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Conditions with aggregates evaluated using gradual numbers*

by

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Abstract: This paper is devoted to the evaluation of conditions involving aggregates in the context of flexible querying of relational databases. An example of such a condition is "the maximum salary of *young* employees is *high*", where the aggregate max applies to a fuzzy set of salaries. At first, we consider the evaluation of quantified statements where the aggregate count (the cardinality) is implicitly used. We extend this result to conditions involving the aggregates average, maximum or minimum. The contribution of this paper is to propose a new theoretical background for their evaluations based on an arithmetic on gradual numbers (N_f,Z_f,Q_f).

Keywords: fuzzy sets, fuzzy condition, linguistic quantifier, quantified statement, aggregate

1. Introduction

Flexible querying of relational databases aims at expressing preferences in queries instead of boolean requirements as is the case for regular (or crisp) querying. As a consequence, a flexible query returns a set of discriminated answers to the user (from the best answers to the less preferred). Many approaches to define flexible queries have been proposed and it has been shown that the fuzzy sets (Zadeh, 1965) based approach is most general (Bosc and Pivert, 1992). An extension of the SQL language (namely SQLf, Bosc and Pivert, 1995) has been proposed to define sophisticated flexible queries using fuzzy sets.

In this context, predicates are defined by fuzzy sets and are called fuzzy predicates. They can be combined using various operators such as generalized conjunctions and generalized disjunctions (respectively expressed by t-norms

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and t-conorms) or using more sophisticated operators such as averages. In addition, linguistic quantifiers (Zadeh, 1983) (which are quantifiers defined by linguistic expressions like *most of* or *around 3*) allow to define a particular type of conditions called quantified statements. Many types of linguistic quantifiers can be found in the literature (Diaz-Hermida, Bugarin and Barro, 2003; Glockner, 1997, 2004; Losada, Diaz-Hermida and Bugarin, 2006; Zadeh, 1983) as semi-fuzzy quantifiers which allow to model expressions like "there are twice as many men as women". We limit this presentation to the original linguistic quantifiers defined by Zadeh (Zadeh, 1983) and the two types of quantified statements he proposes.

A quantified statement of the first type is denoted "Q X are A" where Q is a linguistic quantifier, X is a crisp set and A is a fuzzy predicate. Such a condition means that "the cardinality of the fuzzy set made of elements from X satisfying A is in agreement with Q". An example is provided by "*most of* employees are *young*" where Q is *most of*, X is a set of employees whereas A is the condition *to be young*. In this first type of quantified statements, the referential for the linguistic quantifier is a crisp set (denoted by X, the set of employees when considering the example). In the second type of quantified statements, the quantifier applies to a fuzzy set as in "*most of young* employees are *well-paid*" where the referential for *most of* is a fuzzy set (of *young* employees). This second type of conditions is written "Q B X are A" (in the example, Q is *most of* while B is the predicate *to be young* and A is the predicate *well-paid*). Such a statement means that the proportion of A elements among the B elements is in agreement with the linguistic quantifier Q.

Two kinds of linguistic quantifiers can be distinguished: absolute quantifiers (which refer to an absolute number such as *about 3*, *at least 2* ...) and relative quantifiers (which refer to a proportion such as *about the half*, *at least a quarter* ...). To evaluate a quantified statement is to determine the extent to which it is true and this paper proposes a new theoretical framework for their evaluation. Propositions are based on the handling of gradual integers (N_f, Z_f) (Rocacher, 2003; Rocacher and Bosc, 2003) and gradual rational numbers (Q_f) as defined in Rocacher and Bosc (2005). These specific numbers express well-known but gradual numbers and differ from usual fuzzy numbers which define imprecise (ill-known) numbers. In addition, since their definition is closely related to the concept of cardinality of a fuzzy set, their use to evaluate quantified statements appears to be natural.

We think that the study of quantified statements is a first step to evaluate more complex conditions involving an aggregate. In these conditions, the difficulty is to compute the value of an aggregate on a fuzzy set. As an example, when considering the condition "the maximum salary of *young* employees is *high*" it is necessary to determine the maximum value of a fuzzy set. This paper shows that gradual number theory can be used for this purpose in case of the average, maximum and minimum aggregates.

Section 2 introduces the gradual numbers while Section 3 introduces the definition of linguistic quantifiers and quantified statements of type "Q B X are A" and "Q X are A". Section 4 proposes a gradual truth value as the result of evaluation of a quantified statement. Since a scalar value is mandatory in the context of SQLf, Section 5 proposes two interpretations to obtain two different scalar evaluations from this gradual truth value. The extension of this work to evaluate conditions involving the aggregates average, maximum and minimum is shown in Section 6.

2. Gradual numbers

It has been shown (Rocacher, 2003) that dealing with both quantification and preferences defined by fuzzy sets leads to defining gradual natural integers (elements of N_f) corresponding to fuzzy cardinalities. Then, N_f has been extended to Z_f (the set of gradual relative integers) and Q_f (the set of gradual rationals) in order to deal with queries based on difference or division operations (Rocacher and Bosc, 2005). These new frameworks provide arithmetic foundations where difference or ratio between gradual quantities can be evaluated. As a consequence, gradual numbers are essential, in particular, for dealing with flexible queries using absolute or relative fuzzy quantifiers. This is the reason why this section shortly introduces the set N_f of gradual integers (Subsection 2.1), and its extensions Z_f (Subsection 2.2) and Q_f (Subsection 2.3). In Subsection 2.4, it is shown that applying a fuzzy predicate on a gradual number provides a specific truth value, which is also gradual.

2.1. Gradual natural integers

The fuzzy cardinality $|F|$ of a fuzzy set F, as proposed by Zadeh (Zadeh, 1983) is a fuzzy set on N, called $FGCount(F)$, defined by:

$$\forall \text{ n} \in N,\ \mu_{|F|}(\text{n}) = \sup\{\alpha|\ |F_\alpha| \geq \text{n}\},$$

where F_α denotes the α-cut of fuzzy set F.

The degree $\mu_{|F|}(\text{n})$, associated with a number n in the fuzzy cardinality $|F|$ is interpreted as the extent, to which F has at least n elements. The fuzzy cardinality is a normalized fuzzy set of integers and the associated nonincreasing characteristic function provides the different cardinalities of the different α-cuts.

Example 1 *The fuzzy cardinality of the fuzzy set* $F = \{1/x_1, 1/x_2, 0.8/x_3, 0.6/x_4\}$ *is:* $|F| = \{1/0, 1/1, 1/2, 0.8/3, 0.6/4\}$. *Degree 0.8 is the extent to which* F *contains at least 3 elements.*

It is very important to notice that we do not interpret a fuzzy cardinality as a fuzzy number based on a possibility distribution (which has a disjunctive interpretation). In our case, the knowledge of all the cardinalities of all the different α-cuts of a fuzzy set F provides an exact characterization of the number of elements belonging to F. Consequently, $|F|$ must be viewed as a conjunctive fuzzy set of integers. As matter of fact, the considered fuzzy set F represents a perfectly known collection of data (without uncertainty), so its cardinality $|F|$ is also perfectly known. We think that it is more convenient to qualify such cardinality as a "gradual" number rather than a "fuzzy" number. Other fuzzy cardinalities based on the definition of FGCounts, such as FLCounts or FECounts, have been defined by Zadeh (1983) or Wygralak (1999). Dubois and Prade (1985) and Delgado, Sanchez and Vila (2000) have adopted a possibilistic point of view where fuzzy cardinality is interpreted as a possibility distribution over α-cuts corresponding to a fuzzy number (Dubois and Prade, 1987).

