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Fuzzy certainty on fuzzy values[*][†]

by

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Abstract: Imprecision and uncertainty appear together in many applications of soft computing. Imprecise and uncertain values are usually expressed by means of linguistic terms, specially when they have been provided *by* or *for* a human being. However, in many applications it is desirable that both aspects are combined into a single value that appropriately describes the intended information. In this work, we extend our previous research on this topic and we study how to combine imprecision and uncertainty when both of them are expressed by fuzzy numbers and the final goal is to obtain a normalized fuzzy value that provides the same amount of information about the described fact.

Keywords: fuzzy certainty, uncertainty qualification, fuzzy statements, fuzzy values, information measure.

1. Introduction

Imprecision and uncertainty coexist in many applications. More often than desirable, a given numerical information is affected by imprecision and uncertainty at the same time. In fact, this problem was early stated by Zadeh (1978) in his description of quantification of truth, probability and possibility. For example, optimization methods in fuzzy graphs (Delgado, Verdegay and Vila, 1990) have to deal with certainty values associated to fuzzy values, which are combined using the first ones to truncate the second ones. The same happens in problems related to sensor reliability (Dubois, Prade and Yager, 1999) where imprecision can be represented as a fuzzy value and uncertainty is a measure of the proper sensor performance. Uncertain fuzzy databases (Bordogna and Pasi, 2000) or the application of knowledge discovery techniques in data warehouses are other examples of domains, where uncertainty and imprecision have to be managed

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at the same time. Thus, in many situations, if the reasoning system is based on Possibility Theory, we have to transform all the available information (possibility distribution provided by the fuzzy value and the certainty value) into a single possibility distribution.

Let us suppose that we have a fuzzy value A understood as acting as a fuzzy restriction on the possible values of a variable X, and that this value is affected by an uncertainty degree, say α. Then, the problem is to represent a qualified statement like *it is* α*-certain that* X *is* A, that is, the problem of uncertainty qualification of fuzzy statements already stated by Zadeh (1978). In Bouchon-Meunier et al. (1999), the reader can find a good overview of proposals on how to face the problem.

This situation was formulated in a previous paper (Gonzalez, Pons and Vila, 1999) as a conditional expression using the generalized modus ponens, in the following terms:

- *If the certainty level is 1, then the value is* A*.*
- *If the certainty level is* $\alpha < 1$*, then the value is* A', where A' is a joint value of the original fuzzy set A and its certainty α.

In this way, the qualified statement *it is* α*-certain that* X *is* A is represented as X *is* A'.

A natural way to solve the problem is to consider that the numerical information we are handling is defined as:

$$A'(x) = I(\alpha, A(x))$$

where I is a material implication function which reflects the previous interpretation and $A(x)$ and $A'(x)$ are membership functions.

In the literature, two main approaches exist of dealing with imprecise and uncertain data:

1. To Truncate: If the information is A *with certainty* α, then A' is defined by the membership function $A'(x) = min(\alpha, A(x))$ which directly implies that we are using Mamdani's implication in our reasoning.
2. To Expand: If we assume that α is a necessity, then A' is given by the membership function $A'(x) = max(1 - \alpha, A(x))$, which corresponds to Kleene-Dienes' implication as foundation of our reasoning.

In relation to the second idea, in Yager (1984) a new proposal for certainty qualification is suggested

$$A'(x) = \alpha \otimes A(x) + 1 - \alpha$$

where $\otimes$ is a conjunction operator. Obviously, this model generalizes the use of the Kleene-Dienes' implication, but it maintains the same idea of expanding the imprecision of the fuzzy value in order to incorporate the uncertainty value in its representation.

Although these proposals can be useful in many applications, unfortunately they can also be inappropriate in many others. Mamdani's implication obliges us to work with non-normalized fuzzy values while Kleene-Dienes' implication, and the more general formulation of Yager, obliges to assign the same possibility to all the points of the underlying domain independently of the distance to the support set of the fuzzy value. Therefore, the proposed solutions give rise to a series of inconveniences: the interpretability, in some cases, and those ones derived from the use of non-normalized or non-trapezoidal fuzzy sets, in others[1].

As an alternative proposal, in Gonzalez, Pons and Vila (1999) we propose a certainty qualification method for trapezoidal fuzzy numbers that consists in increasing the imprecision around the support set of value A depending on an uncertainty value. That is, the imprecision is distributed according to a metric that takes into account the nearness to the original information. This proposal is based on the use of information measures that allow us to transform the uncertainty of the fuzzy statement into imprecision. For example, when we have the information that *"X is black"* with certainty α, it is not very convenient to assign a positive possibility to color white, as the expanding based method proposes, but to colors near enough to black depending on value α.

Therefore, the process we proposed in Gonzalez, Pons and Vila (1999) was to obtain A' in two steps:

1. First, considering that the height of a fuzzy number is the certainty degree associated to it (Dubois, 1983; Gonzalez, 1987), we use the certainty degree α associated to the fuzzy value A to truncate it at level α. After this operation, we obtain a non-normalized fuzzy set A^{α}. Nevertheless, the resulting fuzzy value remains trapezoidal.
2. Since, in many applications, non-normalized fuzzy sets give rise to a series of inconveniences, in a second step we normalize it. To do this, we assume that uncertainty is being translated into imprecision under particular conditions. The most important point to be considered is that the amount of information provided by the fuzzy number remains equal before and after the normalization process.

Therefore, with the above mentioned process the problem of uncertainty qualification of fuzzy statements when the uncertainty is expressed by a number in the interval $[0, 1]$ was solved, Gonzalez, Pons and Vila (1999). However, till now, data are expressed by means of an imprecise value A (e.g. represented by a trapezoidal fuzzy number) together with a real certainty level α associated to such fuzzy value.

Unfortunately, there exist many applications where certainty values are not expressed in a precise numeric but in a linguistic way, using terms like *not likely*, *very possible*, *almost impossible*, etc. This situation, called linguistic

[1] For example, when data are going to be stored (Pons et al., 2002), it is of great importance that they be normalized for the sake of simplicity in the representation, management, and understanding.

possibility qualification by Zadeh (1978), is usual not only when data are directly obtained from observations of human beings, but also in those problems where instruments are not as precise and reliable as needed.

More concretely, this paper has been motivated by a problem that scientists of the spatial mission ROSETTA (Colangeli et al., 2004) have been dealing with. ROSETTA includes, among other components, a Grain Impact Analyzer and Dust Accumulator (GIADA). One of the aims of the experiment is to measure the dust flows coming from the comet 67P/Churyumov-Gerasimenko and to control the deterioration of the performance of the instruments on board the orbital. The microbalances are used to weigh the comet dust that reaches Rosetta. In this problem, data are affected by both imprecision and uncertainty:

- Imprecision: The computation of the mass of comet dust during a period of time is made from the values given by the microbalances, whose precision is 10^{-10} gr. This means that particles whose mass is less than this quantity will not be detected by the instrument, until the accretion of them reaches the sensibility limit, introducing in the resulting value some imprecision.
- Uncertainty: The surface of the microbalances and, in general, all the instruments and the solar panels are contaminated along the time with dust, which is a source of uncertainty when a measurement is made. Moreover, as the deposited mass increases, the measurement of the microbalance goes to saturation; at this point, a cleaning of the device is carried out. Unfortunately, after this action, the response of the system will not be the original one because some dust will not be removed. As a consequence of this, the uncertainty level increases with time.

According to the above description and due to the fact that uncertainty affecting the obtained measures varies with every cleaning, it is very important for the performance of the method adopted to combine imprecision and uncertainty. Fig. 1 describes evolution of certainty along the process.

All these considerations claim for a mechanism similar to the one used in Gonzalez, Pons and Vila (1999) that permits us:

- To represent certainty in (linguistic) imprecise terms.
- To combine such certainty with the imprecision of the numerical information.
- To transform the resulting value into an equivalent fuzzy normalized one[2].

More formally, the problem we are tackling is to consider that the certainty value is also an imprecise value represented by a trapezoidal fuzzy set C. Thus, C is now a certainty measure given by a possibility distribution, that is, $\Pi(\alpha) = C(\alpha)$ where $\alpha \in [0,1]$. We extend the case *X is A with certainty* α to the case *X is A with certainty C* and the idea is to give an equivalent expression as *X is A''*. Therefore, our problem is to study how to obtain the A'' and to verify which properties are fulfilled.

