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Persistent bounded disturbance rejection for uncertain time-delay systems*†

by

Man Sun[1] **and Yingmin Jia**[2]

[1]Department of Automation and Engineering,
Northeastern University at Quinhuangdao
Quinhuangdao 066004, Hebei, P.R. China
[2]Department of Systems and Control, Beihang University
Beijing 100083, P.R. China
e-mail: sun7661@126.com, ymjia@buaa.edu.cn

Abstract: This paper considers the problem of persistent bounded disturbance rejection for a class of time-delay systems with parametric uncertainty by Lyapunov function and positively invariant set analysis method. Sufficient conditions for internal stability and L_1-performance analysis are given in terms of linear matrix inequalities (LMIs). Based on the results, a simple approach to the design of a linear state-feedback controller is presented to stabilize robustly the uncertain time-delay systems and achieve a desired level of disturbance attenuation. All the obtained conditions are delay-dependent. A numerical example is included to illustrate the proposed method.

Keywords: persistent disturbance, time-delay system, linear matrix inequality (LMI), robust control.

1. Introduction

The problem of designing robust controllers for time-delay systems with parameter uncertainty attracted much attention in control system literature (Li and Jian, 1999; Kwon et al., 2006; Jeung et al., 1996). From the point of view of robust control design methods, the variable structure control concept has played an important role because of its robustness with respect to parameter uncertainties and external disturbances (Jafarov, 2005; Mahmoud, Shi and Shi, 2003; Szita and Sanathanan, 1997). The conditions in Jafarov (2005) and Szita and Sanathanan (1997) are, however not based on LMIs which can be solved

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very efficiently by various convex optimization algorithms. When the disturbances involved are persistently bounded with size measured in terms of peak time-domain values, it leads to the problem of peak-to-peak gain minimization, i.e., the L_1 or induced L_∞ problem, formulated in Vidyasagar (1986, 1991). It has recently attracted attention because of incorporating time domain specifications directly, see, e.g., Bobillo and Dahleh (1992), Abedor, Nagpal and Poola (1996), Hao et al., (2003), Blanchini and Sznaier (1995), Tang, Zhang and Ma (2004) or Lin, Zhai and Antsaklis (2003), and the references therein. However, among these works, analysis and control synthesis problems of persistent bounded disturbance rejection are studied for systems without time delay.

On the other hand, time delay is, in many cases, a source of instability. Therefore, the control problem for time-delay systems are of theoretical and practical importance. Many proposed methods have presented delay-independent conditions, Li and Jian (1999), Kwon et al. (2006). In general, these conditions are more conservative than the delay-dependent ones when the size of the time delay is small. There have been very few works concerning delay-dependent persistent bounded disturbance rejection problems so far, which motivates the present paper.

In this paper, we investigate the L_1-control problem for parametric uncertain linear time-delay systems subject to persistent bounded disturbances. By using Lyapunov function and invariant-set analysis method, we first establish an LMI condition that ensures internal stability and desired L_1-performance for the systems without uncertainty. Then we extend the result to uncertain time-delay systems. The result suggests a simple approach to the design of a state-feedback controller to robustly stabilize the system and to achieve a desired level of disturbance attenuation. We also give a numerical example to illustrate the theoretical result.

In this paper, $C_{n,d} = C([-d, 0], R^n)$ denotes a Banach space of continuous functions mapping the interval $[-d, 0]$ into R^n with the topology of uniform convergence. Then $x_t \in C_{n,d}$ can be defined by $x_t = x(t+\theta)$, $\theta \in [-d, 0]$. For simplification, we use the following notation. R is the set of all real numbers. R^n is the set of all $n-$tuples of real numbers. BR^l denotes the closed unit ball in the space R^l. The symbol $Sym\{\cdot\}$ denotes $Sym\{X\} \overset{\text{def}}{=} X + X^T$, and the symbol $*$ denotes the the submatrices that lie below the diagonal in a symmetric block matrix.

