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Multiobjective variational programming under generalized (V, ρ)-B-type I functions*

by

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Abstract: In this paper, we generalize the (V, ρ)-invexity defined for nonsmooth multiobjective fractional programming by Mishra, Rueda and Giorgi (2003) to variational programming problems by defining new classes of vector-valued functions called (V, ρ)-B-type I and generalized (V, ρ)-B-type I. Then we use these new classes to derive various sufficient optimality conditions and mixed type duality results.

Keywords: multiobjective variational programming, efficient solution, proper efficient solution, generalized (V, ρ)-B-type I functions, sufficient conditions, mixed type duality rusults.

1. Introduction

In this paper we consider the following multiobjective variational programming problem,

$$(MOP)\quad \begin{cases} Minimize \int_a^b f(t,x,\dot{x})dt = (\int_a^b f_1(t,x,\dot{x})dt, \ldots, \int_a^b f_p(t,x,\dot{x})dt) \\ subject\ to \\ x(a) = \alpha, \quad x(b) = \beta, \\ g(t,x,\dot{x}) \leqq 0, \quad t \in I, \end{cases}$$

where $I = [a, b]$ is a real interval, $f : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^p$, and $g : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^m$ are continuously differentiable functions with respect to each of their arguments, up to the second order.

The field of multiobjective variational programming problems, also known as continuous time programming problems, has grown remarkably in different directions in the setting of optimality conditions and duality theory. It has been enriched by the application of various types of generalizations of convexity theory, with or without differentiability assumptions and in fractional variational

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programming, saddle point theory, symmetric duality, etc. A new reader may like to consult Bhatia and Mehra (1999), Aghezzaf and Khazafi (2004), Bector and Husain (1992) for relatively more exhaustive references in the subject. This development in multiobjective variational programming was originated by the growth of generalizations of invexity, introduced by Hanson (1981) in mathematical programming, to variational programming problems. More specifically, Bhatia and Mehra introduced recently the class of B-type I functions, a generalization of invexity, and derived various sufficient conditions and duality results.

In this paper, we generalize the (V,ρ)-invexity defined for nonsmooth multiobjective fractional programming, Zhou and Wang (2003), to multiobjective variational programming problems by defining new classes of vector-valued functions called (V,ρ)-B-type I and generalized (V,ρ)-B-type I, then we use these new classes to establish various sufficient optimality conditions and mixed type duality results.

2. Preliminaries

Let $I\!R^n$ be n-dimensional Euclidean space, and $I\!R^n_+$ be its nonnegative orthant. Let x and y be in $I\!R^n$, we denote

$$x \leqq y \Longleftrightarrow x_i \leq y_i, \quad for\ i = 1, \ldots, n.$$

$$x \leq y \Longleftrightarrow x \leqq y, \quad but \quad x \neq y.$$

$$x < y \Longleftrightarrow x_i < y_i, \quad for\ i = 1, \ldots, n.$$

In order to consider $f(t,x,\dot{x})$, where $x : I \longrightarrow I\!R^n$ with its derivative $\dot{x}$, denote the $p \times n$ matrices of first partial derivatives of f with respect to x, $\dot{x}$ by f_x and $f_{\dot{x}}$, such that

$$f_{ix} = (\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}) \ \ and \ \ f_{i\dot{x}} = (\frac{\partial f_i}{\partial \dot{x}_1}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n}), \quad i = 1, 2, \ldots, p.$$

Similarly, g_x and $g_{\dot{x}}$ denote the $m \times n$ matrices of first partial derivatives of g with respect to x and $\dot{x}$.

Let $\mathcal{C}(I, I\!R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| := \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$u = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t u(s)ds.$$

Therefore, $D = d/dt$, except at discontinuities.

Let $K := \{\ x \in \mathcal{C}(I, I\!R^n),\ x(a) = \alpha,\ x(b) = \beta,\ g(t,x,\dot{x}) \leqq 0, \quad \forall t \in I\ \}$ be the set of feasible solutions of (MOP).

Definition 2.1 *A point $x^* \in K$ is said to be an efficient (Pareto optimal) solution of (MOP) if there exists no other $x \in K$ such that*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,x^*,\dot{x}^*)dt\,.$$

Definition 2.2 *(Geoffrion, 1968, Bector and Husain, 1992). An efficient solution x^* of (MOP) is said to be properly efficient if there exists a positive number M such that for each i, we have* $\int_a^b f_i(t,x^*,\dot{x}^*)dt - \int_a^b f_i(t,x,\dot{x})dt \leqq M\big(\int_a^b f_j(t,x,\dot{x})dt - \int_a^b f_j(t,x^*,\dot{x}^*)dt\big)$ *for some j such that* $\int_a^b f_j(t,x,\dot{x})dt > \int_a^b f_j(t,x^*,\dot{x}^*)dt$ *whenever $x \in K$ and* $\int_a^b f_i(t,x,\dot{x})dt < \int_a^b f_i(t,x^*,\dot{x}^*)dt$.

