and Mohtadi, 1989), continuous-time MVC and MVC-related strategies (analogous to GMVC, Clarke and Gawthrop, 1975 and 1979 and WMVC, Moir and Grimble, 1986) have been skipped over to immediately arrive at, e.g., robust continuous-time

*Submitted: January 2008; Accepted: June 2009.

GPC (Demircioğlu and Gawthrop, 1991, 1992). Notwithstanding, an important inheritance of the original MVC research has been to redefine the discrete-time *minimum phase* LTI SISO systems as those whose transfer function zeros lie strictly inside the unit disc, or those 'stable invertible', or in, other words, those systems for which MVC is asymptotically stable. This redefinition, probably due to Åström's group (Åström, 1970; Åström and Witternmark, 1973), has soon been extended to the square MIMO case (Borisson, 1979; Keviczky and Hetthessy, 1977; Koivo, 1979), involving the transmission zeros. This has later turned attention to the MVC problem for *nonsquare* LTI MIMO systems, giving rise to the introduction of new 'multivariable' zeros, i.e. the so-called 'control zeros' (Latawiec, 1998; Latawiec, Bańka and Tokarzewski, 2000). The rationale has been that a zero-reference MVC law can just be zeroing a deterministic part of the output predictor, thus resulting in that zero-reference MVC, perfect regulation and *output-zeroing* control laws are all identical.

Control zeros, being an extension of transmission zeros to nonsquare systems, have originally been introduced for discrete-time systems (Latawiec, 1998; Latawiec, Bańka and Tokarzewski, 2000). The continuous-time MVC (CMVC) problem has only recently been effectively tackled (Hunek, 2003; Latawiec, 2004) in order to extend the definition of control zeros to continuous-time LTI MIMO systems. It has turned out that the CMVC problem is more complex (than MVC) to solve and it requires involving the theory of emulation of certain unrealizable operations (like continuous-time output prediction), see Gawthrop (1987, 1990), Demircioğlu (1989), Demircioğlu and Gawthrop (1991, 1992), Gawthrop, Jones and Sbarbaro (1996), Latawiec and Hunek (2002), Hunek (2003), Latawiec (2004). However, a closer inspection of the CMVC law has revealed striking similarities between discrete-time and continuous-time solutions (Latawiec 2004). This makes it possible to present both discrete-time and continuous-time MVC cases in a unified framework and to offer a new, general MVC/CMVC solution. It is interesting that, in the new uniform solution, the complicated emulation theory is entirely circumvented for the continuous-time case.

It is important that our unified MVC framework is based on the assumption that, in the s-domain continuous-time problem, the time-delay term is Padé-approximated right at the beginning of the control synthesis process. This follows the lines of Gawthrop's celebrated continuous-time solution to the GPC problem (Gawthrop, 1987, 1990). It is worth noticing that effective alternative approaches have been offered for s-domain solution of an LQG-type control problem using state space form (Grimble, 1979) and Wiener form (Grimble, 1980). In those control synthesis procedures, a Padé approximation for the delay term is applied at the end of the synthesis process, which can make the approaches and solutions more general. However, applying those approaches to our case cannot take us any further as our unified framework can be obtained only when the Padé approximation is introduced at the beginning of the procedure. This is how our unified MVC framework is of a limited scope. But, on the other hand, it is worth mentioning that, for the simple CMVC problems, the end of the control design procedure is in fact very close to the beginning, which is quite different from the case of the LQG-type control design.

This paper is organized as follows. Having introduced the MVC/CMVC problems, a uniform model valid for both discrete-time and continuous-time systems is presented in Section 2. Sections 3 and 4 give the solutions to the MVC problem for discrete-time and continuous-time systems, respectively, with the reference in the latter case to the output predictor emulation theory. A new unified MVC framework for discrete-time and continuous-time systems is presented in Section 5. A new concept of robust MVC for nonsquare LTI MIMO systems is introduced in Section 6. Simulation examples are provided in Section 7 and new results of the paper are summarized in the conclusions of Section 8.

2. System representations

Consider an n_u-input n_y-output LTI discrete-time system governed by the input-output description

$$\underline{A}(q^{-1})y(t) = q^{-d}\underline{B}(q^{-1})u(t) + \underline{C}(q^{-1})v(t) \tag{1}$$

where $u(t)$ and $y(t)$ are the input and output vectors, respectively, $v(t)$ is the zero-mean uncorrelated disturbance vector, d is the time delay and $\underline{A}(q^{-1})$, $\underline{B}(q^{-1})$ and $\underline{C}(q^{-1})$ are the appropriate matrix polynomials (in the backward shift operator q^{-1}) of orders n, m and l, respectively. As usual, we assume that the leading coefficient of $\underline{A}(q^{-1})$ is equal to the identity matrix. Assume that $\underline{A}(q^{-1})$ and $\underline{B}(q^{-1})$ as well as $\underline{A}(q^{-1})$ and $\underline{C}(q^{-1})$ are left coprime, with $\underline{B}(q^{-1})$ and (stable) $\underline{C}(q^{-1})$ being of full normal rank n_y. For general purposes and for duality with the continuous-time case, we use here the ARMAX model, even though it is well known that the $\underline{C}(q^{-1})$ polynomial matrix of disturbance parameters is usually in control engineering practice unlikely to be effectively estimated (and is often used as a control design, observer polynomial matrix instead).