2. Problem formulation and preliminaries

Consider the following parametric uncertain time-delay system:

$$\begin{aligned}
&\dot{x}(t) = (A + \Delta_A)x(t) + (A_d + \Delta_{A_d})x(t-d) + (B + \Delta_B)u(t) + (B_1 + \Delta_{B_1})w(t) \\
&z(t) = (C + \Delta_C)x(t) + (D + \Delta_D)u(t) \qquad (1)\\
&x(t) = \phi(t), t \in [-d, 0]
\end{aligned}$$

where $x(t) : R \to R^n$, $u(t) : R \to R^m$, $w(t) : R \to R^l$ and $z(t) : R \to R^r$ are the state, the input, external disturbance, and the regulated output vectors, respectively. Further, A, A_d, B, B_1, C, D are known real constant matrices of appropriate dimensions. The time delay $d > 0$ is assumed to be known. $\phi(t) \in C_{n,d}$ is a real-valued continuous initial function on $[-d, 0]$. The parametric uncertainties are described by

$$[\Delta_A, \Delta_{A_d}, \Delta_B, \Delta_{B_1}] = F\Delta(t)[H_A, H_{A_d}, H_B, H_{B_1}] \tag{2}$$

$$[\Delta_C, \Delta_D] = E\Delta(t)[H_C, H_D] \tag{3}$$

where F, H_A, H_{A_d}, H_B, H_{B_1}, E, H_C, H_D are known matrices of appropriate dimensions, and $\Delta(t)$ is Lebesgue measurable and belongs to $\mathbf{\Delta} =: \{\Delta(t) : \Delta^T(t)\Delta(t) \leq I\}$. Assume that the admissible disturbance set is $W := \{w : R \to BR^l, w^T w \leq 1\}$. The L_∞ norm is defined by $\|w\|_\infty =: sup_t \|w(t)\|_2$. In $C_{n,d}$, the trajectory of the system (1) is denoted by x_t^ϕ. We firstly present the following definitions in $C_{n,d}$ which will be used in the development of our main results.

Definition 1 *A set Ξ is said to be positively invariant for dynamical system if the trajectory x_t^ϕ of the system remains in Ξ for all $t > 0$ and $w \in W$ whenever $\phi(t) \in \Xi$. Furthermore, a set Ξ is said to be robustly positively invariant for a dynamical system, if for $\phi(t) \in \Xi$, the trajectory x_t^ϕ of the system remains in Ξ for all $t > 0$, $w \in W$ and $\Delta \in \mathbf{\Delta}$.*

Definition 2 *Consider the case of $\phi(t) = 0$. Then the origin reachable set $R_\infty(0)$ is said to be the set that the trajectory x_t^0 of the system can reach from the origin, that is as follows.*

$$R_\infty(0) = \{x_t^0 | w \in W, \Delta \in \mathbf{\Delta}, t \geq 0\}.$$

It is the minimal closed positively invariant set containing the origin.

Definition 3 *A set Ω is said to be a robust attractor of a system with respect to $w \in W$, if all the state trajectories x_t^ϕ initiating from the exterior of Ω eventually enter and remain in Ω for all $w \in W$. Obviously, a robust attractor is also a positively invariant set.*

Definition 4 *For fixed $\Delta \in \mathbf{\Delta}$ (respectively $\Delta = 0$), define the performance set*

$$\Omega_\Delta(\rho) = \{x_t^\phi | \|z\|_\infty = \sup_{t\geq 0} \|(C + \Delta_C)x(t)\| \leq \rho, \forall w \in W, \Delta \in \mathbf{\Delta}\}$$

$$(\textit{respectively } \Omega(\rho) = \{x_t^\phi | \|z\|_\infty = \sup_{t\geq 0} \|Cx(t)\| \leq \rho, \forall w \in W\})$$

Thus, if $R_\infty(0) \subset \Omega_\Delta(\rho)$ for all $w \in W$, $\Delta \in \mathbf{\Delta}$, then the system has robust $\rho-$performance. Particularly, we say the system without uncertainty has $\rho-$performance for all $w \in W$ when $\Delta(t) \equiv 0$ $(R_\infty(0) \subset \Omega(\rho))$.

The objective of this paper is to find for system (1) a state-feedback control law $u(t) = Kx(t)$ with constant gain matrix $K \in R^{m\times n}$ such that the resulting closed-loop system

$$\begin{aligned}\dot{x}(t) &= (A + BK + \Delta_A + \Delta_B K)x(t) + (B_1 + \Delta_{B_1})w(t)\\ &\quad + (A_d + \Delta_{A_d})x(t-d)\\ z(t) &= (C + DK + \Delta_C + \Delta_D K)x(t)\end{aligned} \tag{4}$$

satisfies the following conditions:

(i) The closed-loop system is robustly internally stable, namely the system without external disturbance (i.e., $w = 0$) is robustly asymptotically stable.