Let $\rho = (\rho^1,\rho^2)$ be a vector in $\mathbb{R}^{p+m}$ where: $\rho^1 = (\rho^1_1,\dots,\rho^1_p)$ is a vector in $\mathbb{R}^p$ and $\rho^2 = (\rho^2_{1+p},\dots,\rho^2_{m+p})$ is a vector in $\mathbb{R}^m$ and let $d : I\times\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$ be a function.

Definition 2.3 *A pair (f,g) is said to be (V,ρ)-B-type I at $u \in \mathcal{C}(I,\mathbb{R}^n)$ with respect to b_0, b_1 and η if there exist functions b_0, $b_1 : \mathcal{C}(I,\mathbb{R}^n)\times\mathcal{C}(I,\mathbb{R}^n) \longrightarrow \mathbb{R}_+$ and $\eta : I\times\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for all $x \in K$,*

$$\begin{aligned} & b_0(x,u)\Big[\int_a^b f(t,x,\dot{x})dt - \int_a^b f(t,u,\dot{u})dt\Big] \\ & \geqq \int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}(t,u,\dot{u})\Big]dt + \rho^1\int_a^b d^2(t,\ x,\ u)dt \qquad (2.1) \\ & \quad -b_1(x,u)\int_a^b g(t,u,\dot{u})dt \\ & \geqq \int_a^b \eta(t,x,u)^t\Big[g_x(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}(t,u,\dot{u})\Big]dt + \rho^2\int_a^b d^2(t,\ x,\ u)dt. \end{aligned}$$

If in the previous definition, (2.1) is satisfied as a strict inequality, then we say that a pair (f,g) is semistrictly (V,ρ)-B-type I at $u \in \mathcal{C}(I,\mathbb{R}^n)$ with respect to b_0, b_1 and η.

If $\rho^1 = \rho^2 = 0$, then $(f,\ g)$ is B-type I (Aghezzaf and Khazafi, 2004) at $u \in \mathcal{C}(I,\mathbb{R}^n)$ with respect to b_0, b_1 and η.

Consider the example given by Bhatia and Mehra (1999).

Example 2.1 *Define functions f, g by :*

$$\begin{aligned} f : I\times[0,1]\times[0,1] &\longrightarrow \mathbb{R} \\ (t,x(t),\dot{x}(t)) &\longmapsto \frac{-x^2(t)t}{b^2-a^2} \\ g : I\times[0,1]\times[0,1] &\longrightarrow \mathbb{R} \\ (t,x(t),\dot{x}(t)) &\longmapsto -\frac{(x^2(t)+1)t}{b^2-a^2}\,. \end{aligned}$$

Neither the function f nor g defined previously are invex at $u(t) = 0$, so neither the function f nor g is convex at u. But the pair $(f,\ g)$ is B-type I at u with *respect to functions* $b_0, b_1 : [0,1] \times [0,1] \longrightarrow I\!R_+$ *and* $\eta : I \times [0,1] \times [0,1] \longrightarrow I\!R$ *defined in the following text,*

$$\eta(t,x,u) = x^3(t) - u^3(t)$$
$$b_0(x,u) = 3u^2(t)$$
$$b_1(x,u) = \begin{cases} x^3(t) - u^3(t), & if \quad u(t) > x(t), \\ 0 & otherwise. \end{cases}$$

Therefore, for $\rho^1 = \rho^2 = 0$*, the pair* $(f,\ g)$ *is* (V,ρ)*-*B*-type I at* u *with respect to the same functions* b_0*,* b_1 *and* η*.*

(V,ρ)-B-type I need not to be B-type I functions as can be seen from the following example.

Example 2.2 *Define functions* f, g *by :*

$$f : I \times [0,1]^2 \times [0,1]^2 \longrightarrow I\!R^2$$
$$(t, x(t), \dot{x}(t)) \longmapsto x(t) - 1$$
$$g : I \times [0,1]^2 \times [0,1]^2 \longrightarrow I\!R$$
$$(t, x(t), \dot{x}(t)) \longmapsto x_1(t) + x_2(t) - 2.$$

For $\rho^1 = (-2,-2)$*,* $\rho^2 = -4$ *and* $d \equiv 1$*,* (f,g) *is* (V,ρ)*-*B*-type I at* $u(t) = (1,1)$ *with respect to functions* b_0*,* $b_1 : \mathcal{C}([0,1], I\!R^2) \times \mathcal{C}([0,1], I\!R^2) \longrightarrow I\!R_+$ *and* $\eta :$ $I \times I\!R^2 \times I\!R^2 \longrightarrow I\!R^2$ *defined as below:*

$$\eta(t,x,u) = x(t) + 1$$
$$b_0(x,u) = b_1(x,u) = 1\,.$$

But (f,g) *is not* B*-type I at* u *with respect to the same* b_0*,* b_1 *and* η*.*

Now we generalize the class of (V,ρ)-B-type I functions in the spirit of generalizations made in Aghezzaf and Khazafi (2004).