The set of all FGCounts is called N_f and represents the set of gradual natural integers. The α-cut x_α of a gradual natural integer x is an integer defined as the highest integer value appearing in the description x associated with a degree at least equal to α:

$$x_\alpha = \max\{c \in N \,|\, \mu_x(c) \geq \alpha\}.$$

When x describes the FGCount of a fuzzy set A, the following equality holds:

$$x_\alpha = |A_\alpha|.$$

This approach is along the line presented by Dubois and Prade (2005) where they introduce the concept of fuzzy element e in a set S, defined as an assignment function a_e from a complete lattice to S. Following this view, a gradual natural integer x belonging to N_f can be defined by an assignment function a_x from $]0,1]$ to N such that:

$$\forall\, \alpha \in]0,1],\ a_x(\alpha) = x_\alpha.$$

If x is identified with a fuzzy cardinality $|F|$ of a fuzzy set F, then $a_x(\alpha)$ is the cardinality of the α level cut of F.

EXAMPLE 2 $|F| = \{1/0, 1/1, 1/2, 0.8/3, 0.6/4\}$ *is a gradual natural integer defined by an assignment a function* $a_{|F|}$ *graphically represented by Fig. 1. From Fig. 1, we get :* $a_{|F|}(0.7) = |F_{0.7}| = 3$.

Any operation # between two natural integers can then be extended to gradual natural integers x and y by defining (Rocacher and Bosc, 2005) the corresponding assignment function $a_{x\#y}$ as follows:

$$\forall\, \alpha \in]0,1],\ a_{x\#y}(\alpha) = a_x(\alpha)\#a_y(\alpha) = x_\alpha\#y_\alpha.$$

Due to the specific characterization of gradual integers, it can easily be shown that N_f is a semi-ring structure. So, the addition and product operations satisfy the following properties: $(N_f, +)$ is a commutative monoïd (+ is closed and associative) with the neutral element $\{1/0\}$; $(N_f, *)$ is a monoïd with the neutral element $\{1/0,\ 1/1\}$; the product is distributive over the addition.

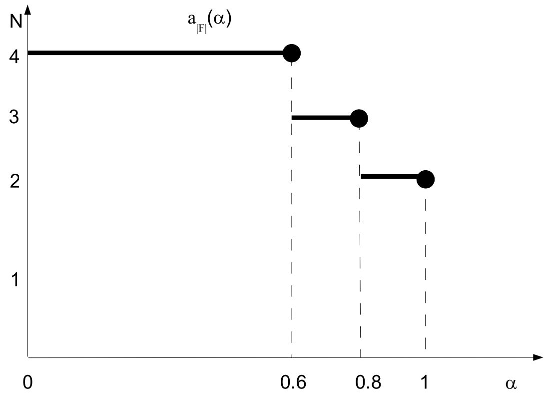

Figure 1. The assignment function of a fuzzy cardinality.

2.2. Gradual relative integers

In N_f the difference between two gradual natural integers may be not defined. As a consequence, N_f has to be extended to Z_f in order to build up a group structure.

The set of gradual relative integers Z_f is defined by the quotient set $(N_f \times N_f)/\Re$ of all equivalence classes on $(N_f \times N_f)$ with regards to $\Re$ the equivalence relation characterized by:

$$\forall(x^+, x^-) \in (N_f \times N_f),\ \ \forall(y^+, y^-) \in (N_f \times N_f),$$
$$(x^+, x^-)\Re(y^+, y^-) \text{ iff } x^+ + y^- = x^- + y^+.$$

The α-cut of a gradual relative integer (x^+, x^-) is defined as the relative integer $x^+_\alpha - x^-_\alpha$. As a consequence, the assignment function a_x of a gradual relative integer x is a function from $]0, 1]$ to Z such that:

$$\forall\alpha \in]0, 1],\ a_x(\alpha) = x^+_\alpha - x^-_\alpha = a_{x^+}(\alpha) - a_{x^-}(\alpha).$$

The assignment function can be also be represented by a unique canonical representative x^c, which enumerates the values of its different α-cuts on Z:

$$x^c = \{\alpha_i/(x^+_{\alpha_i} - x^-_{\alpha_i})\},$$

where the α_is correspond to the different degrees appearing in the representation of x^+ and x^-. The canonical representation gathers the discontinuity points of the assignment function.

Example 3 *The canonical representation of the fuzzy relative* $x = (x^+, x^-)$ *(with:* $x^+ = \{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}$ *and* $x^- = \{1/0, 1/1, 0.9/2\}$*) is:*

$$(x^+, x^-)^c = \{1/0, 0.9/-1, 0.8/0, 0.5/1, 0.2/2\}.$$

For the level of 0.9 we get: $x^+_{0.9} = 1$ *while* $x^-_{0.9} = 2$*. As a consequence, the* α*-cut of* (x^+, x^-) *at level 0.9 is* $x^+_{0.9} - x^-_{0.9} = -1$*. The assignment function of* (x^+, x^-) *is represented by Fig. 2.*

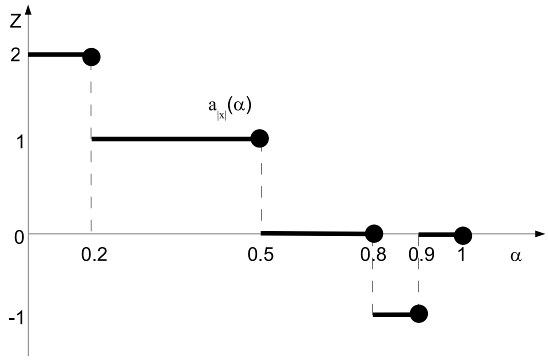

Figure 2. Assignment function of the gradual relative integer (x^+, x^-).

If x and y are two gradual relative integers, the addition and the multiplication are respectively defined by the classes $(x^+ + y^+, x^- + y^-)$ and $((x^+ * y^+) + (x^- * y^-), (x^+ * y^-) + (x^- * y^+))$. The addition is commutative, associative and has a neutral element, denoted by 0_{Z_f}, defined by the class $\{(x, x) | x \in N_f\}$. When considering the assignment functions of two relative integers x and y, the assignment functions of their sum and product are :

$$\forall\, \alpha \in]0, 1], a_{x+y}(\alpha) = a_x(\alpha) + a_y(\alpha) = x_\alpha + y_\alpha.$$
$$\forall\, \alpha \in]0, 1], a_{x*y}(\alpha) = a_x(\alpha) * a_y(\alpha) = x_\alpha * y_\alpha.$$

Each fuzzy relative integer (x^+, x^-) has an opposite, denoted by $-x = (x^-, x^+)$. This is remarkable because in the framework of usual fuzzy numbers this property is not always satisfied. It can be easily checked that the product in Z_f is commutative, associative and distributive over the addition, the neutral element being the fuzzy relative integer $(\{1/0,\ 1/1\},\ \{1/0\})$. Therefore we conclude that $(Z_f, +, *)$ forms a ring.