[2]Here, the meaning of *equivalent* will be understood as *providing the same information.*

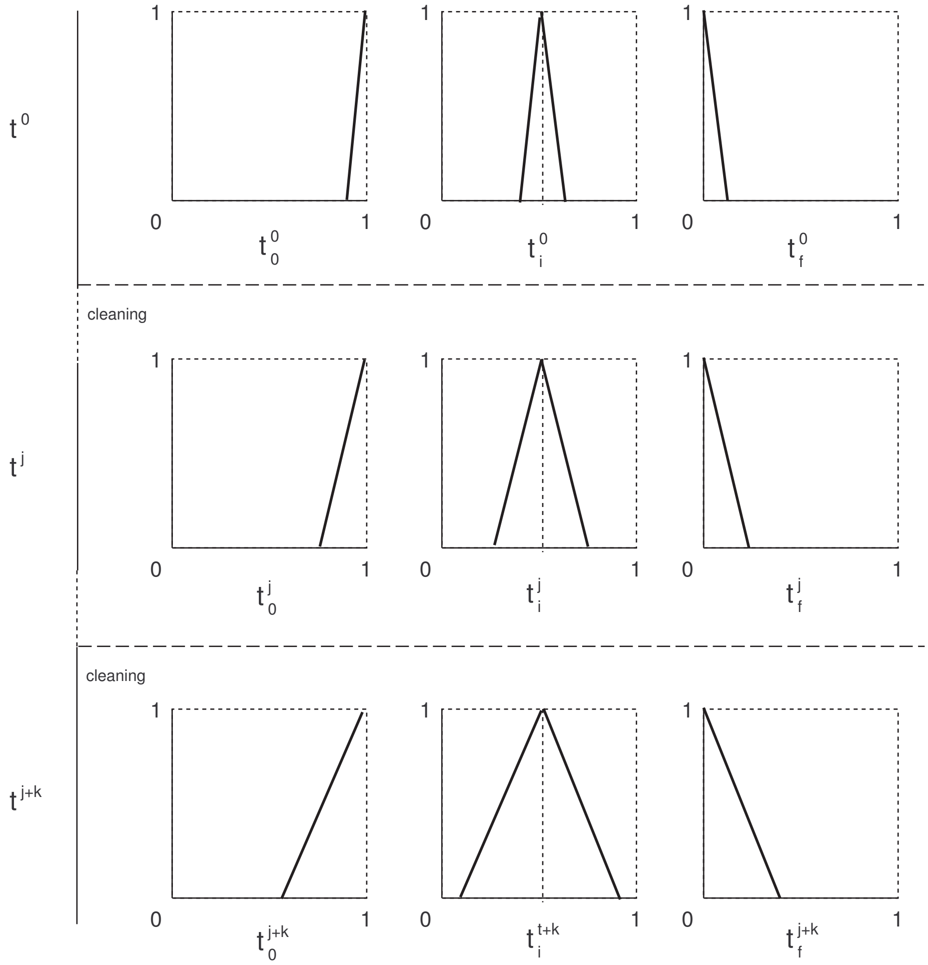

Figure 1. Certainty evolution of the microbalance measurements along time. t^j represents a period of time between two cleaning processes. At any moment t_i^j, the core point of certainty is estimated using a probabilistic model which takes into account the growing presence of dust in the microbalance, and, thus, its value decreases with time within the period t^j. The spread of the certainty label depends on the repetitions on the period t^j and is estimated using another probabilistic model that is built according to the effect that the residual damage produced in the microbalance during the cleaning process (e.g. some impurities cannot be removed) produces in the estimations of the certainty core. This spread increases with every cleaning process until the microbalance is considered to be useless.

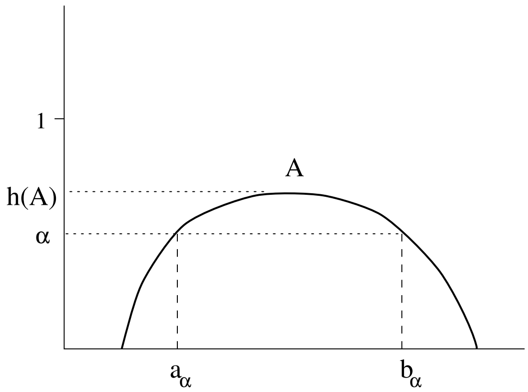

Figure 2. Fuzzy number.

The paper is organized as follows. First, we present a brief summary of the previous results on which our transformation is founded. Then, in Section 3, as the first contribution of this paper, we give the explicit expression of the transformation for LR fuzzy numbers, proving that the transformation of any LR fuzzy number produces also an LR fuzzy number with the same functions. After this, Section 4 addresses the problem of fuzzy uncertainty as the main contribution of the paper. We base our approach on the use of Sugeno's integral together with the possibility and necessity measures (associated to the certainty value C) in order to obtain the joint value of certainty and imprecision. This basic approach generates four possible combinations, which are deeply explained in both mathematical and graphical ways. A study of their properties will give us some criteria for choosing among them. Some concluding remarks and guidelines for further work end the paper.

2. Preliminary concepts and notation

In this section we will introduce some preliminary concepts and notation necessary for the understanding of this paper.

2.1. Fuzzy numbers and LR fuzzy numbers

A fuzzy number is a fuzzy representation of the real value of a property (attribute) when it is not precisely known.

Each interval $[a_\alpha, b_\alpha]$, with $\alpha \in (0, 1]$ (see Fig. 2) is called the α-cut of A. The set $Supp(A) = \{x \in I\!R \mid A(x) > 0\}$ is called the *support set of A*. In this paper, we will assume that a fuzzy number is every fuzzy set of the real line that verifies that its α-cuts are convex sets, its membership function is a continuous function and its support set is a bounded set of $I\!R$. Therefore, fuzzy numbers are fuzzy quantities whose α-cuts are closed and bounded intervals. We will use

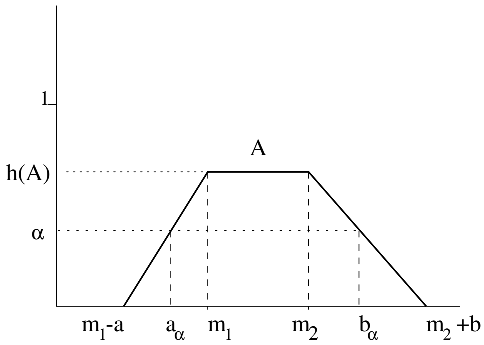

Figure 3. Trapezoidal fuzzy number.

$\widetilde{I\!R}$ to denote the set of fuzzy numbers, and $h(A)$ to denote the height of the fuzzy number A. For the sake of simplicity, we will use capital letters from the beginning of the alphabet to represent fuzzy numbers. If there is, at least, one point x verifying $A(x) = 1$, we say that A is a *normalized* fuzzy number.

Usually, a trapezoidal shape is used in order to represent fuzzy numbers. This representation is very practical as the fuzzy number is completely characterized by four parameters (m_1, m_2, a, b) and the height $h(A)$, as Fig. 3 shows. Other parametrical representations for fuzzy numbers can be found in Delgado, Vila and Voxman (1998). The interval $[m_1, m_2]$ (i.e, the set $\{x \in Supp(A) \mid \forall\, y \in I\!R,\ A(x) \geq A(y)\}$) will be called *modal set*. The values a and b are called left and right *spreads*, respectively.

A generalization of the idea behind a trapezoidal fuzzy number was introduced by Dubois and Prade (1987). Let us remind their parametric definition of an LR fuzzy number:

Definition 1 *Let us consider a function* $L : I\!R^+ \longrightarrow [0,1]$. *We say that* $L(.) \in \mathcal{L}$ *iff* L *verifies:*

1. L *is a decreasing function, that is:* $\forall x, y \in I\!R^+ \ ; \ x \leq y \Rightarrow L(x) \geq L(y)$
2. $L(0) = 1$
3. $\forall u | u > 0, L(u) < 1$
4. $L(1) = 0$, *or* $\forall u, L(u) > 0$ *and* $L(+\infty) = 0$.

Some examples of $\mathcal{L}$ functions are:

- $L(u) = max\{0, (1-u)\}$
- $L(u) = max\{0, (1-u)^p\}\ p > 0$
- $L(u) = max\{0, (1-u^p)\}\ p > 0$
- $L(u) = e^{-u^p}\ p > 0$
- $L(u) = max\{0, \frac{1}{1+u^p}\}\ p > 0$.