Definition 2.4 *A pair* (f,g) *is said to be weak strictly* (V,ρ)*-pseudo-quasi* B*-type I at* $u \in \mathcal{C}(I, I\!R^n)$ *with respect to* b_0*,* b_1 *and* η *if there exist functions* b_0*,* $b_1 : \mathcal{C}(I, I\!R^n) \times \mathcal{C}(I, I\!R^n) \longrightarrow I\!R_+$ *and* $\eta : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^n$ *such that for all* $x \in K$*,*

$$\int_a^b f(t,x,\dot{x})dt \le \int_a^b f(t,u,\dot{u})dt$$
$$\Longrightarrow b_0(x,u)\int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot{u}) - \frac{d}{dt} f_{\dot{x}}(t,u,\dot{u}) \Big] dt$$
$$< -\rho^1 \int_a^b d^2(t,\ x,\ u)dt$$

$$
\begin{aligned}
&-\int_a^b g(t,u,\dot{u})dt \leqq 0\\
&\Longrightarrow b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[g_x(t,u,\dot{u})-\frac{d}{dt}g_{\dot{x}}(t,u,\dot{u})\Big]dt\\
&\quad \leqq -\rho^2\int_a^b d^2(t,\ x,\ u)dt.
\end{aligned}
$$

The class of weak strictly (V,ρ)-pseudo-quasi B-type I does not contain the class of (V,ρ)-B-type I, but does contain the class of semistrictly (V,ρ)-B-type I with $b_0 > 0$.

DEFINITION 2.5 *A pair (f,g) is said to be strong (V,ρ)-pseudo-quasi B-type I at $u \in \mathcal{C}(I,\mathbb{R}^n)$ with respect to b_0, b_1 and η if there exist functions b_0, b_1 : $\mathcal{C}(I,\mathbb{R}^n)\times\mathcal{C}(I,\mathbb{R}^n)\longrightarrow\mathbb{R}_+$ and $\eta : I\times\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that for all $x\in K$,*

$$
\begin{aligned}
&\int_a^b f(t,x,\dot{x})dt \le \int_a^b f(t,u,\dot{u})dt\\
&\Longrightarrow b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u})-\frac{d}{dt}f_{\dot{x}}(t,u,\dot{u})\Big]dt\\
&\quad \le -\rho^1\int_a^b d^2(t,\ x,\ u)dt\\
&-\int_a^b g(t,u,\dot{u})dt \leqq 0\\
&\Longrightarrow b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[g_x(t,u,\dot{u})-\frac{d}{dt}g_{\dot{x}}(t,u,\dot{u})\Big]dt\\
&\quad \leqq -\rho^2\int_a^b d^2(t,\ x,\ u)dt.
\end{aligned}
$$

Instead of the class of weak strictly (V,ρ)-pseudo-quasi B-type I, the class of strong (V,ρ)-pseudo-quasi B-type I does contain the class of (V,ρ)-B-type I with $b_0 > 0$.

We give examples to show that weak strictly (V,ρ)-pseudo-quasi B-type I and strong (V,ρ)-pseudo-quasi B-type I exist.

EXAMPLE 2.3 *Define functions f, g by :*

$$
\begin{aligned}
f : I\times[0,1]^2\times[0,1]^2 &\longrightarrow \mathbb{R}^2\\
(t,x(t),\dot{x}(t)) &\longmapsto x(t)\\
g : I\times[0,1]^2\times[0,1]^2 &\longrightarrow \mathbb{R}\\
(t,x(t),\dot{x}(t)) &\longmapsto x_1(t)+x_2(t).
\end{aligned}
$$

For $\rho^1 = (-3,-3)$*,* $\rho^2 = -4$ *and* $d \equiv 1$*,* (f,g) *is weak strictly* (V,ρ)*-pseudo-quasi* B*-type I at* $u(t) = (1,1)$ *with respect to functions* b_0*,* $b_1 : \mathcal{C}(I,\mathbb{R}^2)\times\mathcal{C}(I,\mathbb{R}^2) \longrightarrow \mathbb{R}_+$ *and* $\eta : I \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined as below:*

$$\eta(t,x,u) = (x_1(t)+1, x_2(t)+1)$$
$$b_0(x,u) = b_1(x,u) = 1.$$

But (f,g) *is not* (V,ρ) B*-type I at* u *with respect to the same* b_0*,* b_1 *and* η*.*

Strong (V,ρ)-pseudo-quasi B-type I need not to be (V,ρ) B-type I with respect to the same b_0, b_1 and η.