Consider also an n_u-input n_y-output LTI continuous-time system governed by the input-output description

$$A(s)Y(s) = e^{-sT_0}B(s)U(s) + C(s)V(s) \tag{2a}$$

where $U(s)$, $Y(s)$ and $V(s)$ are the Laplace transforms of the respective input, output and zero-mean uncorrelated disturbance vectors, T_0 is the time delay and $A(s)$, $B(s)$ and $C(s)$ are the appropriate matrix polynomials in the Laplace operator s, with all the assumptions for the matrix polynomials $\underline{A}(q^{-1})$, $\underline{B}(q^{-1})$ and $\underline{C}(q^{-1})$ remaining valid here for the matrix polynomials $A(s)$, $B(s)$ and $C(s)$.

The continuous-time delay factor will be Padé-approximated as $e^{-sT_0} \approx T(-s)/T(s)$, where $T(s)$ is a stable polynomial of order n_t (Hunek, 2003; Kowalczuk and Suchomski, 1997). (Note: A more general Padé approximation can also be used, with $e^{-sT_0} \approx T_1(-s)/T_2(s)$ and the stable polynomials $T_1(s)$ and $T_2(s)$ of orders n_1 and $n_2 > n_1$, respectively, Holst, 1969 .) Thus we will proceed with the continuous-time system description

$$A(s)Y(s) = \frac{T(-s)}{T(s)}B(s)U(s) + C(s)V(s). \tag{2b}$$

It will be seen from general MVC considerations that system equations (1) and (2b) can be presented in the following general form, valid for *both* discrete-time and continuous-time systems

$$\mathcal{A}(\pi)\mathcal{Y} = \mathcal{D}(\pi)\mathcal{B}(\pi)\mathcal{U} + \mathcal{C}(\pi)\mathcal{V} \tag{3}$$

where for discrete-time systems we have $\pi = q^{-1}$, $\mathcal{A}(.) = \underline{A}(.)$, $\mathcal{B}(.) = \underline{B}(.)$, $\mathcal{C}(.) = \underline{C}(.)$, $\mathcal{Y} = y(t)$, $\mathcal{U} = u(t)$, $\mathcal{V} = v(t)$ and $\mathcal{D}(.) = q^{-d}$, whereas for continuous-time ones there is $\pi = s$, $\mathcal{A}(.) = A(.)$, $\mathcal{B}(.) = B(.)$, $\mathcal{C}(.) = C(.)$, $\mathcal{Y} = Y(s)$, $\mathcal{U} = U(s)$, $\mathcal{V} = V(s)$ and $\mathcal{D}(.) = T(-s)/T(s)$. As usual, we assume from now on that $\mathcal{C}(.)$ is stable and $\mathcal{A}(.)$, $\mathcal{B}(.)$ and $\mathcal{A}(.)$, $\mathcal{C}(.)$ are left coprime. Also, we assume for clarity that time delays with respect to all inputs are equal. The case of different time delays in various inputs is considered in Latawiec (1998).

3. Minimum variance control (discrete-time systems)

THEOREM 1 (MINIMUM VARIANCE CONTROL) (Latawiec, 2004) *Let an LTI discrete-time system be described by the left coprime ARMAX model (1), with* $\underline{B}(q^{-1})$ *and* $\underline{C}(q^{-1})$ *being of full normal rank* n_y. *Then the general MVC law, minimizing* $E\{\|y(t+d) - y_{ref}(t+d)\|^2\}$ *with respect to* $u(t)$, *is of the form*

$$u(t) = \underline{B}^{\mathrm{R}}\tilde{F}^{-1}(q^{-1})[\tilde{\underline{C}}(q^{-1})y_{ref}(t+d) - \tilde{H}(q^{-1})y(t)] \tag{4}$$

where $y_{ref}(t)$ *is the output reference,* $\underline{B}^{\mathrm{R}}(q^{-1})$ *is a right inverse of* $\underline{B}(q^{-1})$ *and the appropriate polynomial matrices* $\tilde{F}^{-1}(q^{-1})$ *and* $\tilde{H}^{-1}(q^{-1})$ *(both of dimensions* $n_y \times n_y$*) are computed from the polynomial matrix identity*

$$\tilde{\underline{C}}(q^{-1}) = \tilde{F}(q^{-1})\underline{A}(q^{-1}) + q^{-d}\tilde{H}(q^{-1}) \tag{5}$$

with

$$\tilde{\underline{C}}(q^{-1})F(q^{-1}) = \tilde{F}(q^{-1})\underline{C}(q^{-1}) \tag{6}$$

and $\tilde{F}(q^{-1}) = I + \tilde{f}_1 q^{-1} + \ldots + \tilde{f}_{d-1}q^{-d+1}$, $\tilde{H}(q^{-1}) = \tilde{h}_0 + \tilde{h}_1 q^{-1} + \ldots + \tilde{h}_{n-1}q^{-n+1}$.

Proof. Since the familiar output predictor

$$y(t+d) = \tilde{\underline{C}}^{-1}(q^{-1})[\tilde{F}(q^{-1})\underline{B}(q^{-1})u(t) + \tilde{H}(q^{-1})y(t)] + F(q^{-1})v(t)$$

is precisely the same as for the square MIMO case (Borisson, 1979; Keviczky and Hetthessy, 1977; Koivo, 1979), the result follows (asymptotically) from the minimization of the performance index. ∎

A general case of non-full normal rank systems is more difficult to tackle due to the generalized inverse of the product $\tilde{F}(q^{-1})\underline{B}(q^{-1})$ then involved in equation (4). Still, recall that the case of non-full normal rank systems is ruled out in practical engineering situations (Latawiec, 1998). Anyway, we will continue with full normal rank systems, implying that, in general, $\mathcal{B}(\pi)$ is of full normal rank.