(ii) For a given scalar $\rho > 0$, the system has robust $\rho-$performance, that is $\|z\|_\infty \le \rho$ for all $w \in W$ and $\Delta \in \boldsymbol{\Delta}$.

In dealing with the above problem, we will make use of the concept of robust attractor of a disturbed dynamical system. A set Ω is said to be a robust attractor of system (4) with respect to $w \in W$, if all the state trajectories of the system initiating from the exterior of Ω eventually enter and remain thereafter in Ω for all $w \in W$. Obviously, a robust attractor is robustly positively invariant.

The following lemmas will be useful in our discussion.

LEMMA 1 *(Khargonekar, Petersen and Zhou, 1990) We are given a symmetric matrix G and any matrices M, N of appropriate dimensions. Then*

$$G + M\Delta N + N^T\Delta^T M^T < 0$$

for all Δ satisfying $\Delta^T\Delta \le I$ if and only if there exists a constant $\varepsilon > 0$ such that

$$G + \varepsilon MM^T + \frac{1}{\varepsilon}N^T N < 0.$$

LEMMA 2 *(Petersen, 1987) Given a symmetric matrix G and any nonzero matrices M, N of appropriate dimensions, the following inequality holds:*

$$G + M\Delta N + N^T\Delta^T M^T \le 0$$

for all Δ satisfying $\Delta^T\Delta \le I$ if and only if there exists a constant $\varepsilon > 0$ such that

$$G + \varepsilon MM^T + \frac{1}{\varepsilon}N^T N \le 0.$$

LEMMA 3 *(Hao et al., 2002) Let P be an $n \times n$ matrix, then, for any scalar $\alpha > 0$, it follows that*

$$2x^T PHw \le \frac{1}{\alpha}x^T PHH^T Px + \alpha w^T w.$$

3. Main results

To formulate our problem, let us consider the following time-delay system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + B_1 w(t) \\ z(t) &= Cx(t) \\ x(t) &= \phi(t), t \in [-d, 0]. \end{aligned} \tag{5}$$

To analyze the stability of the system (5), we assume that A is stable. For a symmetric and positive-definite matrix P, let the ellipsoid $\Omega_P = \{x : x^T P x \leq 1\}$. We first consider system (5) for the above concepts.

THEOREM 1 *For prescribed positive scalars $\rho > 0$, $d > 0$, $\alpha_1 > 0$, $\sigma > 0$, if there exist symmetric and positive-definite matrices $P > 0, Q > 0$ such that*

$$\begin{bmatrix} (1,1) & -PA_d & 0 & d(A+A_d)^T Q & PB_1 \\ * & -Q & 0 & -dA_d^T Q & 0 \\ * & * & -I & dB_1^T Q & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\alpha_1 I \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} -\sigma P & 0 & C^T \\ * & -(\rho^2 - \sigma)I & 0 \\ * & * & -I \end{bmatrix} \leq 0 \tag{7}$$

where $(1,1) = Sym\{P(A+A_d)\} + (\alpha_1+1)P$. Then system (5) is stable and Ω_P is a robust attractor of it with respect to $w \in W$. Moreover, $\Omega_P \subset \Omega(\rho)$, and hence, the system has $\rho-$performance.

Proof. Consider a Lyapunov-Krasovskii functional candidate $V(t)$ as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where

$$\begin{aligned} V_1(t) &= x^T(t) P x(t) \\ V_2(t) &= \int_{t-d}^{t} \left[\int_s^t \dot{x}^T(\theta) d\theta\right] Q \left[\int_s^t \dot{x}(\theta) d\theta\right] ds \\ V_3(t) &= \int_0^d ds \int_{t-s}^{t} (\theta - t + s) \dot{x}^T(\theta) Q \dot{x}(\theta) d\theta. \end{aligned}$$

If the time derivative of $V(t)$ along the trajectory of the system (5) is negative for any $x \notin \Omega_P$, then Ω_P is a robust attractor of (5) with respect to $w \in W$. Note the identity (Leibniz-Newton): $\int_a^b \dot{v}(t)dt = v(b) - v(a)$.