Example 2.4 *Define functions* f, g *by :*

$$f : I \times [0,1]^2 \times [0,1]^2 \longrightarrow \mathbb{R}^2$$
$$(t, x(t), \dot{x}(t)) \longmapsto x(t)$$
$$g : I \times [0,1]^2 \times [0,1]^2 \longrightarrow \mathbb{R}$$
$$(t, x(t), \dot{x}(t)) \longmapsto x_1(t) + x_2(t).$$

For $\rho^1 = (-2,-2)$*,* $\rho^2 = -4$ *and* $d \equiv 1$*,* $(f,\ g)$ *is strong* (V,ρ)*-pseudo-quasi* B*-type I at* $u(t) = (1,1)$ *with respect to functions* b_0*,* $b_1 : \mathcal{C}(I,\mathbb{R}^2) \times \mathcal{C}(I,\mathbb{R}^2) \longrightarrow \mathbb{R}_+$ *and* $\eta : I \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined as below:*

$$\eta(t,x,u) = (x_1(t)+1, x_2(t)+1)$$
$$b_0(x,u) = b_1(x,u) = 1.$$

But (f,g) *is not* (V,ρ) B*-type I at* u *with respect to the same* b_0*,* b_1 *and* η*.*

Definition 2.6 *A pair* $(f,\ g)$ *is said to be weak strictly* (V,ρ)*-pseudo* B*-type I at* $u \in \mathcal{C}(I,\mathbb{R}^n)$ *with respect to* b_0*,* b_1 *and* η *if there exist functions* b_0*,* $b_1 : \mathcal{C}(I,\mathbb{R}^n) \times \mathcal{C}(I,\mathbb{R}^n) \longrightarrow \mathbb{R}_+$ *and* $\eta : I \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *such that for all* $x \in K$*,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})dt$$
$$\Longrightarrow b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}(t,u,\dot{u})\Big]dt$$
$$< -\rho^1\int_a^b d^2(t,\ x,\ u)dt$$
$$-\int_a^b g(t,u,\dot{u})dt \leqq 0$$
$$\Longrightarrow b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[g_x(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}(t,u,\dot{u})\Big]dt$$
$$< -\rho^2\int_a^b d^2(t,\ x,\ u)dt.$$

Example 2.5 *Define functions f, g by :*

$$f : I \times [0,1]^2 \times [0,1]^2 \longrightarrow \mathbb{R}^2$$
$$(t, x(t), \dot{x}(t)) \longmapsto x(t)$$
$$g : I \times [0,1]^2 \times [0,1]^2 \longrightarrow \mathbb{R}$$
$$(t, x(t), \dot{x}(t)) \longmapsto x_1(t) + x_2(t)$$

For $\rho^1 = (-3,-3)$, $\rho^2 = -5$ and $d \equiv 1$, $(f,\ g)$ is weak strictly (V,ρ)-pseudo B-type I at $u(t) = (1,1)$ with respect to functions b_0, $b_1 : \mathcal{C}(I, \mathbb{R}^2) \times \mathcal{C}(I, \mathbb{R}^2) \longrightarrow \mathbb{R}_+$ and $\eta : I \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as below:

$$\eta(t, x, u) = (x_1(t) + 1, x_2(t) + 1)$$
$$b_0(x, u) = b_1(x, u) = 1.$$

But (f, g) is not (V,ρ) B-type I at u with respect to the same b_0, b_1 and η.

3. Sufficient conditions

In this section, we establish various sufficient optimality conditions for (MOP) under (V,ρ)-B-type I and generalized (V,ρ)-B-type I conditions.

Theorem 3.1 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in \mathbb{R}^p, \lambda^* > 0$ and a piecewise smooth function $y^* : I \longrightarrow \mathbb{R}^m$ such that for all $t \in I$,*

$$\lambda^{*t} f_x(t, x^*, \dot{x}^*) + y^*(t)^t g_x(t, x^*, \dot{x}^*)$$
$$= \frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t, x^*, \dot{x}^*) + y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*)\Big), \tag{3.1}$$
$$y^*(t)^t g(t, x^*, \dot{x}^*) = 0, \qquad t \in I, \tag{3.2}$$
$$y^*(t) \geqq 0, \qquad t \in I. \tag{3.3}$$

Further, suppose that $(f, y^*(t)^t g)$ is (V,ρ)-B-type I at x^* with respect to b_0, b_1 and η with $b_0(x, x^*) > 0$ and $\lambda^{*t}\rho^1 + \rho^2 \geqq 0$ for all $x \in K$, then x^* is a proper efficient solution of (MOP) and therefore it is an efficient solution of (MOP).