2.3. Gradual rational numbers

The question is now to define an inverse to each gradual integer and to build up the set of gradual rational numbers. We define Z^*_f as the set of gradual integers, x such that: $\forall \alpha \in]0, 1]$, $a_x(\alpha) \neq 0$ and $\Re$' as the equivalence relation such that:

$$\forall (x, y) \text{ and } (x', y') \in (Z_f \times Z^*_f),$$
$$(x, y)\, \Re' (x', y') \text{ iff } x * y' = x' * y.$$

The set of gradual relational numbers Q_f is defined by the quotient set $(Z_f \times Z_f^*)/\Re'$ of all equivalence classes on $(Z_f \times Z_f^*)$ with regards to $\Re'$. As a consequence, a gradual relational number is defined by a ratio x/y where $x \in Z_f$ and $y \in Z_f^*$. The assignment function $a_{x/y}$ of x/y is a function from]0, 1] to Q defined by:

$$\forall\, \alpha \in]0, 1],\ a_{x/y}(\alpha) = a_x(\alpha)/a_y(\alpha) = x_\alpha/y_\alpha.$$

It can also be represented, thanks to a canonical representation, by enumerating values associated with the different α-cuts, which are rational numbers. When considering the assignment functions of two gradual relational numbers u and v, the assignment functions of their sum and product are:

$$\forall\, \alpha \in]0, 1],\ a_{u+v}(\alpha) = a_u(\alpha) + a_v(\alpha) = u_\alpha + v_\alpha,$$
$$\forall\, \alpha \in]0, 1],\ a_{u*v}(\alpha) = a_u(\alpha) * a_v(\alpha) = u_\alpha * v_\alpha.$$

2.4. Gradual truth value

This section proposes a computation to determine the truth value obtained when applying a fuzzy predicate on a gradual number. Let x be an element of N_f or Z_f or Q_f (its assignment function being a_x) and T a fuzzy predicate. The application of the predicate T on x produces a gradual truth value S defined on the interval $]0, 1]$ characterized by the assignment function defined by:

$$\forall \alpha \in]0, 1],\ a_S(\alpha) = \mu_T(a_x(\alpha)).$$

For a given level α, $a_S(\alpha)$ represents the satisfaction of the corresponding α-cut of the gradual number. In other words, for a given level α, the gradual number satisfies predicate T at degree $a_S(\alpha)$.

EXAMPLE 4 *We consider the fuzzy predicate "*high*" defined by Fig. 3 and condition "the number of* young *employees is* high*", where the number of* young *employees (fuzzy cardinality) is the gradual integer x canonically represented by* $\{1/15, 0.7/20, 0.2/25\}$ *(which means that 15 employees are completely young, 5 employees have the same age and are young at the level 0.7, whereas 5 other people are rather not young since their level of youth is estimated at 0.2).*

The assignment function for the number x is the following:

$$\forall\, \alpha \in]0, 0.2], a_x(\alpha) = 25,$$
$$\forall\, \alpha \in]0.2, 0.7], a_x(\alpha) = 20,$$
$$\forall\, \alpha \in]0.7, 1], a_x(\alpha) = 15.$$

*The application of the predicate "*high*" to the gradual integer x produces a gradual truth value S, whose function of assignment is defined by:*

$$\forall\, \alpha \in]0, 1],\ a_S(\alpha) = \mu_{high}(a_x(\alpha)).$$

We get the gradual truth value given by Fig. 4.

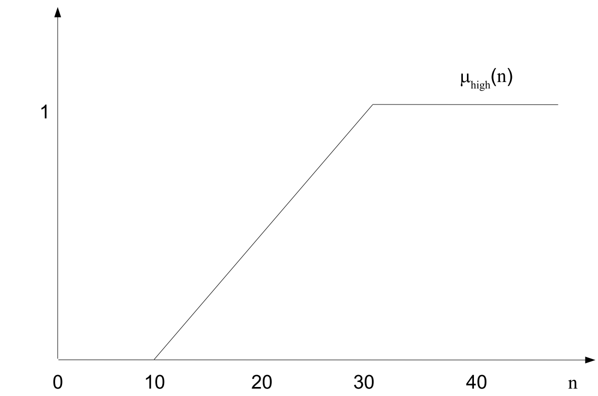

Figure 3. The fuzzy predicate *high*

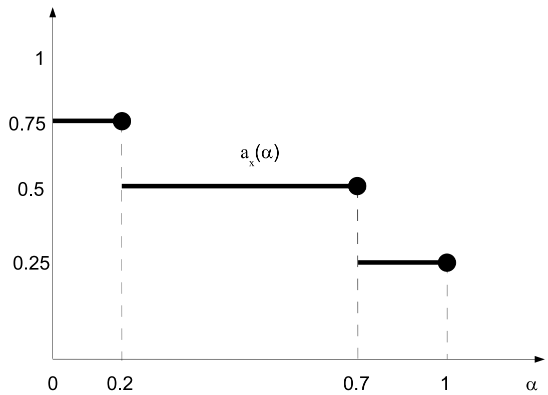

Figure 4. Gradual truth value S

This gradual truth value shows the different results associated to the different α-cuts. When referring to previous example and when considering level 0.8, the fuzzy cardinality x states that the cardinality of this α-cut is 15 ($a_x(0.8) = 15$). Since $\mu_{high}(15) = 0.25$, this cardinality satisfies to be high at degree 0.25. It can be checked that $a_S(0.8) = 0.25$.

3. Linguistic quantifiers

Subsection 3.1 recalls the definition of linguistic quantifiers. Subsection 3.2 introduces the principles advocated in this paper for the evaluation of quantified statements of type "Q X are A" and "Q B X are A".

3.1. Representation of linguistic quantifiers

A first representation for an absolute quantifier (respectively, relative quantifier) is a fuzzy subset Q of the real line (respectively of the unit interval [0,1]). This fuzzy subset is interpreted in terms of a matching between cardinalities (respectively proportions) and degrees of satisfaction. In both cases, μ_Q(j) represents the truth value of the statement "Q X are A" when j elements in X completely satisfy A, whereas A is fully unsatisfied by the others (j being a number or a proportion). The representation of an increasing linguistic quantifier satisfies:

1) $\mu_Q(0) = 0$,
2) $\exists\, k$ such as $\mu_Q(k) = 1$,
3) $\forall\, i, j$ if $i > j$ then $\mu_Q(i) \geq \mu_Q(j)$.

A decreasing linguistic quantifier is defined by:
1) $\mu_Q(0) = 0$,
2) $\exists\, k$ such as $\mu_Q(k) = 1$,
3) $\forall\, i, j$ if $i > j$ then $\mu_Q(i) \geq \mu_Q(j)$.

A unimodal quantifier is a fuzzy subset Q such that:
1) $\mu_Q(0) = 0$,
2) $\exists\, k$ such as $\mu_Q(k) = 1$,
3) $\forall\, i, j$ if $i > j > k$ then $\mu_Q(i) \geq \mu_Q(j)$,
4) $\forall\, i, j$ if $i < j < k$ then $\mu_Q(i) \leq \mu_Q(j)$.

3.2. Quantified statements

This section introduces the basis for the evaluation of "Q X are A" and "Q B X are A" statements (Q being absolute or relative). First, we consider the evaluation of quantified statements in the particular case of crisp predicates. This situation is then adapted to the case of fuzzy predicates in order to propose principles for an interpretation in the general case. It is worth mentioning that, in case of an absolute quantifier, a quantified statement of type "Q B X are A" reverts to a quantified statement of the first type, since it can be rewritten : "Q X are (A and B)". As an example, "*at least 3 young* employees are *well-paid*" is equivalent to "*at least 3* employees are (*young* and *well-paid*)". As a consequence, when dealing with quantified statements of type "Q B X are A", this paper only deals with relative quantifiers.

Obviously, when A is a crisp predicate, the evaluation of "Q X are A", Q being absolute, is given by μ_Q(c) where c is the cardinality of the set made of elements from X which are A. In case of a relative quantifier Q, the evaluation is provided by μ_Q(c/n) where c is the cardinality of the set made of elements from X which are A, while n is the cardinality of set X (c/n being the proportion of A elements in X). When the predicates A and B in the quantified statement "Q B X are A" are crisp (Q being relative), the evaluation is provided by μ_Q(p) where p is the proportion of elements which are A among the elements which are B.