By Lemma 3, we have the following formula for any scalar $\alpha_1 > 0$

$$\begin{aligned}\dot{V}_1(t) &= 2x^T(t)P[(A+A_d)x(t) - A_d\int_{t-d}^{t}\dot{x}(\theta)d\theta + B_1 w(t)]\\ &\le 2x^T(t)P(A+A_d)x(t) - 2x^T(t)PA_d\int_{t-d}^{t}\dot{x}(\theta)d\theta\\ &\quad +\alpha_1^{-1}x^T(t)PB_1B_1^TPx(t) + \alpha_1 w^T(t)w(t)\\ &= \beta^T\Xi_1\beta - (\alpha_1+1)[x^T(t)Px(t) - w^T(t)w(t)]\end{aligned}$$

where

$$\beta = \begin{bmatrix} x^T(t) & \int_{t-d}^{t}\dot{x}^T(\theta)d\theta & w^T(t)\end{bmatrix}^T$$

$$\Xi_1 = \begin{bmatrix} (1,1) & -PA_d & 0\\ * & 0 & 0\\ * & * & -I\end{bmatrix}$$

$$(1,1) = Sym\{P(A+A_d)\} + \alpha_1^{-1}PB_1B_1^TP + (\alpha_1+1)P$$

$$\dot{V}_2(t) = 2\int_{t-d}^{t}(\theta - t + d)\dot{x}^T(t)Q\dot{x}(\theta)d\theta - \left[\int_{t-d}^{t}\dot{x}^T(\theta)d\theta\right]Q\left[\int_{t-d}^{t}\dot{x}(\theta)d\theta\right]$$

$$\dot{V}_3(t) = \tfrac{1}{2}d^2\dot{x}^T(t)Q\dot{x}(t) - \int_{t-d}^{t}(\theta - t + d)\dot{x}^T(\theta)Q\dot{x}(\theta)d\theta.$$

By Lemma 3, it can be shown that

$$2\dot{x}^T(t)Q\dot{x}(\theta) \le \dot{x}^T(t)Q\dot{x}(t) + \dot{x}^T(\theta)Q\dot{x}(\theta).$$

Therefore

$$\begin{aligned}\dot{V}_2(t) &\le \tfrac{1}{2}d^2\dot{x}^T(t)Q\dot{x}(t) + \int_{t-d}^{t}(\theta - t + d)\dot{x}^T(\theta)Q\dot{x}(\theta)d\theta\\ &\quad - \left[\int_{t-d}^{t}\dot{x}^T(\theta)d\theta\right]Q\left[\int_{t-d}^{t}\dot{x}(\theta)d\theta\right].\end{aligned}$$

Then, we can get

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t)\\ &\le -(\alpha_1+1)[x^T(t)Px(t) - w^T(t)w(t)] + \beta^T(\Xi_1 + \Xi_2 + \Xi_3)\beta,\end{aligned}$$

where

$$\Xi_2 = d^2\begin{bmatrix}(A+A_d)^T\\ -A_d^T\\ B_1^T\end{bmatrix}Q\begin{bmatrix} A+A_d & -A_d & B_1\end{bmatrix}$$

$$\Xi_3 = diag\{0, -Q, 0\}.$$

By the Schur complement formula, (6) is equivalent to $\Xi_1+\Xi_2+\Xi_3 < 0$. Therefore, we have $\dot{V}(t) < -(\alpha_1+1)x^T(t)Px(t) < 0$ whenever $w = 0$. Furthermore,

since $x^T(t)Px(t) > 1$ for $x \notin \Omega_P$ and $w^T(t)w(t) \leq 1$ for $w \in W$, we have $\dot{V}(t) < \beta^T(\Xi_1 + \Xi_2 + \Xi_3)\beta < 0$ for any $w \in W$. Therefore, system (1) is stable and Ω_P is a robust attractor of it with respect to $w \in W$.
Again by the Schur complement formula, (7) is equivalent to the following inequality

$$\begin{bmatrix} -\sigma P + C^T C & 0 \\ * & -(\rho^2 - \sigma)I \end{bmatrix} \leq 0.$$

It follows that

$$\sigma x^T(t)Px(t) - x^T(t)C^T Cx(t) + (\rho^2 - \sigma)w^T(t)w(t) \geq 0.$$

Then, we can get

$$(\rho^2 - \sigma)w^T(t)w(t) - \|Cx(t)\|^2\sigma + x^T(t)Px(t) \geq 0.$$

From this, it is clear that if $x(t) \in \Omega_P$ and $w(t) \in W$, $\|z(t)\|^2 = \|Cx(t)\|^2 \leq \sigma + (\rho^2 - \sigma) = \rho^2$. This shows that $\Omega_P \subset \Omega(\rho)$ and hence $R_\infty(0) \subset \Omega_P \subset \Omega(\rho)$. Therefore, system (5) has $\rho-$performance. This completes the proof. ∎

For given positive scalars $\alpha_1, \sigma > 0$, (6) and (7) are LMIs. And it is clear that the results are delay-dependent. Then we give the following result for the uncontrolled (i.e., $u(t) = 0$) uncertain system (1).