THEOREM 2 (STABILITY OF MVC) *Let an LTI discrete-time system be described by the left coprime ARMAX model (1), with $\underline{B}(q^{-1})$ and $\underline{C}(q^{-1})$ being of full normal rank n_y. Then the MVC law (4), where $\tilde{F}(q^{-1})$ and $\tilde{H}(q^{-1})$ are as above, is asymptotically stable iff $\underline{B}(q^{-1})$ is stably (right-)invertible.*

Proof. Combining the model equation (1) with the control law (4) while employing the identity (5) and equation (6) one obtains

$$\{I + F(q^{-1})\underline{C}^{-1}(q^{-1})[\underline{B}(q^{-1})\underline{B}^{\mathrm{R}}(q^{-1}) - I]\underline{A}(q^{-1})\}\hat{y}(t+d) = y_{ref}(t+d),$$

with $y(t+d) = \hat{y}(t+d) + F(q^{-1})v(t+d)$. Since $\underline{C}(q^{-1})$ is stable, the result follows by virtue of the standard stability arguments. ∎

REMARK 1 *Note that MVC is a sort of 'inverse-model control' in that it involves the factor $\underline{B}^{\mathrm{R}}(q^{-1})$ of the inverse system representation $q^d\underline{B}^{\mathrm{R}}(q^{-1})\underline{A}(q^{-1})$. A variety of types of optimal right inverses of $\underline{B}(q^{-1})$ have been introduced in Latawiec (2004), Latawiec, Hunek and Łukaniszyn (2005).*

REMARK 2 *A 'symmetric' (to Theorem 2) result, involving left invertibility of $\underline{B}(q^{-1})$, can be easily obtained. However, in case of left invertible systems, MVC is not quite 'minimum variance' here in that the absolute minimum $E[\|F(q^{-1})v(t)\|^2]$ of the MVC performance index (or zero for perfect regulation) cannot be reached, in general (Latawiec, 1998, 2004).*

4. Continuous-time minimum variance control

We will first tackle the CMVC problem from the original viewpoint of *emulation* of the output predictor transform (Gawthrop 1987, 1990; Demircioğlu 1989; Demircioğlu and Gawthrop, 1991, 1992 and 2002; Gawthrop, Jones and Sbararo, 1996; Kowalczuk and Suchomski, 1997; Latawiec and Hunek, 2002; Hunek, 2003; Latawiec, 2004). Full details of the emulator-based CMVC results can be found in Hunek (2003), Latawiec (2004) and Latawiec and Hunek (2002). Then, we will approach the CMVC problem right in the same way as for discrete-time systems in order to arrive at a new result concerning the uniform MVC treatment for both discrete-time and continuous-time forms.

Consider an LTI MIMO system described by the s-domain model (2a). Again we assume that $A(s)$ and $B(s)$ as well as $A(s)$ and $C(s)$ are left coprime, with $B(s)$ and (stable) $C(s)$ being of full normal rank n_y. In the continuous-time case, the polynomial matrix $C(s)$ (usually considered an observer) cannot be of arbitrary order (l) due to physical realization reasons (Gawthrop, 1987; Demircioğlu, 1989; Demircioğlu and Gawthrop, 1991; Hunek, 2003). In many cases, differentiation of disturbances should be avoided, which can be provided by choosing $l = n - 1$. Also, direct use of the plant output (without its low-pass filtering) is in some cases undesirable, which can be obtained by selecting $l = n$. Thus, we will assume that either $l = n - 1$ or $l = n$.

Referring to the models (2a) and (2b), the transform $Y_T(s) = e^{sT_0}Y(s)$ of the output predictor is now approximately given as

$$Y_T(s) = A^{-1}(s)B(s)U(s) + \frac{T(s)}{T(-s)}A^{-1}(s)C(s)V(s). \tag{7}$$

Accounting for (7), introduce the factorization (Demircioğlu and Gawthrop, 2002)

$$A^{-1}(s)C(s) = \tilde{C}(s)\tilde{A}^{-1}(s) \tag{8}$$

which is quite similar to (6), in terms of matrix incommutability.

Define the matrix polynomial identity (compare Gawthrop, 1987, Demircioğlu, 1989, for the SISO case)

$$\frac{T(s)}{T(-s)}\tilde{C}(s)\tilde{A}^{-1}(s) = \frac{E_T(s)}{T(-s)} + F_T(s)\tilde{A}^{-1}(s) \tag{9}$$

which constitutes a decomposition of the disturbance transform into a strictly proper, realizable part $F_T(s)\tilde{A}^{-1}(s)$ and an unrealizable component $E_T(s)/T(-s)$ (as $T(-s)$ is unstable, Demircioğlu, 1989, Gawthrop, 1987), where $\deg(F_T) = \deg(A) - 1$, $\deg(\tilde{A}) = \deg(A)$, $\deg(\tilde{C}) = \deg(C)$ and $E_T(s)$ is of full normal rank n_y, with $\deg(E_T) = \deg(T)$ or $\deg(E_T) = (\deg(T)) - 1$ for $\deg(C) = \deg(A)$ or $\deg(C) = (\deg(A)) - 1$, respectively.