In the general case, the cardinality c of the previous evaluations is the cardinality of a fuzzy set while proportion p is a ratio between two cardinalities of fuzzy set. We define these cardinalities of fuzzy sets as gradual numbers (FGCount) and we carry out the computation using the extended arithmetic defined on gradual numbers (Rocacher and Bosc, 2005) (see Section 2). In this context, the result can be either a gradual truth value (see Section 4) or a scalar truth value (i.e. a degree set in [0, 1], see Section 5).

4. Evaluation of quantified statements

Section 4.1 considers the evaluation of quantified statements of type "Q X are A", where Q is absolute or relative. Section 4.2 considers the evaluation of quantified statements of type "Q B X are A", where Q is relative.

4.1. Quantified statements of type "Q X are A"

In case of an absolute quantifier, we need to compute $\mu_Q(\mathrm{c})$, where c is the cardinality of the fuzzy set $A(\mathrm{X})$ made of elements from X, which satisfy fuzzy condition A ($\forall \mathrm{e} \in \mathrm{X}$, $\mu_{A(\mathrm{X})}(\mathrm{e}) = \mu_A(\mathrm{e})$). A canonical representation for c is (see Section 2.1):

$$\forall\, \alpha \in]0,1],\ \mathrm{c}(\alpha) = |A(\mathrm{X})_\alpha|.$$

The application of predicate Q on the gradual integer c gives a gradual truth value S defined by (see Section 2.4):

$$\forall\, \alpha \in]0,1],\ a_S(\alpha) = \mu_Q(\mathrm{c}(\alpha)) = \mu_Q(|A(\mathrm{X})_\alpha|).$$

The gradual truth value S expresses the satisfaction of each α-cut of $A(\mathrm{X})$ with respect to the linguistic quantifier Q.

A degree α is viewed as a quality threshold for the satisfactions with respect to A and the value $a_S(\alpha)$ states that: "the quantity of elements which satisfy A at least at level α is in agreement with Q". In other words, $a_S(\alpha)$ represents the truth value for "Q X are A" when considering the α-cut $A(\mathrm{X})_\alpha$ ($a_S(\alpha)$ being the truth value of "Q elements are in $A(\mathrm{X})_\alpha$").

In case of a relative quantifier, we need to compute $\mu_Q(\mathrm{c}/\mathrm{n})$ where c is the cardinality of the fuzzy set $A(\mathrm{X})$, made of elements from X, which satisfy the fuzzy condition A ($\forall\, \mathrm{e} \in \mathrm{X}$, $\mu_{A(\mathrm{X})}(\mathrm{e}) = \mu_A(\mathrm{e})$), and n the cardinality of set X. Similarly to the case of an absolute quantifier, the application of a predicate Q on the gradual number c/n gives a gradual truth value S defined by:

$$\forall\, \alpha \in]0,1],\ a_S(\alpha) = \mu_Q(\mathrm{c}(\alpha)/\mathrm{n}) = \mu_Q(|A(\mathrm{X})_\alpha|/\mathrm{n}).$$

Here again, the gradual truth value S expresses the satisfaction of each α-cut of $A(\mathrm{X})$ with respect to the linguistic quantifier.

Example 5 *We consider the statement "*about 3 X *are* A*" where* $\text{X} = \{x_1, x_2, x_3, x_4\}$ *such that* $\mu_A(x_1) = \mu_A(x_2) = 1$*,* $\mu_A(x_3) = 0.8$*,* $\mu_A(x_4) = 0.6$*. The linguistic quantifier* about 3 *is given by Fig. 5. The gradual truth value for "*about 3 X *are* A*" (defined by:* $\forall\alpha \in]0,1]$*,* $\mu_S(\alpha) = \mu_Q(\text{c}(\alpha))$*) is given by Fig. 6.*

This gradual truth value provides the satisfactions obtained for the different α*-cuts of* $A(\text{X})$ *(set made of elements from* X *which satisfy fuzzy condition* A*). As an example* $\mu_S(0.7) = \mu_Q(|A(\text{X})_{0.7}|) = \mu_Q(3) = 1$*.*

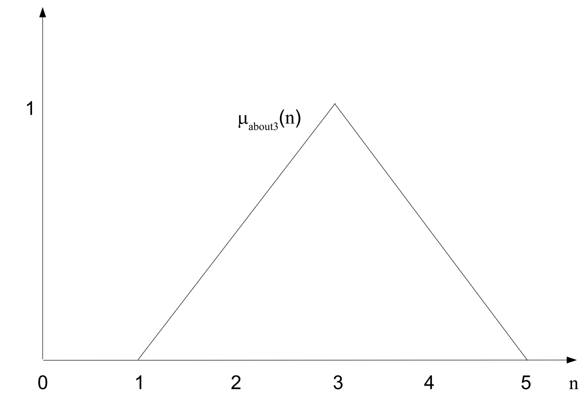

Figure 5. A representation for the quantifier *about 3*

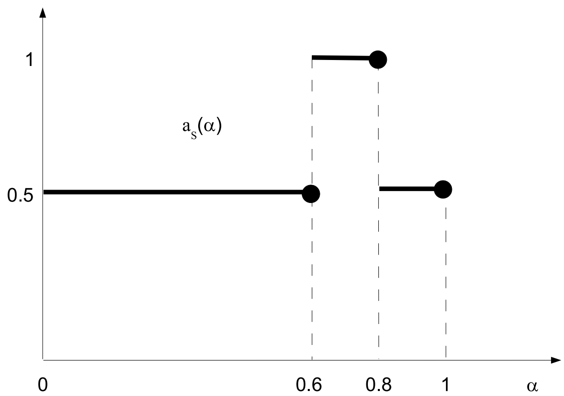

Figure 6. A gradual truth value for "*about 3* X are *A*"

In the two cases, the value $\mu_S(\alpha)$ represents the truth value for the quantified statements when considering the interpretations at the level α of the fuzzy sets. The advantage of this representation is to provide the different results given by the different interpretations of the fuzzy sets. As a consequence, this result has a clear meaning and can be the base for further processing of such as defuzzifications.

4.2. Quantified statements of type "Q B **X** are A" where Q is relative

In this case, we need to compute $\mu_Q(\mathrm{p})$ where $\mathrm{p} = \mathrm{c}/\mathrm{d}$ such that:

- c is the cardinality of the fuzzy set $(A \cap B)(\mathrm{X})$ made of elements from X which satisfy fuzzy condition A and condition B ($\forall x \in \mathrm{X}$, $\mu_{A\cap B(\mathrm{X})}(x) = \min(\mu_A(x), \mu_B(x))$),
- d is the cardinality of the fuzzy set $B(\mathrm{X})$ made of elements from X which satisfy fuzzy condition B.