Theorem 2 *For the given performance level $\rho > 0$, and prescribed positive scalars $d > 0$, $\sigma > 0$, $\alpha_1 > 0$, if there exist positive scalars α_2, α_3, α_4, symmetric and positive-definite matrices $P > 0$, $Q > 0$ such that*

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & PF & 0 \\ * & (2,2) & (2,3) & -A_d^T Q & (2,5) & 0 & 0 \\ * & * & (3,3) & B_1^T Q & 0 & 0 & 0 \\ * & * & * & -d^{-2}Q & 0 & 0 & QF \\ * & * & * & * & (5,5) & 0 & 0 \\ * & * & * & * & * & -\alpha_2 I & 0 \\ * & * & * & * & * & * & -\alpha_3 I \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} -\sigma P + \alpha_4 H_C^T H_C & \alpha_4 H_C^T H_D & C^T & 0 \\ * & -(\rho^2 - \sigma) + \alpha_4 H_D^T H_D & 0 & 0 \\ * & * & -I & E \\ * & * & * & -\alpha_4 I \end{bmatrix} \leq 0 \tag{9}$$

where

$$(1,1) = Sym\{P(A+A_d)\} + (\alpha_1+1)P + (\alpha_2+\alpha_3)(H_A+H_{A_d})^T(H_A+H_{A_d})$$
$$(1,2) = -PA_d - (\alpha_2+\alpha_3)(H_A+H_{A_d})^T H_{A_d}$$
$$(1,3) = \alpha_3(H_A+H_{A_d})^T H_{B_1}$$
$$(1,4) = (A+A_d)^T Q$$
$$(1,5) = PB_1 + \alpha_2(H_A+H_{A_d})^T H_{B_1}$$
$$(2,2) = -Q + (\alpha_2+\alpha_3)H_{A_d}^T H_{A_d}$$
$$(2,3) = -\alpha_3 H_{A_d}^T H_{B_1}$$
$$(2,5) = -\alpha_2 H_{A_d}^T H_{B_1}$$
$$(3,3) = -I + \alpha_3 H_{B_1}^T H_{B_1}$$
$$(5,5) = -\alpha_1 I + \alpha_2 H_{B_1}^T H_{B_1}$$

then the uncontrolled system (1) is robustly internally stable and Ω_P is a robust attractor of it with respect to $w \in W$. Moreover, $\Omega_P \subset \Omega_\Delta(\rho)$ and hence the system has robust $\rho-$performance.

Proof. For the system (1) without control, we can obtain by Theorem 1 that the following inequalities hold for any $\Delta \in \boldsymbol{\Delta}$.

$$\begin{bmatrix} (1,1) & (1,2) & 0 & (1,4) & (1,5) \\ * & -Q & 0 & -(A_d+F\Delta H_{A_d})^T Q & 0 \\ * & * & -I & (B_1+F\Delta H_{B_1})^T Q & 0 \\ * & * & * & -d^{-2}Q & 0 \\ * & * & * & * & -\alpha_1 I \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} -\sigma P & 0 & (C+E\Delta H_C)^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} \leq 0 \tag{11}$$

where

$$(1,1) = Sym\{P(A+A_d+F\Delta H_A+F\Delta H_{A_d})\} + (\alpha_1+1)P$$
$$(1,2) = -P(A_d+F\Delta H_{A_d})$$
$$(1,4) = (A+A_d+F\Delta H_A+F\Delta H_{A_d})^T Q$$
$$(1,5) = P(B_1+F\Delta H_{B_1}.)$$