The transform of the output predictor can now be presented as

$$Y_T(s) = \bar{Y}_T(s) + \bar{E}_T(s) \tag{10}$$

where $\bar{Y}_T(s)$ is the predictor emulator and $\bar{E}_T(s) = \frac{E_T(s)}{T(-s)}V(s)$ represents the emulation error.

The output predictor emulator is now of the form (Hunek, 2003; Latawiec, 2004; Latawiec and Hunek, 2002)

$$\bar{Y}_T(s) = \frac{1}{T(s)}E_T(s)C^{-1}(s)B(s)U(s) + F_T(s)\tilde{C}^{-1}(s)Y(s). \tag{11}$$

Equating, in the standard way, the emulator $\bar{Y}_T(s)$ to the reference transform $Y_{ref}(s)$ we arrive at the CMVC law (Hunek, 2003; Latawiec, 2004; Latawiec and Hunek, 2002)

$$U(s) = T(s)B^{\mathrm{R}}(s)C(s){E_T}^{-1}(s)[Y_{ref}(s) - F_T(s)\tilde{C}^{-1}(s)Y(s)]. \tag{12}$$

Remark 3 *It is interesting to note that the solution (12) can be obtained by quite formal (without any physical interpretation) minimization of the "performance index"*

$$E\{[Y_T(s)-Y_{ref}(s)]^{\mathrm{T}}[Y_T(s)-Y_{ref}(s)]\} \text{ or } E\{[Y_T(s)-Y_{ref}(s)]^{*}[Y_T(s)-Y_{ref}(s)]\}$$

with respect to $U(s)$, where $(.)^{\mathrm{T}}$ and $(.)^{*}$ are regular and conjugate transposes, respectively. In fact, by taking e.g.

$$\frac{\partial E\{[Y_T(s)-Y_{ref}(s)]^{\mathrm{T}}[Y_T(s)-Y_{ref}(s)]\}}{\partial U(s)} = 2E\{[Y_T(s)-Y_{ref}(s)]^{\mathrm{T}}\}\frac{\partial E[Y_T(s)]}{\partial U(s)}$$

and assuming $E[\bar{E}_T(s)] = 0$ (since $v(t)$ is a zero-mean disturbance) we arrive at equation (12). It should be emphasized that the above peculiar stochastic minimization formulation is deprived of any physical interpretation (what is the expectation of a transform?), but still it is strongly supported by 1) MVC/CMVC routine of equating the output predictor/emulator to the reference/reference transform, and 2) right the same derivation process used for disrete-time domain formulated MVC. On the other hand, the peculiar stochastic minimization formulation appears to support the unified discrete-time/continuous-time MVC framework.

Theorem 3 (Stability of CMVC) *Let an LTI MIMO continuous-time system be described by the left coprime model (2b), with $B(s)$ and $C(s)$ being of full normal rank n_y. Then the CMVC law (12), where $E_T(s)$ and $F_T(s)$ are defined in equations (9) and (8), is asymptotically stable iff $B(s)$ is stably (right-) invertible.*

Proof. Similarly to the proof of Theorem 2, combining (2b) and (12), while accounting for (8), (9), (10) and (11) we arrive at $\{I+E_T(s)C^{-1}(s)[B(s)B^{\mathrm{R}}(s)-I]A(s)/T(s)\}\bar{Y}_T(s) = Y_{ref}(s)$. Since $C(s)$ and $T(s)$ are stable, the result follows by virtue of the standard stability arguments. ■

It is interesting that the closed-loop system equation in the proof of Theorem 3 is quite similar to that in the proof of Theorem 2. This indicates that the CMVC control law (12) could be synthesized in an alternative way, just like for discrete-time systems. In such a case, we could formally get around the whole emulation theory, which would be rather surprising.

THEOREM 4 (CONTINUOUS-TIME MINIMUM VARIANCE CONTROL REVISITED) *Let an LTI MIMO continuous-time system be described by the left coprime model (2b), with $B(s)$ and $C(s)$ being of full normal rank n_y. Then the general MVC law, minimizing (quite formally) $E\{[Y_T(s) - Y_{ref}(s)]^{\mathrm{T}}[Y_T(s) - Y_{ref}(s)]\}$ or $E\{[Y_T(s) - Y_{ref}(s)]^{*}[Y_T(s) - Y_{ref}(s)]\}$ with respect to $U(s)$, is of the form*

$$U(s) = T(s)B^{\mathrm{R}}(s)\tilde{F}^{-1}(s)[\tilde{C}(s)Y_{ref}(s) - \tilde{H}(s)Y(s)] \tag{13}$$

where the appropriate polynomial matrices $\tilde{F} \in \mathbb{R}^{n_y \times n_y}[s]$ and $\tilde{H} \in \mathbb{R}^{n_y \times n_y}[s]$ are computed from the polynomial matrix identity

$$\tilde{C}(s) = \tilde{F}(s)A(s)/T(s) + \tilde{H}(s)T(-s)/T(s) \tag{14}$$

with

$$\tilde{C}(s)F(s) = \tilde{F}(s)C(s) \tag{15}$$

and $\tilde{H}(s) = \tilde{h}_0 + \tilde{h}_1 s + ... + \tilde{h}_{n-1}s^{n-1}$, $\tilde{F}(s) = I + \tilde{f}_1 s + ... + \tilde{f}_{n_f}s^{n_f}$, with $n_f = n_t$ $(= deg[(T(s)])$ for $l = n$ or $n_f = n_t - 1$ for $l = n - 1$.