Definition 2 *Let us consider a fuzzy number A. We say that A has an LR-representation iff there are four real numbers, m_1, m_2, a, b (with $a > 0$ and $b > 0$) and two functions $L, R \in \mathcal{L}$ such that:*

$$\forall x \in I\!R, A(x) = \begin{cases} 0 & if\, x \leq m_1 - a \\ L(\frac{m_1 - x}{a}) & if\, m_1 - a \leq x \leq m_1 \\ 1 & if\, m_1 \leq x \leq m_2 \\ R(\frac{x - m_2}{b}) & if\, m_2 \leq x \leq m_2 + b \\ 0 & if\, x \geq m_2 + b \end{cases}$$

We will denote this fact as $A \equiv (m_1, m_2, a, b)_{LR}$.

It is easy to prove that any fuzzy number with continuous membership function admits an LR-representation. In this paper, we will consider LR-fuzzy numbers with finite support set.

The basic idea underlying this work is that when a fuzzy number is not normalized, the situation can be interpreted as a lack of confidence in the information provided by such a number (Dubois, 1983; Gonzalez, 1987). In fact, the height of the fuzzy number could be considered as a certainty degree of the represented value, and this implies that normalized fuzzy numbers represent imprecise quantities on which we have complete certainty.

Since the first step in our proposal is to truncate, we can consider that the truncated fuzzy number represents the imprecise information together with the original uncertainty.

In Gonzalez, Pons and Vila (1999), we show how uncertainty can be translated, using a suitable transformation, into imprecision, taking into account that reducing uncertainty about a fuzzy number implies increasing imprecision of the number. This transformation is made in such a way that the amount of information provided by the fuzzy number is the same before and after the modification. Our idea is to transform the truncated fuzzy number in order to obtain a completely certain fuzzy number. The next sections summarize these transformation process.

2.2. Information measure on fuzzy values

As pointed out in the previous section, we are going to translate fuzzy uncertainty into imprecision under given conditions. The most important of these conditions is that the amount of information provided by the fuzzy number remains equal after the transformation. Therefore, the first step is to define an information function for fuzzy numbers.

In Gonzalez, Pons and Vila (1999), we proposed an axiomatic definition of information, partially inspired by the *generalized information* given by Kampé de Fériet (1974) and that can be related to the precision indexes (Dubois and Prade, 1985) and the specificity concept introduced by Yager (1981).

DEFINITION 3 *Let* $\mathcal{D} \subseteq \widetilde{I\!R} \mid I\!R \subseteq \mathcal{D}$*; we say that* $I : \mathcal{D} \longrightarrow [0,1]$ *is an* **information function** *on* $\mathcal{D}$ *if it verifies:*

1. $I(A) = 1, \ \forall \ A \in I\!R$.
2. $\forall \ A, B \in \mathcal{D} \ \mid \ h(A) = h(B)$ *and* $A \subseteq B \Longrightarrow I(B) \leq I(A)$.

The information about fuzzy numbers may depend on different factors, in particular, on imprecision and certainty. In this work, we focus on general types of information related only to these two factors.

DEFINITION 4 *The* **imprecision** *(Gonzalez, 1987) of a fuzzy number is defined as follows:*

$$\forall \ A \in \widetilde{I\!R}, \ imp(A) = \int_0^{h(A)} (b_\alpha - a_\alpha \ )d\alpha.$$

That is, the imprecision function f coincides with the area below the membership function of the fuzzy value, and can also be expressed as follows:

$$\forall \ A \in \widetilde{I\!R}, \ imp(A) = \int_{m_1-a}^{m_2+b} A(x) \ dx.$$

There are many ways to build information functions but, for our purpose, we use an information that depends only of the height (certainty) and the imprecision of the fuzzy number. This information will permit, subsequently, the definition of transformations that keep constant the amount of information a fuzzy number provides.

In Gonzalez (1987) we defined the function:

$$I : \widetilde{I\!R} \longrightarrow [0,1]$$
$$\forall \ A \in \widetilde{I\!R}, \ I(A) = \frac{h(A)}{k \cdot imp(A) + 1}$$

where $h(A)$ is the height of A, imp(A) is the imprecision associated to A and $k \neq 0$ is a parameter that depends on the domain scale. The interpretation and selection of this parameter was studied in Gonzalez, Pons and Vila (1999). This is the simplest function that verifies the mentioned properties of information functions.

2.3. Transformation under precise certainty

Once we have an information function on fuzzy numbers, we can use it to define transformations that preserve the information amount it provides. The idea is to find an *equivalent* representation of the considered fuzzy number in such a way that we change uncertainty by imprecision keeping constant the balance between them, which is determined by the information function.

The aim of the transformations we are proposing in this section is, basically, to be able to modify the height of a fuzzy number keeping the information contained in it.

The definition of transformation will be obtained from the condition of equality in the information. As a first step, we must establish what we understand as transformation of a fuzzy number on a subset of $\widetilde{\mathbb{R}}$.

DEFINITION 5 *Let us consider* $\alpha \in (0,1]$ *and the class of fuzzy numbers* $D \subseteq \widetilde{\mathbb{R}}$. *We say that*

$$\mathcal{T}_\alpha : D \longrightarrow \widetilde{\mathbb{R}}$$

is a **transformation** *for an information function* I *on* D, *if it verifies that:*

1. $\mathcal{T}_\alpha(A) \in D$
2. $h(\mathcal{T}_\alpha(A)) = \alpha$
3. $I(\mathcal{T}_\alpha(A)) = I(A), \ \forall \ A \in D.$

We will note by λ the class of trapezoidal fuzzy numbers on $\mathbb{R}$. Given a fuzzy number $A \in \lambda$, we are looking for the conditions that another fuzzy number $\mathcal{T}_\alpha(A)$, with fixed height $\alpha \in (0,1]$, must satisfy to have the same information amount as A. Assuming the following conditions:

1. modal imprecision is preserved,
2. the increase/decrease of imprecision is equally distributed in the right and left sides of the fuzzy number independently of its shape,

we proposed in Gonzalez, Pons and Vila (1999) the following transformation:

DEFINITION 6 *Let* $A \in \lambda$ *such that*

$$A = \{(m_1, m_2, a, b), h(A)\}$$

where m_1, m_2, a *and* b *are shown in Fig. 3 and* $h(A)$ *is the height of* A.

Take $\alpha \in (0,1]$. *We will denote* $\Delta = \frac{\alpha - h(A)}{\alpha \cdot h(A)}$ *and define transformation*

$$T_\alpha(A) = \{(m_1, m_2, a + \frac{\Delta}{k}, b + \frac{\Delta}{k}), \alpha\}$$

for α, *for which the transformation makes sense (notice that some values of* α *lower than h(A) could produce negative spreads).*

In Fig. 4 we have graphically represented the behavior of T_α when the height is decreased and, therefore, imprecision is also decreased. On the other hand, in Fig. 5 it is shown how an increment of height produces an increment of imprecision. This result agrees with the following assertion: *"Imprecision and uncertainty can be considered as two antagonistic points of view about the same*

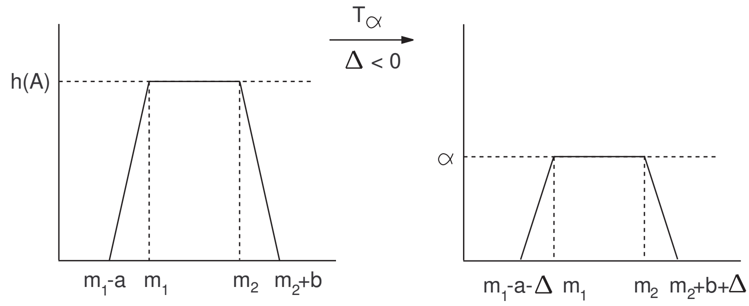

Figure 4. Transformation that decreases imprecision (k=1).

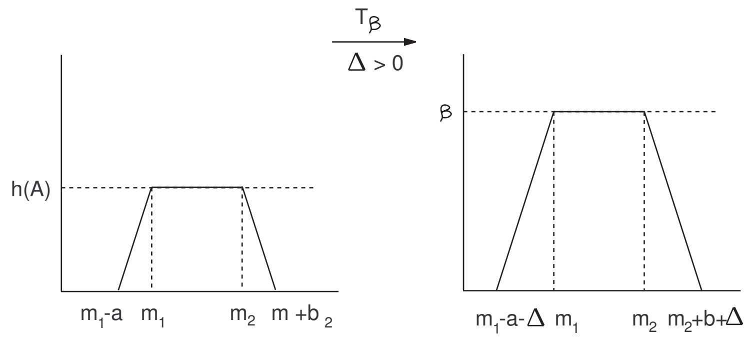

Figure 5. Transformation that increases imprecision (k=1)

reality, which is human imperfection.... and if the contents of a proposition is made more precise, then uncertainty will have to be augmented" (Dubois, 1983), which is a way to enunciate the principle of incompatibility between certainty and precision, established by Zadeh (1973).