Equivalently, the inequality (10) can be rewritten as

$$\Gamma + Sym\{\zeta_1\Delta\zeta_2 + \zeta_3\Delta\zeta_4\} < 0$$

where

$$\Gamma = \begin{bmatrix} (1,1) & -PA_d & 0 & (A+A_d)^T Q & PB_1 \\ * & -Q & 0 & -A_d^T Q & 0 \\ * & * & -I & B_1^T Q & 0 \\ * & * & * & -d^{-2}Q & 0 \\ * & * & * & * & -\alpha_1 I \end{bmatrix}$$

$$(1,1) = Sym\{P(A+A_d)\} + (\alpha_1+1)P$$

$$\zeta_1 = \begin{bmatrix} F^T P & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\zeta_2 = \begin{bmatrix} H_A + H_{A_d} & -H_{A_d} & 0 & 0 & H_{B_1} \end{bmatrix}$$

$$\zeta_3 = \begin{bmatrix} 0 & 0 & 0 & F^T Q & 0 \end{bmatrix}^T$$

$$\zeta_4 = \begin{bmatrix} H_A + H_{A_d} & -H_{A_d} & H_{B_1} & 0 & 0 \end{bmatrix}.$$

By Lemma 1, there exist scalars $\alpha_2 > 0$, $\alpha_3 > 0$ such that

$$\Gamma + \frac{1}{\alpha_2}\zeta_1\zeta_1^T + \alpha_2\zeta_2^T\zeta_2 + \frac{1}{\alpha_3}\zeta_3\zeta_3^T + \alpha_3\zeta_4^T\zeta_4 < 0.$$

By Schur complement formula, this is equivalent to (8).

Similarly, the inequality (11) can be written as

$$\begin{bmatrix} -\sigma P & 0 & C^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} + Sym\left\{ \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix} \Delta \begin{bmatrix} H_C & 0 & 0 \end{bmatrix} \right\} \leq 0.$$

By Lemma 2, there exists a scalar $\alpha_4 > 0$ such that

$$\begin{bmatrix} -\sigma P & 0 & C^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} + \tfrac{1}{\alpha_4} \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix} \begin{bmatrix} 0 & 0 & E^T \end{bmatrix} + \alpha_4 \begin{bmatrix} H_C^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} H_C & 0 & 0 \end{bmatrix} \leq 0.$$

By Schur complement formula, this is equivalent to (9). Thus, if (8) and (9) hold, by Theorem 1, the uncontrolled system (1) is robustly internally stable and Ω_P is a robust attractor of it with respect to $w \in W$. Moreover, $\Omega_P \subset \Omega_\Delta(\rho)$ and hence the system (1) has the robust $\rho-$performance. This completes the proof. ■

THEOREM 3 *For the uncontrolled system (1), a given performance level $\rho > 0$, and prescribed positive scalars $d > 0$, $\sigma > 0$, $\alpha_1 > 0$, if there exist positive scalars $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$, symmetric and positive-definite matrices $X > 0$, $S > 0$*

such that

$$\begin{bmatrix} (1,1) & -A_dS & 0 & (1,4) & B_1 & (1,6) & (1,7) \\ * & -S & 0 & -SA_d^T & 0 & -SH_{A_d}^T & -SH_{A_d}^T \\ * & * & -I & B_1^T & 0 & 0 & H_{B_1}^T \\ * & * & * & (4,4) & 0 & 0 & 0 \\ * & * & * & * & -\alpha_1 I & H_{B_1}^T & 0 \\ * & * & * & * & * & -\alpha_2 I & 0 \\ * & * & * & * & * & * & -\alpha_3 I \end{bmatrix} < 0 \qquad (12)$$

$$\begin{bmatrix} -\sigma X & 0 & XC^T & XH_C^T \\ * & -(\rho^2-\sigma)I & 0 & 0 \\ * & * & -I+\alpha_4 EE^T & 0 \\ * & * & * & -\alpha_4 I \end{bmatrix} \leq 0 \qquad (13)$$

where

$$(1,1) = Sym\{(A+A_d)X\} + (\alpha_1+1)X + \alpha_2 FF^T$$
$$(1,4) = X(A+A_d)^T$$
$$(1,6) = (1,7) = X(H_A + H_{A_d})^T$$
$$(4,4) = -d^{-2}S + \alpha_3 FF^T$$

then the system is robustly internally stable and $\Omega_{X^{-1}}$ is a robust attractor of the system with respect to $w \in W$ and all the parameter uncertainties. Moreover, $\Omega_{X^{-1}} \subset \Omega_\Delta(\rho)$ and hence the system has robust $\rho-$performance.