Proof. Since the output predictor transform can now be easily presented as $Y_T(s) = \tilde{C}^{-1}(s)[\tilde{H}(s)Y(s) + \tilde{F}(s)B(s)U(s)/T(s)] + F(s)V(s)/T(-s)$, the result follows. ■

REMARK 4 *It is striking how similar is the above result to that for the discrete-time case. With $B(s)$ being right-invertible, the whole analysis and synthesis problems are practically identical for discrete-time and continuous-time systems. Of course, the stability analysis is right the same as for Theorem 3, with the closed-loop system equation $\{I + F(s)C^{-1}(s)[B(s)B^{\mathrm{R}}(s) - I]A(s)/T(s)\}\bar{Y}_T(s) = Y_{ref}(s)$, the result being 'almost' the same as for discrete-time systems.*

5. Uniform treatment of discrete-time and continuous-time systems

The above enables to formulate new, general, uniform results valid for both discrete- and continuous-time systems.

THEOREM 5 (DISCRETE-TIME/CONTINUOUS-TIME MINIMUM VARIANCE CONTROL) *Let an LTI discrete-time or continuous-time system be described by the left coprime model (3), with $\mathcal{B}(\pi)$ and $\mathcal{C}(\pi)$ being of full normal rank n_y. Then the general MVC/CMVC law, minimizing (quite formally) $E\{[\mathcal{Y}_+ - \mathcal{Y}_{ref}]^{\mathrm{T}}[\mathcal{Y}_+ - \mathcal{Y}_{ref}]\}$ or $E\{[\mathcal{Y}_+ - \mathcal{Y}_{ref}]^{*}[\mathcal{Y}_+ - \mathcal{Y}_{ref}]\}$ with respect to $\mathcal{U}$, where $\mathcal{Y}_+ = y(t+d)$ and $\mathcal{Y}_{ref} = y_{ref}(t+d)$ for discrete-time and $\mathcal{Y}_+ = Y_T(s)$ and $\mathcal{Y}_{ref} = Y_{ref}(s)$ for continuous-time systems, is of the form*

$$\mathcal{U} = T_+\mathcal{B}^{\mathrm{R}}(\pi)\tilde{\mathcal{F}}^{-1}(\pi)[\tilde{\mathcal{C}}(\pi)\mathcal{Y}_{ref} - \tilde{\mathcal{H}}(\pi)\mathcal{Y}] \tag{16}$$

where the appropriate polynomial matrices $\tilde{\mathcal{F}} \in \mathbb{R}^{n_y \times n_y}[\pi]$ *and* $\tilde{\mathcal{H}} \in \mathbb{R}^{n_y \times n_y}[\pi]$ *are computed from the matrix polynomial identity*

$$\tilde{\mathcal{C}}(\pi) = \tilde{\mathcal{F}}(\pi)\mathcal{A}(\pi)/T_+ + \mathcal{D}(\pi)\tilde{\mathcal{H}}(\pi) \tag{17}$$

with

$$\tilde{\mathcal{C}}(\pi)\mathcal{F}(\pi) = \tilde{\mathcal{F}}(\pi)\mathcal{C}(\pi) \tag{18}$$

where $\tilde{\mathcal{H}}(\pi) = \tilde{h}_0 + \tilde{h}_1\pi + ... + \tilde{h}_{n-1}\pi^{n-1}$*,* $\tilde{\mathcal{F}}(\pi) = I + \tilde{f}_1\pi + ... + \tilde{f}_{n_f}\pi^{n_f}$*, with* $T_+ = 1$ *for discrete-time and* $T_+ = T(s)$ *for continuous-time systems, and* $n_f = d-1$ *for discrete-time and* $n_f = deg[(T(s)]$ *for* $l = n$ *or* $n_f = deg[(T(s)]-1$ *for* $l = n-1$ *for continuous-time systems.*

Proof. Since the output predictor (or output predictor transform) can now be easily presented as $\mathcal{Y}_+ = \tilde{\mathcal{C}}^{-1}(\pi)[\tilde{\mathcal{H}}(\pi)\mathcal{Y} + \tilde{\mathcal{F}}(\pi)\mathcal{B}(\pi)\mathcal{U}/T_+] + \mathcal{F}(\pi)\mathcal{V}/T_-$, with $T_- = 1$ for discrete-time and $T_- = T(-s)$ for continuous-time systems, the result follows. ■

THEOREM 6 (STABILITY OF MVC/CMVC) *Let an LTI discrete-time or continuous-time system be described by the left coprime model (3), with* $\mathcal{B}(\pi)$ *and* $\mathcal{C}(\pi)$ *being of full normal rank* n_y*. Then the MVC/CMVC law (16), where* $\tilde{\mathcal{F}}(\pi)$ *and* $\tilde{\mathcal{H}}(\pi)$ *are defined as above, is asymptotically stable iff* $\mathcal{B}(\pi)$ *is stably (right-)invertible.*