For a deeper study of this transformation and its properties see Gonzalez, Pons and Vila (1999).

This transformation can be directly used to define the joint value that combines imprecision and certainty in a single value.

DEFINITION 7 *Let $A \in \lambda$ and $\alpha \in I\!R$. The joint value of A with certainty α preserving the information is called i-$\bowtie(A,\alpha)$ and is computed as*

$$i\text{-}\bowtie(A,\alpha) = T_1(A^\alpha),$$

where A^α is the truncation at level α of the original value A, and T_1 is the transformation that normalizes this truncated fuzzy number.

As we intended, the obtained i-$\bowtie(A,\alpha)$ remains a normalized trapezoidal fuzzy number.

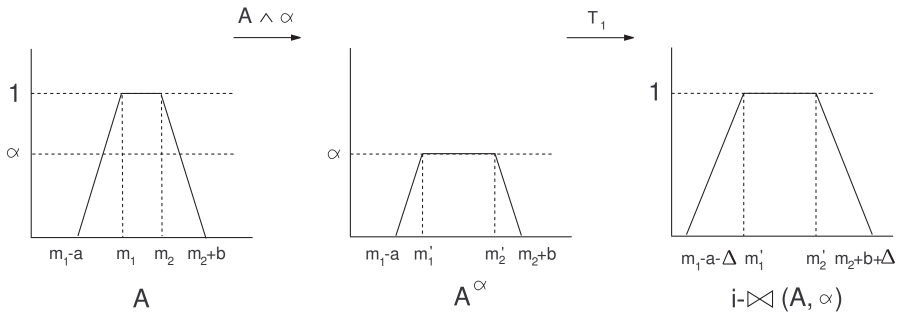

Figure 6. Computation of i-$\bowtie(A,\alpha)$.

Fig. 6 depicts the process of obtaining i-$\bowtie(A,\alpha)$ with both the truncation and transformation steps.

3. The transformation for LR fuzzy numbers

In the previous section, we have briefly introduced an approach to combine imprecision and certainty for trapezoidal fuzzy numbers. As we have previously mentioned, the proposed transformation was introduced in Gonzalez, Pons and Vila (1999). The first contribution of this paper is to analyze the previous approach in the more general case of LR fuzzy numbers.

As we will see, an interesting property of the proposed approach is that it keeps the LR representation of fuzzy numbers; that is, given an LR fuzzy number the result obtained after the combination of imprecision and certainty is another LR fuzzy number.

PROPERTY 1 *Let $A \equiv (m_1, m_2, a, b)_{LR}$ with a certainty value h, then the transformed i-$\bowtie(A,h)$ has also an LR representation $(m_1^{\mathcal{T}}, m_2^{\mathcal{T}}, a^{\mathcal{T}}, b^{\mathcal{T}})_{LR}$.*

Let $A \equiv (m_1, m_2, a, b)_{LR}$ be any LR fuzzy number and let us consider that we have truncated A at the level $h \in [0,1]$. Then the imprecision associated to A is:

$$imp(A) = \int_0^h (b_\alpha - a_\alpha) d\alpha$$

where

$$[a_\alpha, b_\alpha] = \{x | A(x) \geq \alpha\}$$

and the information it provides is:

$$I(A) = \frac{h}{k \cdot imp(A) + 1} \ .$$

Let us consider that A is transformed into i-$\bowtie(A, h)$ with height 1, both providing the same amount of information. We will prove that i-$\bowtie(A, h)$ also admits an LR-representation $(m_1^{\mathcal{T}}, m_2^{\mathcal{T}}, a^{\mathcal{T}}, b^{\mathcal{T}})_{LR}$ with the same L and R functions as A.

This proof consists of finding the four parameters of i-$\bowtie(A, h)$, taking into account that:

$$I(\text{i-}\bowtie(A, h)) = \frac{1}{k \cdot imp(\text{i-}\bowtie(A, h)) + 1} = \frac{h}{k \cdot imp(A) + 1} = I(A)$$

with

$$imp(\text{i-}\bowtie(A, h)) = \int_0^1 (b_\alpha^{\mathcal{T}} - a_\alpha^{\mathcal{T}}) d\alpha$$

and

$$[a_\alpha^{\mathcal{T}}, b_\alpha^{\mathcal{T}}] = \{x | \text{i-}\bowtie(A, h)(x) \geq \alpha\}.$$

According to the LR-representation we have:

$$a_\alpha = m_1 - aL^{-1}(\alpha) \text{ and } b_\alpha = m_2 + bR^{-1}(\alpha),$$
$$a_\alpha^{\mathcal{T}} = m_1^{\mathcal{T}} - a^{\mathcal{T}} L^{-1}(\alpha) \text{ and } b_\alpha^{\mathcal{T}} = m_2^{\mathcal{T}} + b^{\mathcal{T}} R^{-1}(\alpha).$$

Moreover, since i-$\bowtie(A, h)$ is the result of transforming the truncation of A, we have:

$$m_1^{\mathcal{T}}(h) = m_1 - aL^{-1}(h) \text{ and } m_2^{\mathcal{T}}(h) = m_2 + bR^{-1}(h).$$

According to the information equality, we have:

$$h + h \cdot k \cdot imp(\text{i-}\bowtie(A, h)) = 1 + k \cdot imp(A),$$

and, if we denote

$$l = \int_0^1 L^{-1}(\alpha) d\alpha, \;\; r = \int_0^1 R^{-1}(\alpha) d\alpha$$

using the previous expression we obtain the following result:

$$\begin{aligned} hk(b^{\mathcal{T}} r + a^{\mathcal{T}} l) =& ak\left(\int_0^h L^{-1}(\alpha) d\alpha - hL^{-1}(h)\right) \\ &+ bk\left(\int_0^h R^{-1}(\alpha) d\alpha - hR^{-1}(h)\right) + 1 - h. \end{aligned}$$

According to the properties of Riemann's integral we have

$$\int_0^h L^{-1}(\alpha) d\alpha = hL^{-1}(p) \text{ being } 0 \leq p \leq h.$$

The same reasoning can be applied to

$$\int_0^h R^{-1}(\alpha)d\alpha = hL^{-1}(q) \text{ being } 0 \le q \le h,$$

and finally we have:

$$b^{\mathcal{T}} r + a^{\mathcal{T}} l = a(L^{-1}(p) - L^{-1}(h)) + b(R^{-1}(q) - R^{-1}(h)) + \frac{1-h}{kh}$$

with $p, q \in [0, h]$.

In relation to $b^{\mathcal{T}}$ and $a^{\mathcal{T}}$ we can assume that:

$$a^{\mathcal{T}} = a\frac{L^{-1}(p) - L^{-1}(h)}{l} + \frac{1-h}{2lhk}$$
$$b^{\mathcal{T}} = b\frac{R^{-1}(q) - R^{-1}(h)}{r} + \frac{1-h}{2rhk}$$

and the equality of information is verified for A and i-$\bowtie(A, \alpha)$.

It should be remarked that, by the $\mathcal{L}$ definition we can assume that both L^{-1} and R^{-1} are non increasing functions, and, therefore, the above formula can be expressed as:

$$a^{\mathcal{T}} = a\gamma(h) + \delta(h)$$
$$b^{\mathcal{T}} = b\zeta(h) + \eta(h)$$

with $\gamma(h) > 0$ and $\zeta(h) > 0$.

In summary, we can conclude that i-$\bowtie(A, \alpha)$ has an LR representation, which is given by

$$(m_1^{\mathcal{T}}, m_2^{\mathcal{T}}, a^{\mathcal{T}}, b^{\mathcal{T}})_{LR}.$$

In the case of trapezoidal fuzzy numbers, where $L(u) = R(u) = 1 - u$, we have $l = r = \int_0^1 (1-u)du = 1/2$ and

$$\gamma(h) = \zeta(h) = \frac{(1 - h/2) - (1 - h)}{1/2} = h.$$

Therefore, $a^{\mathcal{T}} = ha + (1-h)/h$ and $b^{\mathcal{T}} = hb + (1-h)/h$, which coincides with the results presented in the previous Section.

According to these results, we can conclude that Definition 7 can be extended to the more general case in which A is an LR-fuzzy number.

4. A new approach to linguistic possibility qualification rules

Once we know how to solve the qualification problem when the uncertainty is represented as a real value, the problem is to suitably generalize this process for the case where the uncertainty is represented as a fuzzy value.