For the proportion c/d to be defined, d has to belong to Z_f^*, which means $\forall\alpha \in]0, 1]$, $a_d \neq 0$, which implies $\forall\alpha \in]0, 1]$, $|B(\mathrm{X})_\alpha| \neq 0$ (in other words, fuzzy set $B(\mathrm{X})$ is normalized). The gradual rational number c/d is defined by the couple (c, d). A canonical representation for c/d is (see Subsection 2.3):

$$\forall\, \alpha \in]0, 1],\ \mathrm{p}(\alpha) = \mathrm{c}(\alpha)/\mathrm{d}(\alpha).$$

The cardinality c (respectively d) being that of the fuzzy set $A \cap B(\mathrm{X})$ (respectively $B(\mathrm{X})$), we get:

$$\forall\, \alpha \in]0, 1],\ \mathrm{p}(\alpha) = |(A \cap B(\mathrm{X}))_\alpha|)/|B(\mathrm{X})_\alpha|.$$

The application of a predicate Q on a gradual rational number such as p gives a gradual truth value S defined by (see Subsection 2.4):

$$\forall\, \alpha \in]0, 1],\ a_S(\alpha) = \mu_Q(\mathrm{p}(\alpha)/\mathrm{n}) = \mu_Q(|(A \cap B(\mathrm{X}))_\alpha|/|B(\mathrm{X})_\alpha|).$$

The fuzzy truth value S expresses the satisfaction of each α-cut of $A(\mathrm{X})$ and $A \cap B(\mathrm{X})$ with respect to the linguistic quantifier.

In case of a not normalized fuzzy set $B(\mathrm{X})$, one may consider that empty α-cuts receive an evaluation of 0:

$$\begin{aligned}&\forall\, \alpha \in]0, \max_{x\in X} \mu_B(x)],\\ &\qquad a_S(\alpha) = \mu_Q(\mathrm{p}(\alpha)/\mathrm{c}(\alpha)) = \mu_Q(|(A \cap B(\mathrm{X}))_\alpha|/|B(\mathrm{X})_\alpha|).\\ &\forall\, \alpha \in]\max_{x\in X} \mu_B(x), 1], a_S(\alpha) = 0.\end{aligned}$$

The value α is viewed as a quality threshold for the satisfactions with respect to A and B. When the minimum is chosen as a t-norm to define $A \cap B(\mathrm{X})$, the value $\mu_S(\alpha)$ is the degree of truth of the statement : "among the elements which satisfy B at least at level α, the proportion of elements x with $\mu_A(x) \geq \alpha$, is in agreement with Q" (since we have $(A \cap B(\mathrm{X}))_\alpha = A(\mathrm{X})_\alpha \cap B(\mathrm{X})_\alpha$). In other words, $\mu_S(\alpha)$ represents the truth value for "Q B X are A" when considering the two α-cuts $B(\mathrm{X})_\alpha$ and $A(\mathrm{X})_\alpha$) ($\mu_S(\alpha)$ is the truth value of "Q elements in $B(\mathrm{X})_\alpha$ are in $A(\mathrm{X})_\alpha$"). The value $\mu_S(\alpha)$ represents the truth value for the quantified statement when considering the interpretations at level α of the two fuzzy sets. The gradual truth value provides the different results given by the different interpretations of the fuzzy sets $A(\mathrm{X})$ and $B(\mathrm{X})$.

Example 6 *We consider the statement* "about half B X *are* A" *where* $\mathrm{X} = \{x_1, x_2, x_3, x_4\}$ *and the linguistic quantifier* about half *from Fig. 7.*

The different satisfactions with respect to B and A are described by Fig. 8 and we obtain the fuzzy truth value given by Fig. 9. As an example, we get $a_S(0.6) = 1/3$ *because* $|(A \cap B(\mathrm{X}))_{0.6}|/|B(\mathrm{X})_{0.6}| = 2/3$ *and* $\mu_Q(2/3) = 1/3$. *The truth value of the statement* "about half *elements in* $\{x$ *such that* $\mu_B(x) \geq 0.6\}$ *are in* $\{x$ *such that* $\mu_B(x) \geq 0.6\}$" *is 1/3.*

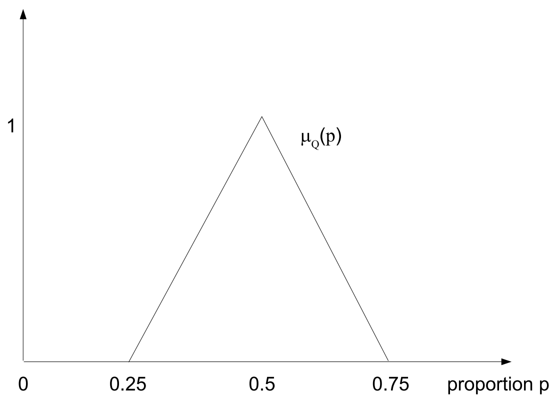

Figure 7. The quantifier *about half*

	x_1	x_2	x_3	x'
$\mu_B(x_i)$	1	0.9	0.7	0.3
$\mu_A(x_i)$	0.8	0.3	1	1
$\mu_{A\cap B}(x_i)$	0.8	0.3	0.7	0.3

Figure 8. The satisfaction with respect to B and A

5. A scalar value for the evaluation

The gradual truth value S computed in the previous section represents the different satisfactions of the different α-cuts with respect to the linguistic quantifier.

This gradual truth value can be defuzzified in order to obtain a scalar evaluation (set in $[0, 1]$). Various interpretations can be associated to this defuzzification and we consider the following one (since it is more natural):

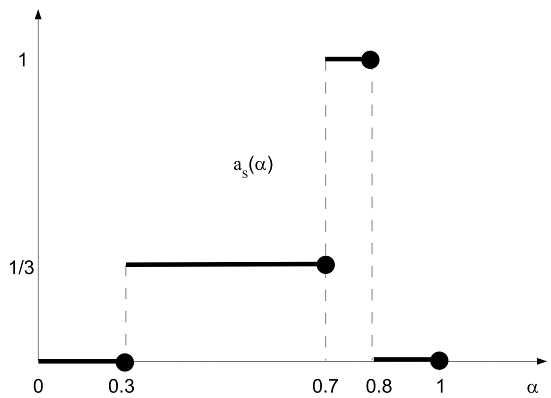

Figure 9. The fuzzy truth value for "*about half* B X are A"

"the higher the scalar interpretation, the more levels α have a high truth value $\mu_S(\alpha)$".

In case of a gradual truth value provided by a quantified statement, we get:

"the higher the scalar interpretation, the more α-cuts satisfies the constraint defined by the linguistic quantifier".

A scalar interpretation of 1 for "Q X are A" (respectively "Q B X are A") means that whatever is the chosen interpretation for $A(x)$ (respectively $A(x)$ and $B(x)$) the cardinality of elements A (respectively the proportion of elements A among the B elements) is fully in agreement with Q. Otherwise, the higher the scalar evaluation, the more there exists interpretations such that the cardinality (respectively the proportion) highly satisfies Q.

In Section 5.1, we consider a quantitative defuzzification (since based on an additive measure (a surface)), while in Section 5.2 we consider a qualitative defuzzification (based on a non additive computation).

5.1. A quantitative approach

In this approach, the surface of the fuzzy truth value is delivered to the user. The scalar interpretation is then:

$$\delta = \int_{\alpha=0}^{1} a_S(\alpha) d\alpha.$$

Value δ is the area delimited by function a_S. Since this function is a stepwise function, we get:

$$\delta = (\alpha_1 - 0) * a_S(\alpha_1) + (\alpha_2 - \alpha_1) * a_S(\alpha_2) + \ldots + (\alpha_n - \alpha_{n-1}) * a_S(\alpha_n),$$

where the discontinuity points are $(\alpha_1, a_S(\alpha_1))$, $(\alpha_2, a_S(\alpha_2))$, …, $(\alpha_n, a_S(\alpha_n))$, with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. When dealing with "$Q$ X are A" statements, it has been shown (Lietard and Rocacher, 2005) that this approach is a generalization of the OWA based interpretation.