Proof. Similarly to the proofs of Theorems 2 and 3, combining equations (3) and (16), while accounting for (17) and (18) as well as for the predictor equation $\mathcal{Y}_+ = \hat{\mathcal{Y}}_+ + \mathcal{F}(\pi)\mathcal{V}/T_-$, where $\hat{\mathcal{Y}}_+ = \tilde{\mathcal{C}}^{-1}(\pi)[\tilde{\mathcal{H}}(\pi)\mathcal{Y} + \tilde{\mathcal{F}}(\pi)\mathcal{B}(\pi)\mathcal{U}/T_+]$, we arrive at $\{I + \mathcal{F}(\pi)\mathcal{C}^{-1}(\pi)[\mathcal{B}(\pi)\mathcal{B}^{\mathrm{R}}(\pi) - I]\mathcal{A}(\pi)/T_+\}\hat{\mathcal{Y}}_+ = \mathcal{Y}_{ref}$, with T_+ and T_- as above. Since $\mathcal{C}(\pi)$ and T_+ are stable, the result follows by virtue of the standard stability arguments. ■

REMARK 5 *It is worth mentioning that solutions to the Generalized MVC (GMVC) problems have been given both for discrete-time (Latawiec, 1998) and continuous-time systems (Hunek, 2003). Those control problems can as well be given a uniform formulation, easily extending the above MVC/CMVC results. However, much more interesting would be an extension of the unifying framework to discrete-time/continuous-time GPC, which is left for future research.*

6. Robust MVC for nonsquare LTI MIMO systems

A variety of types of optimal (minimum norm) right inverses for $\mathcal{B}(\pi)$ have given rise to defining of new 'multivariable' zeros, being poles of $\mathcal{B}^{\mathrm{R}}(\pi)$ and called control zeros (type 1 and type 2) (Latawiec, 1998, 2004, 2005; Latawiec and Hunek, 2002; Hunek 2003; Latawiec, Hunek and Łukaniszyn, 2004, 2005).

As a matter of fact, we are in a position to offer an infinite number of optimal right inverses for $\mathcal{B}(\pi)$ (Latawiec, 2004), from which we can select such inverse(s) that can provide robustness of MVC/CMVC (Latawiec, Hunek and Adamek, 2005; Hunek, Latawiec and Łukaniszyn, 2006; Hunek, Latawiec and Stanisławski, 2007; Hunek, 2008). In particular, stable MVC/CMVC can be obtained even for the case when all channels of a nonsquare LTI MIMO system are nonminimum phase (Hunek, Latawiec and Łukaniszyn, 2006; Hunek, 2008), this result being rather surprising. This result will be illustrated on simple examples below. Moreover, our new results concerning the stability of MVC/CMVC for nonsquare LTI MIMO systems without transmission zeros show that we can *always* find a no-pole solution for $\mathcal{B}^{\mathrm{R}}(\pi)$, thus providing the tools for generic treatment of *any* nonsquare LTI MIMO system as minimum phase (Hunek, Latawiec and Łukaniszyn, 2006; Hunek, Latawiec and Stanisławski, 2007; Hunek, 2008). Since nonsquare systems generically have no transmission zeros, this implies that the *structural stability* of MVC/CMVC can generically be obtained for nonsquare LTI MIMO systems, the feature being characteristic of, e.g., the LQR strategy. This can offer a new tool for design of robust MVC/CMVC for nonsquare LTI MIMO systems (Hunek, 2008), thus obviating, at least partly, the need for application of some other powerful (but computationally involving) optimal control strategies. Preliminary results on the issue of robustness of MVC/CMVC for nonsquare LTI MIMO systems, including selection of an order of a no-pole inverse $\mathcal{B}^{\mathrm{R}}(\pi)$, have been presented in Hunek, Latawiec and Łukaniszyn (2006), and Hunek, Latawiec and Stanisławski (2007). More complete results, beyond the scope of this paper, have been presented in Hunek (2008). Those results confirm the significance of (not quite well-known) control zeros, whose contribution to design of robust MVC is crucial.

7. Simulation examples

EXAMPLE 1 Consider a simple, two-input one-output continuous-time system governed by the model (2b), with $B(s) = [\ s - 1.2 \quad s\ ]$, $A(s) = s + 0.5$, $C(s) = s + 0.3$, $T_0 = 1$, $n_t = 3$, $\mathrm{var}\{v(t)\} = 1e - 3$. Notice that both channels of the system are nonminimum phase. Using the methods presented in Latawiec, Hunek and Adamek (2005), Hunek, Latawiec and Łukaniszyn (2006) we can find the no-pole inverse $B^{\mathrm{R}}(s) = \left[\ -\frac{25}{86}s - \frac{5}{6} \quad \frac{25}{86}s + \frac{125}{258}\ \right]^{\mathrm{T}}$ and an appropriate CMVC law according to (12) or (13) (for single-output systems it does not matter which equation is used). Fig. 1 presents the performance of CMVC as compared with perfect control ($v(t) = 0$). The performance is quite negligibly affected by the choice of n_t. Using the zero-order hold with the sampling period of 0.1 s, the following ARMAX model is obtained: $\underline{B}(q^{-1}) = [\ 1 - 1.117q^{-1} \quad 1 - q^{-1}\ ]$, $\underline{A}(q^{-1}) = 1 - 0.9512q^{-1}$, $\underline{C}(q^{-1}) = 1 - 0.9707q^{-1}$, with $d = 10$. Under a certain no-pole inverse (omitted for brevity) we arrive at the MVC law (4), whose performance is presented in Fig. 2. It is seen that the performance of MVC is better than that for CMVC