Proof. For system (5), by Theorem 1, premultiplying and postmultiplying $diag\{P^{-1}, Q^{-1}, I, I, I\}$ on both sides of (6) and taking $X = P^{-1}$, $S = Q^{-1}$, we can obtain the following inequality.

$$\begin{bmatrix} (1,1) & -A_dS & 0 & X(A+A_d)^T & B_1 \\ * & -S & 0 & -SA_d^T & 0 \\ * & * & -I & B_1^T & 0 \\ * & * & * & -d^{-2}S & 0 \\ * & * & * & * & -\alpha_1 I \end{bmatrix} < 0 \qquad (14)$$

where $(1,1) = Sym\{(A+A_d)X\} + (\alpha_1+1)X$.

Similarly, (7) is equivalent to the following inequality.

$$\begin{bmatrix} -\sigma X & 0 & XC^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} \leq 0. \qquad (15)$$

For the uncontrolled system (1), we have

$$\Sigma + Sym\{\xi_1 \Delta \xi_2 + \xi_3 \Delta \xi_4\} < 0$$

$$\begin{bmatrix} -\sigma X & 0 & XC^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} + Sym\left\{ \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix} \Delta \begin{bmatrix} H_C X & 0 & 0 \end{bmatrix} \right\} \leq 0$$

where

$$\Sigma = \begin{bmatrix} (1,1) & -A_dS & 0 & X(A+A_d)^T & B_1 \\ * & -S & 0 & -SA_d^T & 0 \\ * & * & -I & B_1^T & 0 \\ * & * & * & -d^{-2}S & 0 \\ * & * & * & * & -\alpha_1 I \end{bmatrix}$$

$$(1,1) = Sym\{(A+A_d)X\} + (\alpha_1+1)X$$

$$\xi_1 = \begin{bmatrix} F^T & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\xi_2 = \begin{bmatrix} (H_A + H_{A_d})X & -H_{A_d}S & 0 & 0 & H_{B_1} \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} 0 & 0 & 0 & F^T & 0 \end{bmatrix}^T$$

$$\xi_4 = \begin{bmatrix} (H_A + H_{A_d})X & -H_{A_d}S & H_{B_1} & 0 & 0 \end{bmatrix}.$$

By Lemma 1, there exist scalars $\alpha_2 > 0$, $\alpha_3 > 0$ such that

$$\Sigma + \alpha_2\xi_1\xi_1^T + \frac{1}{\alpha_2}\xi_2^T\xi_2 + \alpha_3\xi_3\xi_3^T + \frac{1}{\alpha_3}\xi_4^T\xi_4 < 0 \tag{16}$$

$$\begin{bmatrix} -\sigma X & 0 & XC^T \\ * & -(\rho^2-\sigma)I & 0 \\ * & * & -I \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ E \end{bmatrix} \begin{bmatrix} 0 & 0 & E^T \end{bmatrix} + \tfrac{1}{\alpha_4} \begin{bmatrix} XH_C^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} H_C X & 0 & 0 \end{bmatrix} \leq 0. \tag{17}$$

By Schur complement formula, (16) and (17) are equivalent to (12) and (13) respectively. This completes the proof. ■

THEOREM 4 *For a given performance level $\rho > 0$, and prescribed positive scalars $d > 0$, $\sigma > 0$, $\alpha_1 > 0$, if there exist positive scalars $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$, symmetric and positive-definite matrices $X > 0$, $S > 0$ and a matrix L such that the LMIs (18) and (19) hold:*

$$\begin{bmatrix} (1,1) & -A_dS & 0 & X(A+A_d)^T + L^TB^T & B_1 & X(H_A+H_{A_d})^T + L^TH_B^T & X(H_A+H_{A_d})^T + L^TH_B^T \\ * & -S & 0 & -SA_d^T & 0 & -SH_{A_d}^T & -SH_{A_d}^T \\ * & * & -I & B_1^T & 0 & 0 & H_{B_1}^T \\ * & * & * & -d^{-2}S + \alpha_3 FF^T & 0 & 0 & 0 \\ * & * & * & * & -\alpha_1 I & H_{B_1}^T & 0 \\ * & * & * & * & * & -\alpha_2 I & 0 \\ * & * & * & * & * & * & -\alpha_3 I \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} -\sigma X & 0 & XC^T + L^T D^T & XH_C^T + L^T H_D^T \\ * & -(\rho^2 - \sigma)I & 0 & 0 \\ * & * & -I + \alpha_4 EE^T & 0 \\ * & * & * & -\alpha_4 I \end{bmatrix} < 0 \qquad (19)$$

where

$$(1,1) = Sym\{(A + A_d)X + BL\} + (\alpha_1 + 1)X + \alpha_2 FF^T$$

then there exists a state-feedback controller $u(t) = Kx(t)$ *with*

$$K = XL^{-1} \qquad (20)$$

such that the closed-loop system (4) is robustly internally stable and $\Omega_{X^{-1}}$ *is a robust attractor of (4) with respect to* $w \in W$ *and the parameter uncertainties specified by* $\mathbf{\Delta}$.