EXAMPLE 7 *We consider the statement* "about half B X *are* A" *of Example 6 and the gradual truth value given by Fig. 9. We compute:*

$$\delta = (0.7 - 0.3) * 1/3 + (0.8 - 0.7) * 1 \approx 0.233.$$

This result is in accordance with our intuition since it seems that the proportion of elements which are A among the B elements is near to 2/3 (with $\mu_Q(2/3) = 1/3$*).*

5.2. A qualitative approach

According to this approach, the scalar interpretation takes into consideration two aspects:

- a guaranteed (minimal) satisfaction value β associated to the α-cuts (β must be as high as possible),
- the repartition of β among the α-cuts (β should be attained by the most possible α-cuts).

Obviously, these two aspects are in opposition since, in general, the higher β, the smaller the repartition. The scalar interpretation δ reflects a compromise between these two aspects and we get:

$$\delta = \max_{\beta \in]0,1]} \min(\beta, \text{each}\,(\beta)),$$

where each(β) means "for each level α, $a_S(\alpha) \geq \beta$". The truth value for each(β) can be a matter of degree and we propose to sum the lengths of intervals (of levels) where the threshold β is reached:

$$\text{each}\,(\beta) = \sum_{]\alpha_i,\alpha_j] \text{ such that } \forall \alpha \in]\alpha_i,\alpha_j],\, a_S(\alpha) \geq \beta} (\alpha_j - \alpha_i).$$

The higher each(β), the more numerous the levels α for which $\mu_S(\alpha) \geq \beta$. In particular, each(β) equal 1 means that for each level α, $a_S(\alpha)$ is larger than (or equal to) β.

When dealing with "Q X are A" statements, it has been shown (Bosc and Lietard, 2005) that this approach is a generalization of the Sugeno fuzzy integral based interpretation. In addition, from the computational point of view, the definition of δ needs to handle an infinity of values β. However, it is possible (Bosc and Lietard, 2005) to restrict computations to β values belonging to the set of "effective" $a_S(\alpha)$ values:

$$\delta = \max_{\{\beta \mid \exists \alpha \text{ such that } \beta = a_S(\alpha)\}} \min(\beta, \text{each}\,(\beta)).$$

EXAMPLE 8 *We consider the statement "*about half B X *are* A*" of Example 7 and the fuzzy truth value given by Fig. 9. The values* β *to be considered are 1/3 and 1. Furthermore* $\text{each}\,(1/3) = 0.5$ *and* $\text{each}\,(1) = 0.1$. *We get* $\delta = \max(\min(1/3, 0.5), \min(1, 0.1)) = 1/3$. *As shown in Example 7, a truth value of 1/3 for "*about half B X *are* A*" is coherent.*

6. Evaluation of conditions involving an aggregate

The evaluation of quantified statements is based on the expression of cardinalities of fuzzy sets, which justifies the use of gradual number theory. This section shows that it is possible to go far beyond the evaluation of quantified statements, in particular - to evaluate conditions calling on an average value of a fuzzy set, as in the example "the average salary of *young* employees is *high*" where *young* and *high* are two vague conditions defined by fuzzy sets. To evaluate this condition reverts to compute the average salary of a fuzzy set of salaries — salaries of *young* employees — and to confront this average value with the fuzzy condition *high*. The computation of the average can be achieved thanks to the gradual number theory (the addition and division being defined), the average being represented by a gradual number. This gradual number is confronted to condition *high*, which leads to a gradual truth value, which can be defuzzified in order to provide a scalar evaluation of the condition.

In the same spirit, it is also possible to evaluate others kinds of conditions, involving the aggregate max or min, as in "the maximum salary of *young* employees is *high*". The evaluation of these conditions needs to define the application of the max and min operators to gradual numbers.

Section 6.1 shows how to evaluate conditions of the type "avg(A) is C" where the average value of fuzzy set A is confronted to the fuzzy predicate C. Section 6.2 proposes a computation to evaluate conditions of type "max(A) is C" and "min(A) is C" where A is a fuzzy set and C a fuzzy condition.

6.1. Evaluation of conditions of type "avg (A) is C"

Let A be a fuzzy set of numerical values. The average value of fuzzy set A is given by the following ratio:

$$\text{Avg}\,(A) = \frac{\text{Sum}\,(A)}{\text{Card}\,(A)},$$

where $\text{Sum}\,(A)$ is the sum of elements from A while $\text{Card}\,(A)$ is the gradual integer representing its cardinality (its FGCount). For the computation to be founded, it is necessary for $\text{Card}\,(A)$ to belong to N_f^*, which implies that its assignment function satisfies $\forall\alpha \in]0,1]$, $a_{\text{Card}\,(A)}(\alpha) \neq 0$ which implies $\forall\alpha \in]0,1]$, $|A_\alpha| \neq 0$ (A is a normalized fuzzy set).

We propose to represent $\text{Sum}\,(A)$ by a gradual number and, as a consequence, the average can be computed in the form of a gradual number. To compute

Sum (A), it is necessary to define the fuzzy set A as the union of several fuzzy singletons $\{\mu_A(x)/x\}$, where x belongs to the support of the fuzzy set. Since A is defined on a numerical universe, each fuzzy singleton $\{\mu_A(x)/x\}$ represents a gradual number and we get:

$$\text{Sum}\,(A) = \textstyle\sum_{x \in Support(A)} \{\mu_A(x)/x\}$$

where each gradual number $\{\mu_A(x)/x\}$ is defined by the following assignment function $a_x(\alpha) = x$ where $\alpha \leq \mu_A(x)$ and 0 elsewhere (see Fig. 10). With this definition for Sum (A), we get the following result :

$$\forall\, \alpha \in]0,1],\ a_{\text{Avg}\,(A)}(\alpha) = \frac{\sum_{x \in A_\alpha} x}{|A_\alpha|} = \text{avg}\,(A_\alpha).$$

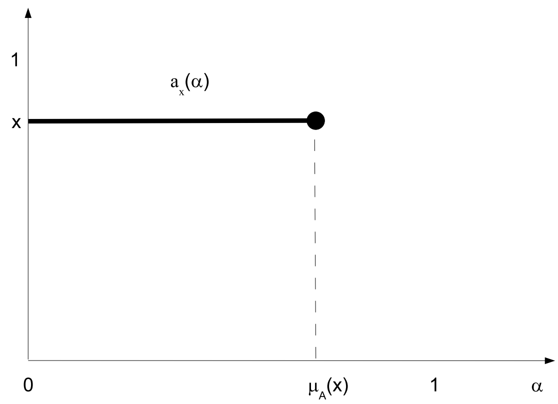

Figure 10. The assignment function of a singleton $\{\mu_A(x)/x\}$

In other words, the assignment function $a_{Avg(A)}$ provides the average value of the different α-cuts of the fuzzy set A.