Proof. Because $(f, y^*(t)^t g)$ is (V,ρ)-B-type I at $x^* \in \mathcal{C}(I, \mathbb{R}^n)$ with respect to b_0, b_1 and η, therefore

$$b_0(x, x^*)\Big[\int_a^b f(t, x, \dot{x})dt - \int_a^b f(t, x^*, \dot{x}^*)dt\Big]$$
$$\geqq \int_a^b \eta(t, x, x^*)^t\Big[f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)\Big]dt$$
$$+\rho^1 \int_a^b d^2(t,\ x,\ x^*)dt \tag{3.4}$$

$$
\begin{aligned}
&-b_1(x,x^*)\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*)dt\\
&\geqq \int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*)-\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt\\
&\quad +\rho^2\int_a^b d^2(t,\ x,\ x^*)dt. \qquad (3.5)
\end{aligned}
$$

Multiplying (3.4) by the nonnegative vector λ^*, we get

$$
\begin{aligned}
&b_0(x,x^*)\Big[\int_a^b \lambda^{*t} f(t,x,\dot{x})dt-\int_a^b \lambda^{*t} f(t,x^*,\dot{x}^*)dt\Big]\\
&\geqq \int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*)-\frac{d}{dt}\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt\\
&\quad +\lambda^{*t}\rho^1\int_a^b d^2(t,\ x,\ x^*)dt. \qquad (3.6)
\end{aligned}
$$

In view of (3.2), (3.5) can be rewritten as

$$
\begin{aligned}
&0\geqq \int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*)-\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt\\
&\quad +\rho^2\int_a^b d^2(t,\ x,\ x^*)dt. \qquad (3.7)
\end{aligned}
$$

Adding (3.6) and (3.7), we obtain

$$
\begin{aligned}
&b_0(x,x^*)\Big[\int_a^b \lambda^{*t} f(t,x,\dot{x})dt-\int_a^b \lambda^{*t} f(t,x^*,\dot{x}^*)dt\Big]\\
&\geqq \int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*)+y^*(t)^t g_x(t,x^*,\dot{x}^*)\\
&\quad -\frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*)+y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt\\
&\quad +(\lambda^{*t}\rho^1+\rho^2)\int_a^b d^2(t,\ x,\ x^*)dt. \qquad (3.8)
\end{aligned}
$$

Inequality (3.8) along with (3.1) yields

$$
\begin{aligned}
&b_0(x,x^*)\Big[\int_a^b \lambda^{*t} f(t,x,\dot{x})dt-\int_a^b \lambda^{*t} f(t,x^*,\dot{x}^*)dt\Big]\\
&\geqq(\lambda^{*t}\rho^1+\rho^2)\int_a^b d^2(t,\ x,\ x^*)dt.
\end{aligned}
$$

Since $\lambda^{*t}\rho^1+\rho^2\geqq 0$, we get

$$
b_0(x,x^*)\Big[\int_a^b \lambda^{*t} f(t,x,\dot{x})dt-\int_a^b \lambda^{*t} f(t,x^*,\dot{x}^*)dt\Big]\geqq 0. \qquad (3.9)
$$

Because $b_0(x, x^*) > 0$ for all $x \in K$, (3.9) gives

$$\int_a^b \lambda^{*t} f(t, x, \dot{x}) dt \geqq \int_a^b \lambda^{*t} f(t, x^*, \dot{x}^*) dt$$

which implies that x^* minimizes $\int_a^b \lambda^{*t} f(t, x, \dot{x}) dt$ over K with $\lambda^* > 0$. Hence , x^* is a properly efficient solution for (MOP) on account of Theorem 1 of Bector and Husain (1992) and therefore x^* is an efficient solution of (MOP). ∎

Theorem 3.2 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in \mathbb{R}^p, \lambda^* \geq 0$ and a piecewise smooth function $y^* : I \longrightarrow \mathbb{R}^m$ such that for all $t \in I$, (x^*, λ^*, y^*) satisfy (3.1) – (3.3) of Theorem 3.1.*
Further, suppose that $(f, y^*(t)^t g)$ is semistrictly (V, ρ)-B-type I at x^* with respect to b_0, b_1 and η with $\lambda^{*t}\rho^1 + \rho^2 \geqq 0$ for all $x \in K$, then x^* is an efficient solution of (MOP).

Proof. If x^* is not an efficient solution of (MOP), then there exists an $x \in K$ such that

$$\int_a^b f(t, x, \dot{x}) dt \leq \int_a^b f(t, x^*, \dot{x}^*) dt.$$

Since $b_0(x, x^*) \geqq 0$, we obtain

$$b_0(x, x^*) \Big[\int_a^b f(t, x, \dot{x}) dt - \int_a^b f(t, x^*, \dot{x}^*) dt \Big] \leqq 0. \tag{3.10}$$

Using (3.2), we get

$$b_1(x, x^*) \int_a^b y^*(t)^t g(t, x^*, \dot{x}^*) dt = 0. \tag{3.11}$$

Equations (3.10) and (3.11), together with the fact that $(f, y^*(t)^t g)$ is semistrictly (V, ρ)-B-type I at x^* with respect to b_0, b_1 and η, lead to

$$\int_a^b \eta(t, x, x^*)^t \Big[f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*) \Big] dt < -\rho^1 \int_a^b d^2(t,\ x,\ x^*) dt$$

$$\int_a^b \eta(t, x, x^*)^t \Big[y^*(t)^t g_x(t, x^*, \dot{x}^*) - \frac{d}{dt} y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*) \Big] dt \leqq -\rho^2 \int_a^b d^2(t, x, x^*) dt. \tag{3.12}$$