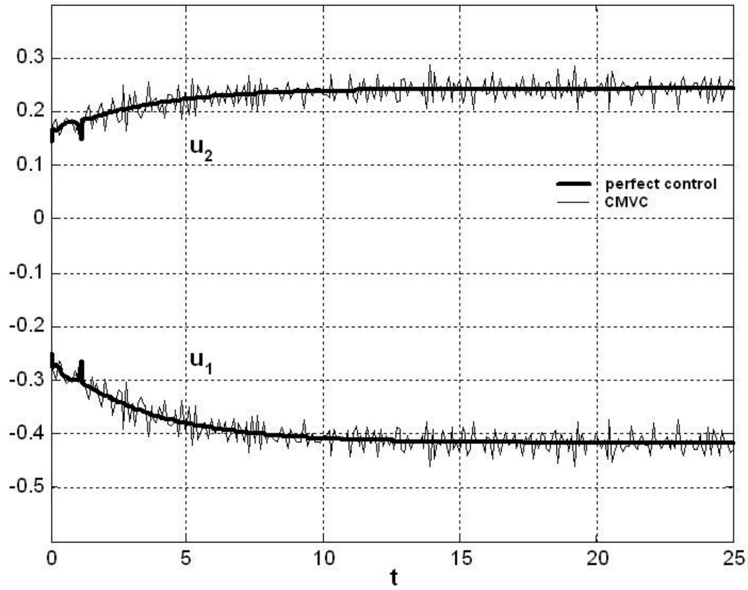

Figure1a. CMVC vs. perfect control: plots of the input signals u_1 and u_2 for Example 1.

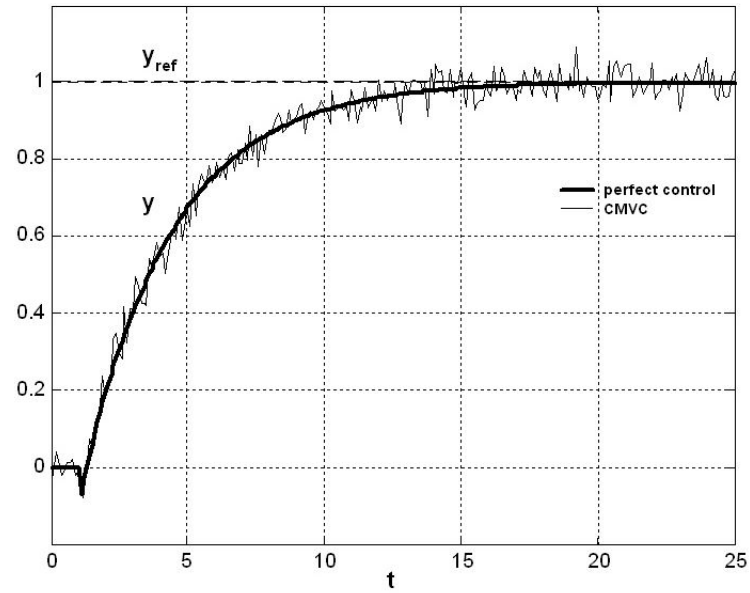

Figure1b. CMVC vs. perfect control: plots of the output and output reference signals, y and y_{ref}, respectively, for Example 1.

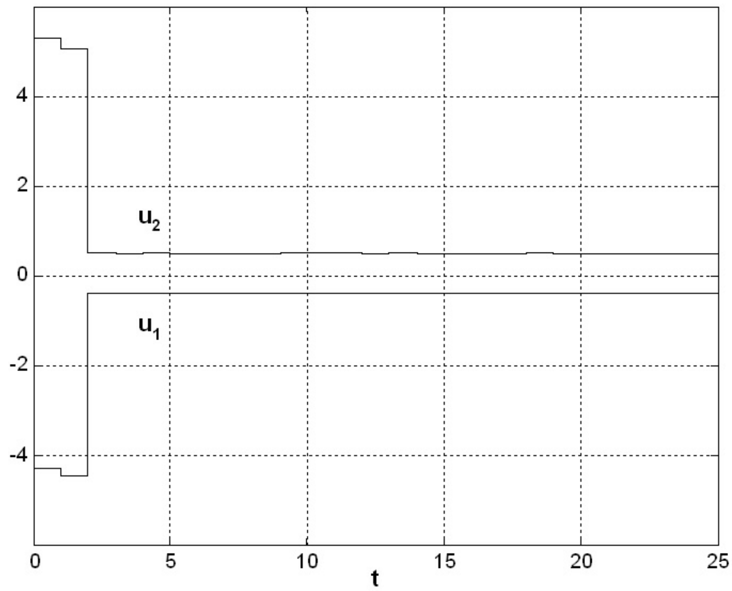

Figure 2a. MVC: plots of the input signals u_1 and u_2 for Example 1.