Proof. The starting point is:

$$\text{Avg}\,(A) = \frac{\sum_{x \in Support(A)} \{\mu_A(x)/x\}}{\text{Card}\,(A)}\,.$$

From the definition of the division and addition of gradual numbers we get:

$$\forall\, \alpha \in]0,1],\ a_{\text{Avg}\,(A)}(\alpha) = \frac{\sum_{x \in Support(A)} a_x(\alpha)}{a_{\text{Card}\,(A)}(\alpha)}.$$

Since:

$$a_{\text{Card}\,(A)}(\alpha) = |A_\alpha|$$

and $a_x(\alpha) = x$ where $\alpha \leq \mu_A(x)$ and 0 elsewhere we get:

$$\forall\, \alpha \in]0,1],\ a_{\text{Avg}\,(A)}(\alpha) = \frac{\sum_{x \in Support(A) \text{ and } \mu_A(x) \geq \alpha} x}{|A_\alpha|}$$
$$= \frac{\sum_{x \in A_\alpha} x}{|A_\alpha|} = \text{avg}\,(A_\alpha).$$

The assignment function of the average value being defined, it is possible to compute the gradual truth value S of condition "avg (A) is C":

$$\forall \alpha \in]0,1], a_{\text{Avg}\,(A)}(\alpha) = \mu_C(a_{\text{Avg}\,(A)}(\alpha))$$
$$= \mu_C(\frac{\sum_{x \in A_{alpha}} x}{|A_\alpha|}) = \mu_C(\text{avg}\,(A_\alpha).$$

This gradual truth value provides the satisfaction of each α-cut with respect to condition "its average value is C".

However, this approach is limited to fuzzy sets A such that $\forall \alpha \in]0,1]$, $|A_\alpha| \neq 0$. One may think of extending this result by considering that empty α-cuts get the satisfaction of 0:

$$\forall \alpha \in]0,1],\ a_S(\alpha) = \mu_C(\frac{\sum_{x \in A_\alpha} x}{|A_\alpha|})$$

when $|A_\alpha| \neq 0$ and 0 otherwise.

Example 9 *The statement "*avg (A) *is* high*" is considered with the following fuzzy set A:*

$$A = 0.1/1 + 0.1/2 + 0.1/3 + 0.1/4 + 0.1/5 + 0.1/15 + 0.2/200$$
$$+ 0.5/700 + 0.8/500 + 1/600.$$

It is assumed that: $\mu_{high}(203) = 0.2$, $\mu_{high}(500) = 0.8$, $\mu_{high}(550) = 0.9$ *and* $\mu_{high}(600) = 1$. *The different values taken by the average are provided by Fig. 11 and the gradual truth value for condition "*avg (A) *is* high*" is given by Fig. 12.*

α	0.1	0.2	0.5	0.8	1
avg(A_α)	203	500	600	550	600
$\mu_{high}(avg(A_\alpha))$	0.2	0.8	1	0.9	1

Figure 11. The truth values of "avg (A_α) is *high*"

This gradual truth value can be defuzzified according to the two methods introduced in Section 5. As a consequence, we obtain two scalar values, a

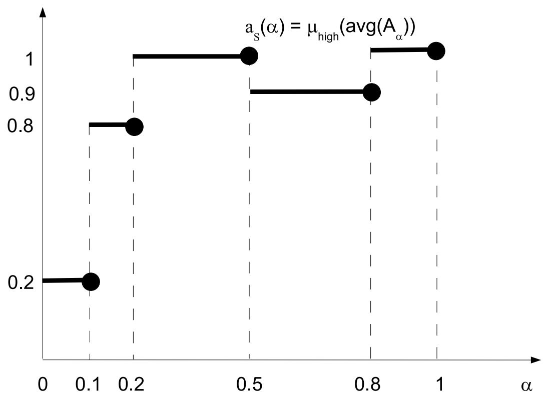

Figure 12. Gradual truth value for "avg (A_α) is *high*"

quantitative scalar value which represents the surface delimited by the gradual truth value:

$$\delta = \int_{\alpha=0}^{1} a_S(\alpha) d\alpha$$

and a qualitative scalar value:

$$\delta = \max_{\{\beta \mid \exists \alpha \text{ such that } \beta = a_S(\alpha)\}} \min(\beta, \text{each}\,(\beta))$$

where each (β) is defined by:

$$\text{each}\,(\beta) = \sum_{]\alpha_i, \alpha_j] \text{ such that } \forall\, \alpha \in\,]\alpha_i, \alpha_j]\ a_S(\alpha) \geq \beta} (\alpha_j - \alpha_i).$$

Example 10 *We consider the previous example, and the gradual truth value provided by Fig. 12. The quantitative scalar value is:*

$$\delta = (0.1 * 0.2) + (0.1 * 0.8) + (0.3 * 1) + (0.3 * 0.9) + (0.2 * 1) = 0.87.$$

To compute the qualitative scalar value we need to determine the values for function each. Fig. 12 gives: each $(0.2) = 1$, each $(0.8) = 0.9$, each $(0.9) = 0.8$ *and* each $(1) = 0.5$. *The qualitative scalar value is then:*

$$\delta = \max(\min(0.2, 1), \min(0.8, 0.9), \min(0.9, 0.8), \min(1, 0.5)) = 0.8.$$

*The statement "*avg (A) *is* high*" is rather true (at degree 0.8 or 0.87) since every interpretation of fuzzy set A (except for the lowest ones) strongly satisfies condition "the average is* high*" (see Fig. 12).*

6.2. Evaluation of conditions of type "max(A) is C" and "min(A) is C"

As explained in Subsection 6.1, the fuzzy set A can be described by the union of several fuzzy singletons $\{\mu_A(x)/x\}$ where x belongs to the support of the fuzzy set (A being defined on a numerical universe, each of these fuzzy singleton is a gradual number). The maximum (respectively minimum) value of the fuzzy set A is the maximum (respectively minimum) value among the different $\{\mu_A(x)/x\}$:

$$\text{Max}\,(A) = \max_{x\in Support(A)}\{\mu_A(x)/x\}$$
$$(\text{respectively Min}\,(A) = \min_{x\in Support(A)}\{\mu_A(x)/x\}).$$

The maximum (respectively minimum) of two gradual numbers x and y (their assignment function being respectively a_x and a_y) is given by (Dubois and Prade, 2005):

$$\forall\, \alpha \in]0,1],\ a_{\max(x,y)}(\alpha) = \max(a_x(\alpha), a_y(\alpha)),$$
$$(\text{respectively}\, a_{\min(x,y)}(\alpha) = \min(a_x(\alpha), a_y(\alpha))).$$

Finally, the maximum value (respectively minimum value) of fuzzy A is a gradual number whose assignment function is:

$$\forall\, \alpha \in]0,1],\ a_{\text{Max}\,(A)}(\alpha) = \max_{x\in\text{support}\,(A)} a_x(\alpha),$$
$$(\text{respectively}\, \forall\, \alpha \in]0,1],\ a_{\text{Min}\,(A)}(\alpha) = \min_{x\in\text{support}\,(A)} a_x(\alpha)).$$

It is obvious to show that the assignment function of the maximum value (respectively minimum value) describes the different maximum values (respectively minimum values) taken by the different α-cuts (assuming that the minimum and maximum of an empty set is value 0):

$$\forall\, \alpha \in]0,1],\ a_{\text{Max}\,(A)}(\alpha) = \max\{x \in |A_\alpha|\},$$
$$(\text{respectively}\, \forall\, \alpha \in]0,1],\ a_{\text{Min}\,(A)}(\alpha) = \min\{x \in |A_\alpha|\}).$$

The gradual truth value S of condition "max(A) is C" (respectively "min(A) is C") is:

$$\forall\, \alpha \in]0,1],\ a_S(\alpha) = \mu_C(\max\{x \in |A_\alpha|\})$$
$$(\text{respectively}\, \forall\, \alpha \in]0,1],\ a_S(\alpha) = \mu_C(\min\{x \in |A_\alpha|\}).)$$

This gradual truth value provides the satisfaction of each α-cut with respect to condition "its maximum values is C" (respectively "its minimum value is C"), assuming that the maximum (respectively minimum) value of an empty set is 0. Here again, this gradual truth value can be defuzzified according to the two methods introduced in Section 5.