Since $\lambda^* \geq 0$, we get

$$\int_a^b \eta(t, x, x^*)^t \Big[\lambda^{*t} f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} \lambda^{*t} f_{\dot{x}}(t, x^*, \dot{x}^*) \Big] dt < -\lambda^{*t} \rho^1 \int_a^b d^2(t, x, x^*) dt. \tag{3.13}$$

Adding (3.12) and (3.13), we get

$$\int_a^b \eta(t,x,x^*)^t \Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*) + y^*(t)^t g_x(t,x^*,\dot{x}^*) \\ -\frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*) + y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big] dt \\ < -(\lambda^{*t}\rho^1 + \rho^2) \int_a^b d^2(t,\ x,\ x^*) dt \leqq 0$$

which contradicts (3.1). Hence x^* is an efficient solution for (MOP) and the proof is complete. ■

THEOREM 3.3 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in I\!R^p, \lambda^* > 0$ and a piecewise smooth function $y^* : I \longrightarrow I\!R^m$ such that for all $t \in I$, (x^*, λ^*, y^*) satisfy $(3.1) - (3.3)$ of Theorem 3.1.*
Further, suppose that $(f, y^*(t)^t g)$ is strong (V,ρ)-pseudo-quasi B-type I at x^* with respect to b_0, b_1 and η with $b_1(x,x^*) > 0$ and $\lambda^{*t}\rho^1 + \frac{b_0(x,x^*)\rho^2}{b_1(x,x^*)} \geqq 0$ for all $x \in K$, then x^* is an efficient solution of (MOP).

Proof. If x^* is not an efficient solution of (MOP), then there exists an $x \in K$ such that

$$\int_a^b f(t,x,\dot{x}) dt \leq \int_a^b f(t,x^*,\dot{x}^*) dt.$$

From (3.2), we have

$$-\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*) dt = 0.$$

Since $(f, y^*(t)^t g)$ is strong (V,ρ)-pseudo-quasi B-type I at x^* with respect to b_0, b_1 and η, therefore

$$b_0(x,x^*) \int_a^b \eta(t,x,x^*)^t \Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t,x^*,\dot{x}^*)\Big] dt \\ \leq -\rho^1 \int_a^b d^2(t,\ x,\ x^*) dt$$

$$b_1(x,x^*) \int_a^b \eta(t,x,x^*)^t \Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt} y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big] dt \\ \leqq -\rho^2 \int_a^b d^2(t,\ x,\ x^*) dt.$$

Since $\lambda^* > 0$ and $b_1(x,x^*)$ is positive , we get

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t}f_x(t,x^*,\dot{x}^*)-\frac{d}{dt}\lambda^{*t}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$< -\lambda^{*t}\rho^1\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.14}$$

$$\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*)-\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$\leqq -\frac{\rho^2}{b_1(x,x^*)}\int_a^b d^2(t,\ x,\ x^*)dt. \tag{3.15}$$

Multiplying (3.16) by $b_0(x,x^*) \geqq 0$, we get

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*)-\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$\leqq -\frac{b_0(x,x^*)\rho^2}{b_1(x,x^*)}\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.16}$$

Adding (3.14) and (3.16), we obtain

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t}f_x(t,x^*,\dot{x}^*)+y^*(t)^t g_x(t,x^*,\dot{x}^*)$$
$$-\frac{d}{dt}\Big(\lambda^{*t}f_{\dot{x}}(t,x^*,\dot{x}^*)+y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt$$
$$< -(\lambda^{*t}\rho^1+\frac{b_0(x,x^*)\rho^2}{b_1(x,x^*)})\int_a^b d^2(t,\ x,\ x^*)dt \leqq 0$$

which contradicts (3.1). Hence x^* is an efficient solution for (MOP) and the proof is complete. ■

In the next theorem, we replace the strong (V,ρ)-pseudo-quasi B-type I by the weak strictly (V,ρ)-pseudo-quasi B-type I of $(f, y^*(t)^t g)$.

Theorem 3.4 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in \mathbb{R}^p$, $\lambda^* \geq 0$ and a piecewise smooth function $y^* : I \longrightarrow \mathbb{R}^m$ such that for all $t \in I$, (x^*, λ^*, y^*) satisfy (3.1) − (3.3) of Theorem 3.1.*
Further, suppose that $(f, y^*(t)^t g)$ is weak strictly (V,ρ)-pseudo-quasi B-type I at x^* with respect to b_0, b_1 and η with $b_1(x,x^*) > 0$ and $\lambda^{*t}\rho^1 + \frac{b_0(x,x^*)\rho^2}{b_1(x,x^*)} \geqq 0$ for all $x \in K$, then x^* is an efficient solution of (MOP).