4.1. Approach

The linguistic qualification of certainty is not a new problem and has already been addressed by some authors under different names.

In López de Mántaras (1990), the concept of an ordered family of fuzzy truth values is introduced. In that paper, fuzzy truth values act as modifiers of the fuzzy (trapezoidal) values, to which they are applied. From the composition of the two membership functions, a new trapezoidal fuzzy set is obtained (which is a very important property from the storage point of view). However, the support set of the original fuzzy value is not extended (moreover, sometimes it is reduced) as a consequence of the presence of uncertainty. This is a counter-intuitive result as the set of possible values is expected to be larger after addition of uncertainty.

In Bordogna and Pasi (2000) linguistic qualifiers of uncertainty were also studied from a similar point of view. The qualifier acts as a modifier of the original fuzzy value (also trapezoidal) but considering a maximum violation level associated to every qualifier. As a consequence of the composition of the two functions (together with the maximum violation value), the support set of the original fuzzy value is extended (as expected from the semantic point of view) but the resulting fuzzy set is not a trapezoidal one. This is an important drawback as the four-parameters representation cannot be used.

Our approach starts from a different point of view: the membership function of the fuzzy certainty $C(.)$ will be used in a similar way as we did in the crisp certainty case. Thus, we want to translate the information X *is* A *is* C, when C is modelled by a normalized fuzzy trapezoidal number, into X *is* A''.

The difficulty is now to give a suitable procedure for computing A''. We will consider that, for any possible truncation level α, the membership function of the linguistic label modifies in a particular way the certainty level. In fact, we can assume that:

$$\text{It is } C \text{ that } X \text{ is } A \longleftrightarrow \forall\alpha \in [0,1], \text{ it is } C(\alpha)\text{-certain that } X \text{ is } A.$$

Fig. 7 depicts the general problem we are trying to explain.

A possible way to solve this problem is to define A'' in such a way that it summarizes the right side of the above sentence by means of some average. It should be remarked that the membership function $C(.)$ induces two fuzzy measures (possibility/necessity) on the [0,1] interval and that the membership function of any fuzzy number transformed at certainty level α can be considered as a function depending on both $\alpha \in [0,1]$ and x, which ranges over another real interval.

In the literature, we can find two main ways for computing the mean value of a function on any kind of fuzzy measure. These two methods are Sugeno integral (Sugeno, 1974; Murofushi and Sugeno, 1989) and Choquet's integral (Huber, 1973; Murofushi and Sugeno, 1989).

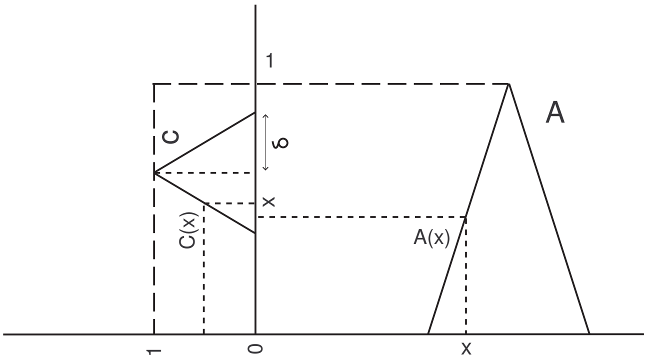

Figure 7. Fuzzy uncertainty on a fuzzy value: the scale used in both axes is not the same for the sake of clarity.

It is well known that the first one is particularly suitable in the case the fuzzy measure is a possibility/necessity measure, which is our case. Therefore, we are going to use Sugeno's integral in all the possible approaches to find A''.

4.2. Mean-based proposals

Sugeno (1974) introduced the concept of fuzzy integral of a fuzzy measure as a way to compute some kind of average value of a function in terms of the underlying fuzzy measure. Obviously, fuzzy measures formally include possibility/necessity measures as special cases. Fuzzy integrals are interpreted as subjective evaluations of objects where subjectivity is represented by means of fuzzy measures.

The fuzzy integral over a referential set X of a function $f(x)$ with respect to a fuzzy measure g is defined as follows:

$$\int_X f(x) \circ g(.) = sup_{\alpha \in [0,1]} \{\alpha \wedge g(F_\alpha)\}$$

where $F_\alpha = \{x | f(x) \geq \alpha\}$.

In the case that the measure is a *possibility* defined by means of the membership function of a fuzzy set $\mu(x)$ with referential X, the Sugeno's integral has the following expression (Vila and Delgado, 1983):

$$\int_X f(x) \circ g(.) = sup_{x \in X}(f(x) \wedge \mu(x)).$$

On the other hand, if we assume that the considered fuzzy measure is a *necessity* induced by the fuzzy set $\mu(x)$, then we have the following expression:

$$\int_X f(x) \circ g(.) = inf_{x \in X}(f(x) \vee (1 - \mu(x))).$$

This expression can be proved from the fact that Sugeno's fuzzy integral is a semiconormed fuzzy integral (Suárez García and Gil Alvarez, 1986).

As we have stated above, the basic idea of our approaches is to use the fuzzy measures (possibility, necessity) induced by the membership function $C(.)$ of the linguistic evaluation of certainty, to compute the *average* of the transformed fuzzy number, by means of Sugeno's integral.

At this point, it is necessary to remind that the transformation process of any fuzzy number A with *crisp certainty* value α has two steps:

(i) Truncating the fuzzy number at the level α, obtaining a non-normalized fuzzy number A^α.

(ii) Transforming A^α into a normalized fuzzy number i-$\bowtie(A, \alpha)$ with the same information amount, that is, i-$\bowtie(A, \alpha) = T_1(A^\alpha)$.

Consequently, depending on the step where the average is computed, we have two different approaches. If we denote as avg_α^C the operator of average (Sugeno's integral) in α with respect to a measure generated by C, then both methods can be described as:

Mean of truncated values. In this case, after step i, we apply Sugeno's integral to the function $A^\alpha(x)$ with respect to the α variable, obtaining a possibly non-normalized fuzzy number. This fuzzy number will be transformed into a normalized one in step ii, that is,

$$A'' = T_1(avg_\alpha^C(A^\alpha)).$$

Mean of transformed values. In this case we perform the averaging procedure after step ii, applying the Sugeno's integral to the membership function of i-$\bowtie(A, \alpha)$, that is,

$$A'' = avg_\alpha^C(\text{i-}\bowtie(A, \alpha)) = avg_\alpha^C(T_1(A^\alpha)).$$

Both approaches have, in turn, two different versions, depending on whether we use the possibility or the necessity measures to apply the integral. This fact will lead us to the definition of four new joint values, namely, i-$\bowtie_P^1(A, C)$, i-$\bowtie_N^1(A, C)$, i-$\bowtie_P^2(A, C)$, and i-$\bowtie_N^2(A, C)$, according to the four possible combinations we can perform, that is,

- i-$\bowtie_P^1(A, C)$ being the joint value of A with certainty C, using the mean of truncated values approach and the interpretation as possibility measure.
- i-$\bowtie_N^1(A, C)$ being the joint value of A with certainty C, using the mean of truncated values approach and the interpretation as necessity measure.

- i-$\bowtie^2_P(A, C)$ being the joint value of A with certainty C, using the mean of transformed values approach and the interpretation as possibility measure.
- i-$\bowtie^2_N(A, C)$ being the joint value of A with certainty C, using the mean of transformed values approach and the interpretation as necessity measure.

In the following paragraphs we will offer the detailed development of each approach as well as a comparison between them.

4.3. Mean of truncated values

Let us begin with the mean of truncated values.

4.3.1. Upper measure: possibility

Let $\Pi_C(.)$ be the possibility measure induced by the fuzzy number C and $T_p(.)$ be the mean of the truncated fuzzy numbers. Then we have:

$$T_p(x) = \int_{[0,1]} A^\alpha(x) \circ \Pi_C(\alpha) = sup_{\alpha\in[0,1]}(A^\alpha(x) \wedge C(\alpha)) =$$
$$= sup_{\alpha\in[0,1]}((A(x) \wedge \alpha) \wedge C(\alpha)) = A(x) \wedge sup_{\alpha\in[0,1]}(\alpha \wedge C(\alpha))$$

If $C_p = sup_{\alpha\in[0,1]}(\alpha \wedge C(\alpha))$, then we finally have:

$$T_p(x) = A(x) \wedge C_p$$

which indicates that, in the case of the possibility measure, the mean of truncated values is the result of truncating with a specific value which only depends on the linguistic label $C(.)$.