Moreover, $\Omega_{X^{-1}} \subset \Omega_{\Delta}(\rho)$ *and hence the closed-loop system (4) has robust* $\rho-$*performance.*

Proof. By Theorem 3, it is easy to prove the result.

4. Illustrative example

Consider the system (4) with the following parameters:

$$A = \begin{bmatrix} -1 & 0 \\ 0.2 & 0.1 \end{bmatrix},\ A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.1 \end{bmatrix},\ C = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 1.8 \end{bmatrix},\ D = \begin{bmatrix} 0.4 \\ 0.7 \end{bmatrix}$$

$$H_B = H_D = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ H_{A_d} = \begin{bmatrix} 0.1 & 0 \\ 0.3 & 0.1 \end{bmatrix},\ F = 0.05I$$

$$E = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},\ H_A = H_{B_1} = H_C = I.$$

Notice that the system matrix A is unstable, and the pair (A, B) is not controllable. Here, we use Theorem 4 to find a linear state-feedback controller to stabilize the system and guarantee the closed loop system to have the ρ-performance with $\rho = 0.9$.

Assume $d = 1, \alpha_1 = 0.1, \sigma = 0.8$, we can solve the LMIs in Theorem 4 and obtain a state-feedback controller $u = Kx$ for the system where the gain matrix is

$$K = \begin{bmatrix} -0.6 & -0.8 \end{bmatrix}.$$

Furthermore, let the external bounded disturbance

$$w = \begin{bmatrix} (1/\sqrt{2})\sin(3\pi t + 1) & (1/\sqrt{2})\cos(6\pi t + 1) \end{bmatrix}^T.$$

The numerical simulation of the state response of the system without disturbances is shown in Fig. 1, and that of the system involving the disturbance effects – in Fig. 2. The initial states are both chosen to be (0.6, -0.2).

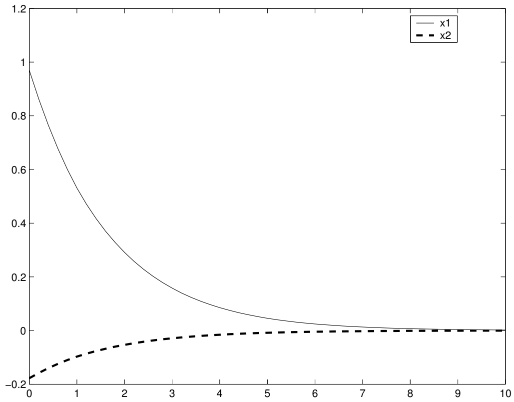

Figure 1. State response of the system without disturbances

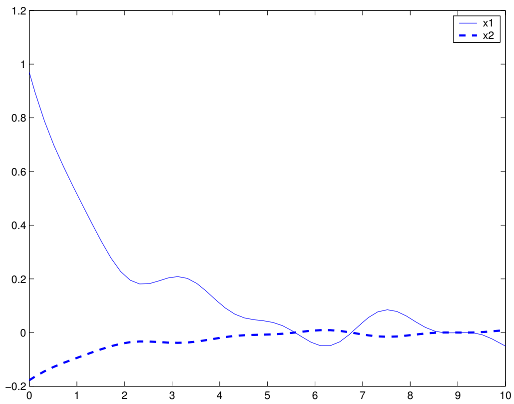

Figure 2. State response of the system with external disturbances

5. Conclusions

We have discussed the problem of persistent bounded disturbance rejection for uncertain time-delay systems by using the concept of positive invariant set and

Lyapunov function method. Delay-dependent sufficient conditions are derived that ensure the internal stability and desired level of persistent bounded disturbances. Then the state-feedback controller is designed to achieve both robust stability and a desired performance level of disturbance rejection for a perturbed time-delay system. A numerical example shows the effectiveness of the method.

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