Proof. If x^* is not an efficient solution of (MOP), then there exists an $x \in K$ such that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,x^*,\dot{x}^*)dt.$$

From (3.2), we have

$$-\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*)dt = 0.$$

Since $(f, y^*(t)^t g)$ is weak strictly (V,ρ) pseudo-quasi B-type I at x^* with respect to b_0, b_1 and η, therefore

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$< -\rho^1\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.17}$$
$$b_1(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$\leqq -\rho^2\int_a^b d^2(t,\ x,\ x^*)dt.$$

Multiplying (3.17) by $\lambda^* \geq 0$, we get

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t}f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}\lambda^{*t}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt$$
$$< -\lambda^{*t}\rho^1\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.18}$$

and now the proof is similar to that of Theorem 3.3. ∎

In our final sufficiency result below, we invoke the weak strictly (V,ρ)-pseudo B-type I of $(f, y^*(t)^t g)$.

THEOREM 3.5 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in I\!R^p, \lambda^* \geqq 0$ and a piecewise smooth function $y^* : I \longrightarrow I\!R^m$ such that for all $t \in I$, (x^*,λ^*,y^*) satisfy (3.1) – (3.3) of Theorem 3.1.*
Further, suppose that $(f, y^*(t)^t g)$ is weak strictly (V,ρ)-pseudo B-type I at x^* with respect to b_0, b_1 and η and $\frac{\lambda^{*t}\rho^1}{b_0(x,x^*)} + \frac{\rho^2}{b_1(x,x^*)} \geqq 0$ for all $x \in K$, then x^* is an efficient solution of (MOP).

Proof. If x^* is not an efficient solution of (MOP), then there exists an $x \in K$ such that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,x^*,\dot{x}^*)dt.$$

From (3.2), we have

$$-\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*)dt = 0.$$

Since $(f, y^*(t)^t g)$ is weak strictly (V,ρ)-pseudo B-type I at x^* with respect to b_0, b_1 and η, therefore

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < -\rho^1\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.19}$$

$$b_1(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < -\rho^2\int_a^b d^2(t,\ x,\ x^*)dt. \tag{3.20}$$

From (3.19) and (3.20), we have $b_0(x,x^*)\neq 0$ and $b_1(x,x^*)\neq 0$, which imply that

$$\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < -\frac{\rho^1}{b_0(x,x^*)}\int_a^b d^2(t,\ x,\ x^*)dt \tag{3.21}$$

$$\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < -\frac{\rho^2}{b_1(x,x^*)}\int_a^b d^2(t,\ x,\ x^*)dt. \tag{3.22}$$

Multiplying (3.21) by $\lambda^* \geqq 0$, we get

$$\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt \leqq -\frac{\lambda^{*t}\rho^1}{b_0(x,x^*)}\int_a^b d^2(t,\ x,\ x^*)dt. \tag{3.23}$$

Adding (3.22) and (3.23), we get

$$\begin{aligned}\int_a^b \eta(t,x,x^*)^t\Big[&\lambda^{*t} f_x(t,x^*,\dot{x}^*) + y^*(t)^t g_x(t,x^*,\dot{x}^*)\\ &-\frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*) + y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt\\ &< -\Big(\frac{\lambda^{*t}\rho^1}{b_0(x,x^*)} + \frac{\rho^2}{b_1(x,x^*)}\Big)\int_a^b d^2(t,\ x,\ x^*)dt \leqq 0\end{aligned}$$

which contradicts (3.1). Hence the result. ∎

4. Mixed type duality

Let J_1 be a subset of M, $J_2 = M \backslash J_1$, and e be the vector of $I\!R^p$ whose components are all ones. We consider the following mixed type dual for (MOP),

$$(XMOP) \quad Maximize \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt$$

$$subject\ to$$

$$u(a) = \alpha, \quad u(b) = \beta,$$

$$\lambda^t f_x(t,u,\dot{u}) + y^*(t)^t g_x(t,u,\dot{u}) = \frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y^*(t)^t g_{\dot{x}}(t,u,\dot{u})\Big), \quad t \in I, \tag{4.1}$$

$$y_{J_2}(t)^t g_{J_2}(t,u,\dot{u}) \geqq 0, \quad t \in I, \tag{4.2}$$

$$y(t) \geqq 0, t \in I, \tag{4.3}$$

$$\lambda \in I\!R^p, \quad \lambda \geqq 0, \quad \lambda^t e = 1, \quad e = (1,\ldots,1) \in I\!R^p. \tag{4.4}$$

We note that we get a Mond-Weir dual for $J_1 = \emptyset$ and a Wolfe dual for $J_2 = \emptyset$ in $(XMOP)$, respectively.

We shall prove various duality results for (MOP) and $(XMOP)$ under generalized (V,ρ)-B-type I conditions.