4.3.2. Lower measure: necessity

Let $N_C(.)$ be the necessity measure induced by the fuzzy number C and $T_n(.)$ be the mean of the truncated fuzzy numbers. Then we have:

$$T_n(x) = \int_{[0,1]} A^\alpha(x) \circ N_C(\alpha) = inf_{\alpha\in[0,1]}(A^\alpha(x) \vee (1 - C(\alpha))) =$$
$$= inf_{\alpha\in[0,1]}((A(x) \wedge \alpha) \vee (1 - C(\alpha))) =$$
$$= inf_{\alpha\in[0,1]}((A(x) \vee (1 - C(\alpha))) \wedge (\alpha \vee (1 - C(\alpha)))) =$$
$$= inf_{\alpha\in[0,1]}(A(x) \vee (1 - C(\alpha))) \wedge inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha))).$$

As C is a normalized fuzzy number then

$$inf_{\alpha\in[0,1]}((A(x) \vee (1 - C(\alpha))) = A(x) \vee (inf_{\alpha\in[0,1]}(1 - C(\alpha))) = A(x)$$

since

$$inf_{\alpha\in[0,1]}(1 - C(\alpha)) = 0.$$

Therefore

$$T_n(x) = A(x) \wedge inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha))).$$

If $C_n = inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha)))$, then we finally have:

$$T_n(x) = A(x) \wedge C_n$$

which indicates that, also in the case of the necessity measure, the mean of truncated values is the result of truncating with a specific value which only depends on the linguistic label $C(.)$

4.3.3. Interpretation of the results

In the two previous subsections, we have shown that the upper and lower measures permit us to obtain a *mean* truncated value that will be used to truncate finally the original fuzzy value. Therefore, at this point, we have got two levels for making this truncation or, what is the same, we obtain two different fuzzy values with the following associated functions:

$$T_p(x) = A(x) \wedge C_p,$$

which corresponds to the interpretation of certainty from the possibility point of view, and

$$T_n(x) = A(x) \wedge C_n,$$

which corresponds to the interpretation of certainty from the necessity point of view.

As it happens with all dual measures, the expert can choose either to work with both of them or to decide which one is the most suitable for the purpose of the system considered. In Fig. 8 we graphically show the results obtained, considering that the linguistic label C has a trapezoidal membership function.

After the truncation, it is necessary to perform the corresponding transformations in order to obtain a normalized fuzzy number. $T_N(.), T_P(.)$ will stand for the transformed $T_n(.)$ and $T_p(.)$.

Taking this into account, we can introduce the two following definitions.

Definition 8 *Let $A, C \in \widetilde{I\!R}$. The joint value of A with certainty C as mean of truncated values with the certainty interpreted as possibility is called $i\text{-}\bowtie^1_P(A, C)$ and is computed as follows:*

$$i\text{-}\bowtie^1_P(A, C) = T_P = T_1(T_p).$$

Definition 9 *Let $A, C \in \widetilde{I\!R}$. The joint value of A with certainty C as mean of truncated values with the certainty interpreted as necessity is called $i\text{-}\bowtie^1_N(A, C)$ and is computed as follows:*

$$i\text{-}\bowtie^1_P(A, C) = T_N = T_1(T_n).$$

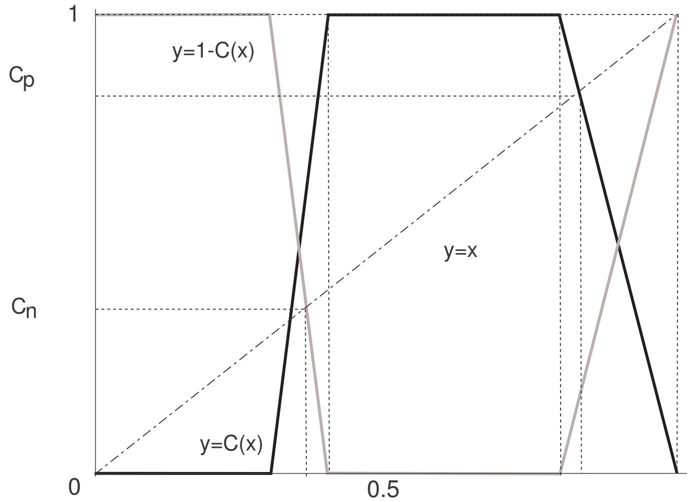

Figure 8. Upper and lower measures.

It should be remarked that these approaches finally lead to a *crisp certainty* process. Therefore, the property concerning LR fuzzy numbers, given in Section 3, also holds in theses cases. This is the formal expression of such a property.

PROPERTY 2 *If* $A = (m_1, m_2, a, b)_{LR}$ *then* $T_P = (m_1^{\mathcal{T}}(C_p), m_2^{\mathcal{T}}(C_p), a^{\mathcal{T}}(C_p), b^{\mathcal{T}}(C_p))_{LR}$ *and* $T_N = (m_1^{\mathcal{T}}(C_n), m_2^{\mathcal{T}}(C_n), a^{\mathcal{T}}(C_n), b^{\mathcal{T}}(C_n))_{LR}$ *where the concrete expressions of* $m_1^{\mathcal{T}}(.), m_2^{\mathcal{T}}(.), a^{\mathcal{T}}(.), b^{\mathcal{T}}(.)$ *can be found in Section 3 and obviously depend on* C_p *and* C_n*, respectively.*

Consequently, the transformation of any LR fuzzy number is also an LR fuzzy number with an explicit and easy computational representation. This is particularly important for the sake of simplicity in representing the transformation of any fuzzy quantity in any intelligent information system.

4.4. Mean of the transformed values

In this case, for each α we calculate A^{α}, that is, the truncated fuzzy number at level α, next we transform it and we obtain i-$\bowtie(A, \alpha)$, a normalized fuzzy number with the same information as the original one, but in order to include the fuzzy certainty C we calculate the average of the i-$\bowtie(A, \alpha)$ using the two induced measures associated to C. Let us denote the membership function of i-$\bowtie(A, \alpha)$ as $T_A(\alpha, x)$, and then we have two options to define the membership function of A'':

Upper mean $S_P(x)$

$$S_P(x) = \int_{[0,1]} T_A(\alpha, x) \circ \Pi_C(\alpha) = sup_{\alpha\in[0,1]}(T_A(\alpha, x) \wedge C(\alpha)).$$

Lower mean $S_N(x)$

$$S_N(x) = \int_{[0,1]} T_A(\alpha, x) \circ N_C(\alpha) = inf_{\alpha\in[0,1]}(T_A(\alpha, x) \vee (1 - C(\alpha))).$$

According to these, we can complete our approach with the two last definitions:

Definition 10 *Let* $A, C \in \widetilde{I\!R}$. *The joint value of A with certainty C as mean of the transformed values with the certainty interpreted as possibility is called* $i\text{-}\bowtie^2_P(A, C)$ *and is computed as follows:*

$$i\text{-}\bowtie^2_P(A, C) = S_P.$$

Definition 11 *Let* $A, C \in \widetilde{I\!R}$. *The joint value of A with certainty C as mean of the transformed values with the certainty interpreted as necessity is called* $i\text{-}\bowtie^2_N(A, C)$ *and is computed as follows:*

$$i\text{-}\bowtie^2_P(A, C) = S_N.$$

The main problem of this option is that now we cannot have algebraic expression of the transformed fuzzy numbers as in the previous case, but we can study some interesting properties of both transformations.

4.5. Some properties of the transformations

Once the new approaches are established, it is necessary to carry out some comparisons between these new ones and the previous ones in order to have criteria for choosing one of them in practical cases.

The first property associated to the mean of truncated values is straightforward, since $C_n \geq C_p$.

Property 3 $\forall x \in Supp(A)$ *we have that* $T_P(x) \leq T_N(x)$, *that is,* $T_P \subseteq T_N$.

We can conclude that T_N offers us a transformed fuzzy number more imprecise than T_P.

In relation to the properties of the mean of the transformed values, and to obtain some more concrete expressions, we will first analyze some properties of $T_A(.,.)$.

Property 4 $\forall x$ $T_A(\alpha, x) = 1$ *if* $\alpha \leq A(x)$.

Proof. This property is a direct consequence of definition of $T(\alpha, x)$ which implies truncating $A(x)$ at the α level and obtaining a new fuzzy value, whose height is equal to one and which carries the same information. Let us suppose that $A = (m_1, m_2, a, b)_{LR}$; then:

$$\forall x \text{ if } m_1 - aL^{-1}(\alpha) \leq x \leq m_2 + bR^{-1}(\alpha) \text{ then } T_A(\alpha, x) = 1;$$

therefore:

$$\alpha \leq L(\frac{m_1 - x}{a}) \text{ and } \alpha \leq R(\frac{x - m_2}{b})$$

and the property holds. ∎

Property 5 $\forall x \; T_A(1, x) = A(x)$.