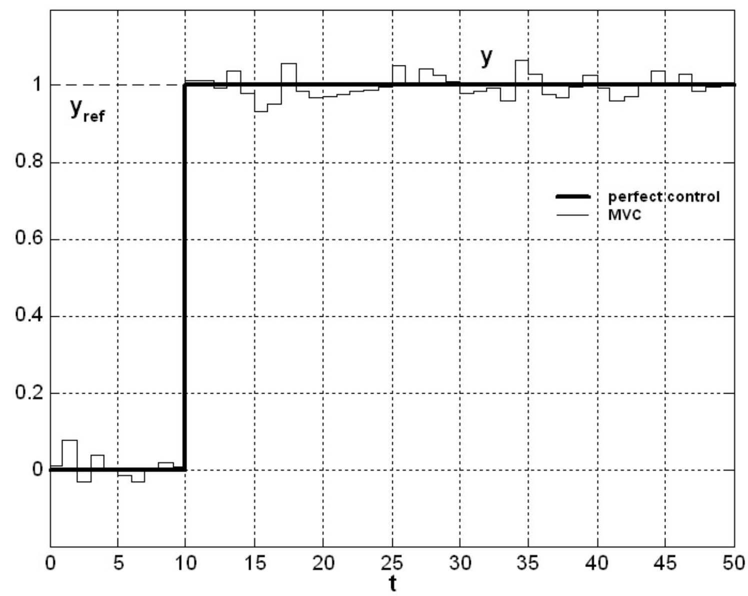

Figure 2b. MVC vs. perfect control: plots of the output and output reference signals, y and y_{ref}, respectively, for Example 1.

and this is inevitably caused by a rather poor quality of the Padé approximation for the time delay. We would like to additionally advocate the use of MVC rather than CMVC by noting that the Matlab/Simulink tools (version 2007a) for solving the continuous-time problems are sensitive to disturbances and open-loop unstable systems, the issues nonexisting for discrete-time systems. In order to illustrate the problem we consider Example 2.

Example 2 A two-input one-output unstable continuous-time system is described with $B(s) = [s^2 - \; 1.2s + 3 \quad -s^2 + 0.5s \;]$, $A(s) = s^2 + 0.6s - 0.8$, $C(s) = s^2 + 0.5s + 0.2$, $T_0 = 0.2$, $n_t = 3$, $\operatorname{var}\{v(t)\} = 1e - 3$. Again, both channels are nonminimum phase. Using the available Matlab/Simulink tools, it is not possible to find a numerically stable solution for CMVC, nor even for perfect control. In contrast, a discrete-time version of the system, with

$$\underline{B}(q^{-1}) = [\; 1 - 2.1058q^{-1} + 1.1350q^{-2} \quad -1 + 2.0525q^{-1} - 1.0525q^{-2} \;],$$
$$\underline{A}(q^{-1}) = 1 - 1.9495q^{-1} + 0.9418q^{-2},$$
$$\underline{C}(q^{-1}) = 1 - 1.9543q^{-1} + 0.9563q^{-2}$$

and $d = 2$, can be effectively stabilized under MVC with a certain no-pole inverse $\underline{B}^{\mathrm{R}}(q^{-1})$, which can be appreciated from Fig. 3.

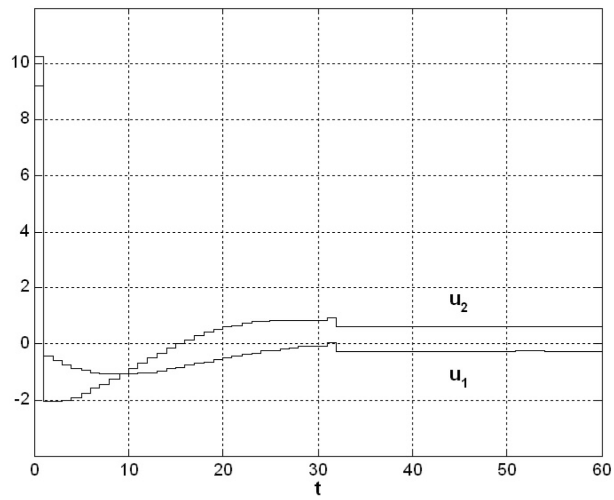

Figure 3a. MVC: plots of the input signals u_1 and u_2 for Example 2.

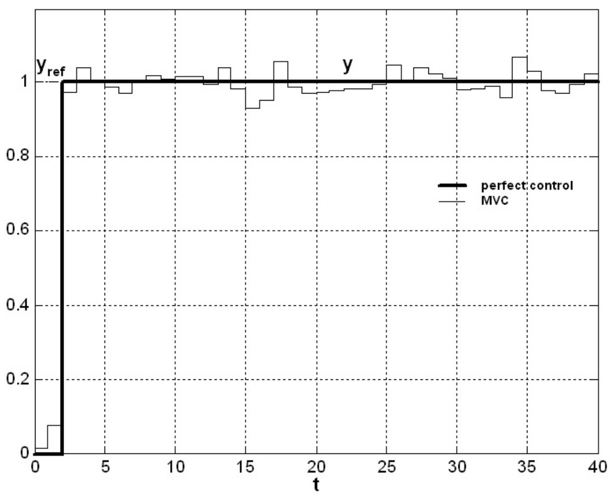

Figure 3b. MVC vs. perfect control: plots of the output and output reference signals, y and y_{ref}, respectively, for Example 2.

8. Conclusions

The minimum variance control problem for possibly nonsquare LTI MIMO systems has been presented in a new unifying framework, covering both discrete-time and continuous-time systems in a joint, compact manner. The unifying framework is valid under one limiting assumption made: the continuous-time delay term is Padé-approximated. It is surprising that in the uniform MVC solution valid for both discrete-time and continuous-time systems, no reference for the latter case is necessary to be made to the celebrated theory of emulation of the output predictor. The unifying framework allows for analyzing and synthesizing the MVC for both discrete-time and continuous-time systems within a uniform methodology, with stimulating implications for possible unification of more advanced control strategies. Also, new ideas for robust MVC for nonsquare LTI MIMO systems have preliminary been introduced on the basis of the MVC-related concept of control zeros.

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