Example 11 *The statement "*max(A) *is* high*" is considered with the following fuzzy set A:*

$$A = 0.2/200 + 0.5/700 + 0.8/400 + 1/500.$$

If we assume that $\mu_{high}(500) = 0.8$ *and* $\mu_{high}(700) = 1$, *we get the results provided in Fig. 13.*

α	0.2	0.5	0.8	1
$\max(A_\alpha)$	700	700	500	500
$\mu_{high}(max(A_\alpha))$	1	1	0.8	0.8

Figure 13. The truth values of "max(A_α) is *high*"

*The gradual truth value for condition "*avg(A) *is* high*" is given by Fig. 14.*

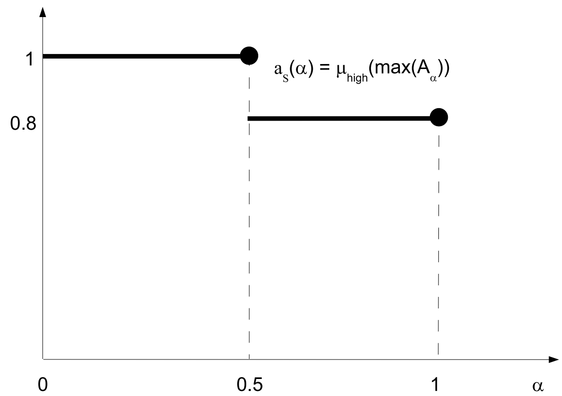

Figure 14. Gradual truth value for "max(A_α) is *high*"

This gradual truth value can be defuzzified according to the two methods introduced in Section 5. As a consequence, we obtain two scalar values, a quantitative scalar value which represents the surface delimited by the gradual truth value:

$$\delta = (0.5 * 1) + (0.5 * 0.8) = 0.9,$$

and a qualitative scalar value:

$$\delta = \max_{\{\beta \mid \exists \alpha \text{ such that } \beta = a_S(\alpha)\}} \min(\beta, \text{each}(\beta))$$

where each(β) *is defined by:*

$$\text{each}(\beta) = \sum_{]\alpha_i, \alpha_j] \text{ such that } \forall \alpha \in]\alpha_i, \alpha_j],\, a_S(\alpha) \geq \beta} (\alpha_j - \alpha_i).$$

Fig. 14 gives: $\text{each}\,(0.8) = 1$*,* $\text{each}\,(1) = 0.5$*. The qualitative scalar value is then:*

$$\delta = \max(\min(0.8, 1), \min(1, 0.5)) = 0.8.$$

The statement "$\max(A)$ *is high" is rather true (at degree 0.8 or 0.9) since every interpretation of fuzzy set A strongly satisfies condition "the maximum value is high" (see Fig. 14).*

7. Conclusion

This paper is situated at the junction of the evaluation of complex fuzzy conditions and fuzzy arithmetic introduced in Rocacher and Bosc (2003, 2005). Gradual numbers provide a new framework, where operations such as difference, division, minimum and maximum can be exactly evaluated. These operations can be used to define complex conditions, where their results are confronted with a fuzzy predicate. As a consequence, arithmetic on gradual numbers allows for evaluating quantified statements and conditions of type "agg(A) is C" where A is a fuzzy set, C a fuzzy predicate and agg is either the aggregate average, minimum or maximum.

Such an evaluation provides a gradual truth value, which represents the different interpretations of the result, each interpretation being an exact evaluation of the predicate "the aggregate satisfies C" computed on an α-cut. A defuzzification process can be applied on the gradual truth value in order to obtain a scalar result, which can be viewed as a kind of summary. In this paper two types of scalar values can be distinguished: the first one corresponds to a quantitative view of the fuzzy value, the second one is a qualitative view (many other defuzzification strategies can also be investigated depending on application domains).

This work is a first attempt to set the evaluation of conditions involving an aggregate in the framework of an extended arithmetic and algebra. This aspect is very important since properties provided by the algebraic framework hold. Further studies may concern the comparison and generalization of the qualitative and quantitative approaches. In a further step, complementary studies have to be conducted to set an algebraic framework to define logical operations between gradual truth values.

References

Bosc, P. and Lietard, L. (2005) A general technique to measure gradual properties of fuzzy sets. *Proc. of the 11th World Congress of the International Fuzzy Systems Association (IFSA05)*, Beijing, China, 485-490.

Bosc, P. and Pivert, O. (1992) Some Approaches for Relational Databases Flexible Querying. *Journal of Intelligent Information Systems* **1**, 323-354.

Bosc, P. and Pivert, O. (1995) SQLf: A relational database language for fuzzy querying. *IEEE Transactions on Fuzzy Systems* **3**, 1-17.

Delgado, M., Sanchez, D. and Vila, M.P. (2000) Fuzzy cardinality based evaluation of quantified sentences. *International Journal of Approximate Reasoning* **23**, 23-66.

Diaz-Hermida, F., Bugarin, A. and Barro, S. (2003) Definition and classification of semi-fuzzy quantifiers for the evaluation fuzzy quantified sentences. *International Journal of Approximate Reasoning* **34**, 49-88.

Dubois, D. and Prade, H. (1985) Fuzzy cardinality and the modeling of imprecise quantification. *Fuzzy Sets and Systems* **16**, 199-230.

Dubois, D. and Prade, H. (1987) Fuzzy numbers: an overview. Analysis of fuzzy information. *Mathematics and Logics* **I**, 3-39.

Dubois, D. and Prade, H. (2005) Fuzzy elements in a fuzzy set. In: *Proceedings of the 11th World Congress of the Inter. Fuzzy Systems Assoc. (IFSA'05)* , Beijing, China.

Glockner, I. (1997) DFS–An axiomatic approach to fuzzy quantification. Technical Report TR97-06, University of Bielefeld.

Glockner, I. (2004) *Fuzzy Quantifiers in Natural Language Semantics and Computational Models*. Der Andere Verlag (Germany).

Glockner, I. (2004) Evaluation of quantified propositions in generalized models of fuzzy quantification. *International Journal of Approximate Reasoning* **37**, 93-126.

Lietard, L. and Rocacher, D. (2005) A generalization of the OWA operator to evaluate non monotonic quantifiers. *Proc. of the 4th Conference of the European Society for Fuzzy Logic and Technologie and 14^{éme} Rencontres francophones sur la logique floues et ses applications* (EUSFLAT05 and LFA05), 183-188.

Losada, D.E., Diaz-Hermida, F. and Bugarin, A. (2006) Semi-fuzzy Quantifiers for Information Retrieval. In: E. Herrera-Viedma, G. Pasi and F. Crestani, eds., *Soft Computing in Web Information Retrieval*. Springer Berlin, Heidelberg, 195-220.

Rocacher, D. (2003) On fuzzy bags and their application to flexible querying. *Fuzzy Sets and Systems* **140** (1), 93-110.

Rocacher, D. and Bosc, P. (2003) About Zf, the set of fuzzy relative integers, and the definition of fuzzy bags on Zf. **LNCS 2715**, Springer-Verlag, 95-102.

Rocacher, D. and Bosc, P. (2005) The set of fuzzy rational numbers and flexible querying. *Fuzzy Sets and Systems* **155** (3), 317-339.

Wygralak, M. (1999) Questions of cardinality of finite fuzzy sets. *Fuzzy Sets and Systems* **102**, 185-210.

Zadeh, L.A. (1965) Fuzzy sets. *Inform. Contr.* **8**, 338-353.

Zadeh, L.A. (1983) A computational approach to fuzzy quantifiers in natural languages. *Computer Mathematics with Applications* **9**, 149-183.