Property 6 *If the LR functions, which define the fuzzy number A have the shape* $L(u) = max\{0, (1-u)^p\}$*, then* $\forall x$ *and* $\forall \alpha, \alpha' \in [0,1]$ *such that* $\alpha \leq \alpha'$, $T_A(\alpha, x) \geq T_A(\alpha', x)$. *That is,* $T_A(\alpha, x)$ *is non increasing with respect to* α.

Proof. Let us consider x belonging to the left side of A (we can proceed with the right side in a similar way). With this assumption:

$$T_A(\alpha, x) = L(\frac{m_1 - aL^{-1}(\alpha) - x}{a^{\mathcal{T}}(\alpha)})$$

and to prove $T_A(\alpha, x) \geq T_A(\alpha', x)$ we must assure:

$$\frac{m_1 - aL^{-1}(\alpha) - x}{a^{\mathcal{T}}(\alpha)} \leq \frac{m_1 - aL^{-1}(\alpha') - x}{a^{\mathcal{T}}(\alpha')},$$

since $L(.)$ is decreasing. There is no problem with respect to the numerator of the fraction:

$$\forall x \; m_1 - aL^{-1}(\alpha) - x \leq m_1 - aL^{-1}(\alpha') - x$$

because $L^{-1}(.)$ is decreasing too and $\alpha \leq \alpha'$.

Regarding the divisor, we have to prove that $a^{\mathcal{T}}(\alpha) \geq a^{\mathcal{T}}(\alpha')$ if $\alpha \leq \alpha'$, i.e., that $a^{\mathcal{T}}(.)$ is a decreasing function. To do it, we will prove that its first derivative is negative.

In the case of that $L(u) = (1-u)^p$ we have:

$$a^{\mathcal{T}}(\alpha) = a\alpha^{\frac{1}{p}} + (p+1)\frac{1-\alpha}{2\alpha}$$

and

$$a'^{\mathcal{T}}(\alpha) = \frac{1-p}{p} a\alpha^{\frac{1}{p}-1} - \frac{1}{2h^2}$$

which proves the property. ∎

It is important to note that Property 6 does not hold for fuzzy numbers whose LR functions are not of type $L(u) = max\{0, (1-u)^p\}$. Nevertheless, we have to take into account that most usual LR fuzzy number classes (e.g. trapezoidal, triangular,...) rely on this type. From now on, we will consider that LR functions are of this kind.

Once we have got the main properties of $T_A(\alpha, x)$, we can analyze those of $S_P(x)$ and $S_N(x)$

PROPERTY 7 *Let $C = (c_0, c_1, d_0, d_1)_{LR}$. If $A(x) \geq c_0$, then $S_P(x) = 1$.*

Proof. Let us remember that $T_A(\alpha, x) = 1$ if $A(x) \geq \alpha$ and $C(\alpha) = 0$ if $\alpha \leq c_0 - d_0$ or $\alpha \geq c_1 + d_1$.

Therefore

$$S_P(x) = sup_{\alpha \in [0,A(x)]}(1 \wedge C(\alpha)) \vee sup_{\alpha \in [A(x),c_1+d_1]}(T_A(\alpha, x) \wedge C(\alpha))\,.$$

In the case that $A(x) \geq c_0$, we have $c_0 \in [0, A(x)]$ and then $sup_{[0,A(x)]}(1 \wedge C(\alpha) = 1$. Finally, $S_P(x) = 1$. ∎

This is a very interesting property, because what we intend is to *soften* the uncertainty value c_0 (lower modal value of the linguistic label) and the behavior of the combination is coherent with the intuition that those values of x, whose membership in A is greater or equal to 0 will have a membership degree 1, the same as the case of a fixed value α. That is, the modal value is the limit to put up the new fuzzy value and this one does not depend on the shape of the fuzzy number.

PROPERTY 8 *Let $C = (c_0, c_1, d_0, d_1)_{LR}$. If $A(x) \geq c_1 + d_1$, then $S_N(x) = 1$.*

Proof. Let us remember that $T_A(\alpha, x) = 1$ if $A(x) \geq \alpha$ and $1 - C(\alpha) = 1$ if $\alpha \leq c_0 - d_0$ or $\alpha \geq c_1 + d_1$.

Therefore:

$$\begin{aligned}
&S_N(x) = inf_{\alpha \in [0,A(x)]}(1 \vee 1 - C(\alpha)) \bigwedge \\
&\bigwedge inf_{\alpha \in [A(x),c_1+d_1]}(T(\alpha, x) \vee 1 - C(\alpha)) \bigwedge \\
&\bigwedge inf_{\alpha \in [c_1+d_1],1}(T(\alpha, x) \vee 1) = \\
&= 1 \bigwedge inf_{\alpha \in [A(x),c_1+d_1]}(T(\alpha, x) \vee 1 - C(\alpha)) = 1
\end{aligned}$$

since $A(x) \geq c_1 + d_1$. ∎

PROPERTY 9 *Let $C = (c_0, c_1, d_0, d_1)_{LR}$. If $T_A(., x)$ is non increasing and continuous, then*

$$S_P(x) = \begin{cases} 1 & \text{if } A(x) \geq c_0 \\ T_A(\widehat{\alpha}, x) & \text{otherwise} \end{cases}$$

or, equivalently,

$$S_P(x) = \begin{cases} 1 & \textit{if } A(x) \geq c_0 \\ C(\widehat{\alpha}) & \textit{otherwise} \end{cases}$$

since $T_A(\widehat{\alpha}, x) = C(\widehat{\alpha})$; $\widehat{\alpha} \in [A(x), c]$.

Proof. The first part of the expression is the Property 7. To prove the second part we assume that $A(x) \leq c_0$ and we have:

$$\begin{aligned} & S_P(x) = sup_{\alpha \in [A(x), c_1+d_1]}(T_A(\alpha, x) \wedge C(\alpha)) = \\ & = sup_{\alpha \in [A(x), c_0]}(T_A(\alpha, x) \wedge C(\alpha)) \bigvee \\ & \bigvee sup_{\alpha \in [c_0, c_1]}(T_A(\alpha, x)) \bigvee \\ & \bigvee sup_{\alpha \in [c_1, c_1+d_1]}(T_A(\alpha, x) \wedge C(\alpha)). \end{aligned}$$

Taking into account that $T_A(.,.)$ is non increasing, the second and third terms of the expression are upper bounded by $T_A(c_0, x)$ and $T_A(c_1, x)$, respectively, and $T_A(c_0, x) \geq T_A(c_1, x)$. Thus, we finally obtain:

$$S_P(x) = sup_{\alpha \in [A(x), c_0]}(T_A(\alpha, x) \wedge C(\alpha)).$$

Since T_A is non increasing and continuous and C is increasing and continuous in $[A(x), c_0]$, we can assure that:

$$\exists \widehat{\alpha} \in [A(x), c_0] \text{ such that } S_P(x) = C(\widehat{\alpha}) = T_A(\widehat{\alpha}, x)$$

and, therefore, the property holds. ∎

From the above proof, it is also straightforward to deduce:

If $A(x) \leq c_1 - d_0$ then $\widehat{\alpha} \in [c_0 - d_0, c_0]$.

PROPERTY 10 *Let* $C = (c_0, c_1, d_0, d_1)_{LR}$. *If* $T_A(., x)$ *is non increasing and continuous, then*

$$S_N(x) = \begin{cases} 1 & \textit{if } A(x) \geq c_1 + d_1 \\ T_A(\bar{\alpha}, x) & \textit{otherwise} \end{cases}$$

or, equivalently,

$$S_N(x) = \begin{cases} 1 & \textit{if } A(x) \geq c_1 + d_1 \\ C(\bar{\alpha}) & \textit{otherwise} \end{cases}$$

since $T_A(\bar{\alpha}, x) = C(\bar{\alpha})$; $\bar{\alpha} \in [c_1, c_1 + d_1]$.

Proof. The first part of the expression is the Property 8. To prove the second part we assume that $A(x) \leq c_0$. Then, we have:

$$\begin{aligned}
&S_N(x) = inf_{\alpha\in[A(x),c_1+d_1]}(T_A(\alpha,x) \vee 1 - C(\alpha)) = \\
&= inf_{\alpha\in[A(x),c_0]}(T_A(\alpha,x) \vee 1 - C(\alpha)) \bigwedge \\
&\bigwedge inf_{\alpha\in[c_0,c_1]}(T_A(\alpha,x)) \bigwedge \\
&\bigwedge inf_{\alpha\in[c_1,c_1+d_1]}(T_A(\alpha,x) \vee 1 - C(\alpha)).
\end{aligned}$$

Since both $T_A(.,.)$ and $1-C(.)$ are non increasing in $[A(x), c_0]$, the first and the second terms of the expression are lower bounded by $T_A(c_0, x)$ and $T_A(c_1, x)$, respectively, and $T_A(c_0, x) \geq T_A(c_1, x)$. Thus, we finally obtain:

$$S_N(x) = inf_{\alpha\in[c_1,c_1+d_1]}(T_A(\alpha,x) \vee C(\alpha)).$$

Since T_A is non increasing and continuous and $1 - C$ is increasing and continuous in $[c1, c_1 + d_1]$, we can assure that:

$$\exists\bar{\alpha} \in [c_1, c_1 + d_1] \text{ such that } S_N(x) = 1 - C(\bar{\alpha}) = T_A(\bar{\alpha}, x)$$

and, therefore, the property holds. ∎

From the above proof, it is also straightforward to deduce:

If $A(x) \geq c_1$ then $\bar{\alpha} \in [A(x), c_1 + d_1]$.

These properties allow us to make comparisons between the different approaches, since for all x in the support of C:

$$\begin{aligned}
&T_P(x) = T(C_p, x) \text{ with } C_p \in [c_1, c_1 + d_1] \\
&S_P(x) = T(\widehat{\alpha}, x) \text{ with } \widehat{\alpha} \in [c_0 - d_0, c_0] \\
&T_N(x) = T(C_n, x) \text{ with } C_n \in [c_0 - d_0, c_0] \\
&S_N(x) = T(\bar{\alpha}, x) \text{ with } \bar{\alpha} \in [c_1, c_1 + d_1].
\end{aligned}$$

Finally, for all x in the support of C we have:

$$\begin{aligned}
&T_N(x) \geq max(S_N(x), T_P(x)) \Rightarrow S_N \subseteq T_N \text{ and } T_P \subseteq T_N \\
&S_P(x) \geq max(S_N(x), T_P(x)) \Rightarrow S_N \subseteq S_P \text{ and } T_P \subseteq S_P.
\end{aligned}$$

From the previous results, we have that T_N and S_P produce less precise values (i.e. they are *pessimistic*), while S_N and T_P deliver more precise values (i.e. they can be considered *optimistic*). Thus, according to the amount of imprecision we are able to accept (or how optimistic/pessimistic we are), we can choose the appropriate method.

Additionally, taking into account that T_N and T_P are easier to compute because they have a defined algebraic expression, we can conclude that these two methods based on the mean of truncated values are preferable to the two methods based on the mean of the transformed values (i.e. S_N and S_P).

5. An example

Using the four proposals presented in this paper, we can transform the original measure so that it incorporates the given fuzzy certainty value. In order to illustrate the application of these results, let us return to our example of the ROSETTA spatial mission and the GIADA instrument. As we know, the micro-balance gives us a measure of the grain weight, that is subject to some imprecision due to the balance threshold sensitivity (around 10^{-10} g.). On the other hand, the balance accumulates some dust from one weighting to another, until the cleaning-by-heating process is triggered. According to the micro-balance status, the measurement has to be done with the uncertainty due to the cleaning process that, as we explained in the introduction, does not guarantee a perfect result. Fig. 9 shows the behavior of our four approximations, starting from an almost clean balance.

In Fig. 9(a) shows the certainty value, associated to the current status of the instrument. In Fig. 9(b) the initial weight is shown with the attached imprecision, and in Figs. 9(c), 9(d), 9(e), 9(f) we show the results obtained by applying the mean of truncated values considering the possibility, the mean of truncated values considering the necessity, the mean of the transformed values considering the possibility, and the mean of the transformed values considering the necessity, respectively.

In Fig. 10 we present the same example but at some later, when the balance is not as clean as in the previous case.

As can be seen in the figures, the results obtained after the application of the four approaches fulfill the constraints of shape posed in the previous sections. The choice between the possibility and the necessity approaches must be done depending on the imprecision we can admit in the system. Methods based on necessity are more restrictive and, in consequence, they produce more imprecise results. On the contrary, the use of the possibility measure leads us to a more precise result, since it is a more optimistic approach. In our case, for the ROSETTA mission, we have opted for the application of T_P (joint value based on the use of the mean of truncated values approach and the interpretation as possibility measure) since the system requires the use of fuzzy numbers as precise as possible to be used in the deduction process that follows this transformation.

6. Conclusions

We have addressed the problem of dealing with linguistic uncertainty associated with a fuzzy quantity. With the basic idea of transforming uncertainty into imprecision, four possible approaches have been presented; all of them give transformations of the initial fuzzy number that lead to normalized fuzzy numbers. We have also studied the properties of the different approaches and, from the attained results, a rule for a possible choice has been developed.

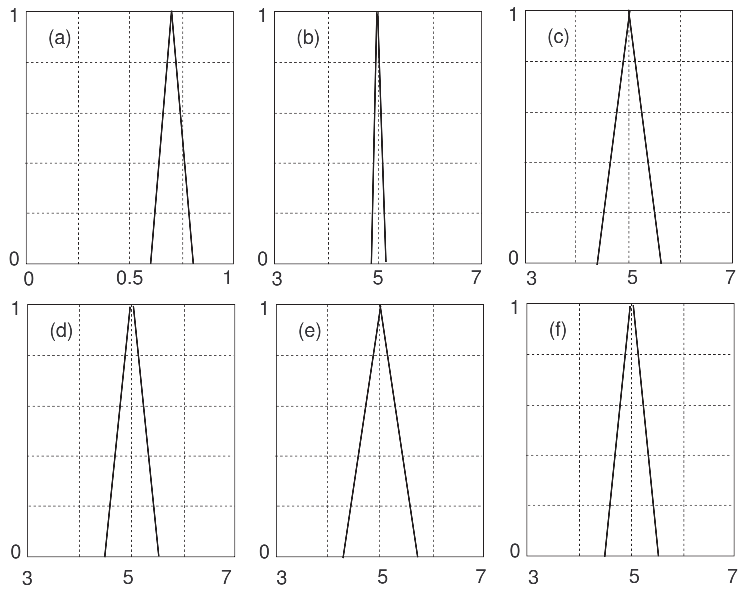

Figure 9. Results obtained by our approaches considering an initial certainty value near to 1.

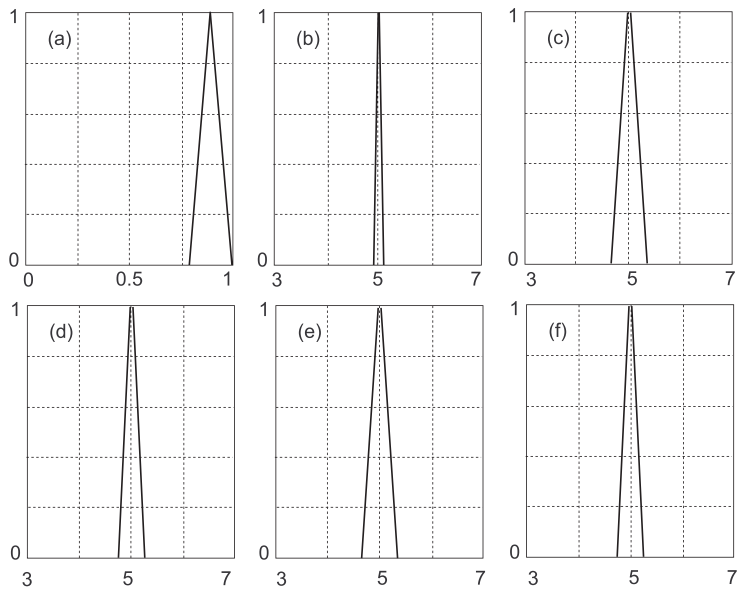

Figure 10. Results obtained by our approaches considering a certainty value obtained just before the cleaning.

One of the significant achievements of our approach is that, if the initial fuzzy number has an LR representation, the final result gives us transformed fuzzy numbers which keep the LR shape of the initial one. Explicit expressions of such transformed fuzzy numbers have also been obtained. This is a particularly useful property from the storage point of view (e.g. within the database world or in a data warehousing context), since it provides us with a simple and unified representation for both original and transformed fuzzy values.

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