THEOREM 4.1 *(Weak duality) If for all feasible x of (MOP) and all feasible (u,λ,y) of $(XMOP)$, any of the following conditions holds:*

(a) $\lambda > 0$, $(f + y_{J_1}(t)^t g_{J_1} e, y_{J_2}(t)^t g_{J_2})$ is strong (V,ρ)-pseudo-quasi B-type I at u with respect to b_0, b_1 and η with $b_1(x,u) > 0$, for all $x \in K$, suppose also that $\lambda^* \rho^1 + \frac{b_0(x,u)\rho^2}{b_1(x,u)} \geqq 0$,

(b) $(f + y_{J_1}(t)^t g_{J_1} e, y_{J_2}(t)^t g_{J_2})$ is weak strictly (V,ρ)-pseudo-quasi B-type I at u with respect to b_0, b_1 and η with $b_1(x,u) > 0$ for all $x \in K$, suppose also that $\lambda^* \rho^1 + \frac{b_0(x,u)\rho^2}{b_1(x,u)} \geqq 0$,

(c) $(f + y_{J_1}(t)^t g_{J_1} e, y_{J_2}(t)^t g_{J_2})$ is weak strictly (V,ρ)-pseudo B-type I at u with respect to b_0, b_1 and η, suppose also that $\frac{\lambda^* \rho^1}{b_0(x,u)} + \frac{\rho^2}{b_1(x,u)} \geqq 0$,

then the following cannot hold

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt.$$

Proof. Let x be feasible for (MOP) and (u,λ,y) feasible for $(XMOP)$. Suppose that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt.$$

Since x is feasible for (MOP) and (u,λ,y) is feasible for $(XMOP)$, we have

$$\int_a^b \{f(t,x,\dot{x}) + [y_{J_1}(t)^t g_{J_1}(t,x,\dot{x})]e\}dt$$
$$\leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt. \tag{4.5}$$

From (4.2), we have

$$-\int_a^b y_{J_2}(t)^t g_{J_2}(t,u,\dot{u})dt \leqq 0. \tag{4.6}$$

Since $(f + y_{J_1}(t)^t g_{J_1}e, y_{J_2}(t)^t g_{J_2})$ is strong (V,ρ)-pseudo-quasi B-type I at u with respect to b_0, b_1 and η, therefore

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_{1x}}(t,u,\dot{u})$$
$$-\frac{d}{dt}\Big(f_{\dot{x}}(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_{1\dot{x}}}(t,u,\dot{u})\Big)\Big]dt \leq -\rho^1\int_a^b d^2(t,\ x,\ u)dt$$
$$b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_{2x}}(t,u,\dot{u}) - \frac{d}{dt}y_{J_2}(t)^t g_{J_{2\dot{x}}}(t,u,\dot{u})\Big]dt$$
$$\leqq -\rho^2\int_a^b d^2(t,\ x,\ u)dt.$$

Since $b_1(x,u)$ is positive and $\lambda > 0$, we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_{1x}}(t,u,\dot{u})$$
$$-\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_{1\dot{x}}}(t,u,\dot{u})\Big)\Big]dt < -\lambda^t\rho^1\int_a^b d^2(t,\ x,\ u)dt, \tag{4.7}$$

$$\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_{2x}}(t,u,\dot{u}) - \frac{d}{dt}y_{J_2}(t)^t g_{J_{2\dot{x}}}(t,u,\dot{u})\Big]dt$$
$$\leqq -\frac{\rho^2}{b_1(x,u)}\int_a^b d^2(t,\ x,\ u)dt. \tag{4.8}$$

Multiplying (4.8) by $b_0(x,u) \geqq 0$, we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_{2x}}(t,u,\dot{u}) - \frac{d}{dt}y_{J_2}(t)^t g_{J_{2\dot{x}}}(t,u,\dot{u})\Big]dt$$
$$\leqq -\frac{b_0(x,u)\rho^2}{b_1(x,u)}\int_a^b d^2(t,\ x,\ u)dt. \tag{4.9}$$

Adding (4.7) and (4.9), we obtain

$$
\begin{aligned}
& b_0(x, x^*) \int_a^b \eta(t,x,u)^t \Big[\lambda^t f_x(t,u,\dot u) + y^*(t)^t g_x(t,u,\dot u) \\
& \quad - \frac{d}{dt}\Big(\lambda^t f_{\dot x}(t,u,\dot u) + y^*(t)^t g_{\dot x}(t,u,\dot x^*)\Big)\Big] dt \\
& \quad < -(\lambda^t \rho^1 + \frac{b_0(x,u)\rho^2}{b_1(x,u)}) \int_a^b d^2(t,\ x,\ u) dt \leqq 0 \qquad (4.10)
\end{aligned}
$$

which contradicts (4.1).

Now, by hypothesis (b) and from (4.2) and (4.5), we get

$$
\begin{aligned}
& b_0(x,u) \int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 x}(t,u,\dot u) \\
& \quad - \frac{d}{dt}\Big(f_{\dot x}(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 \dot x}\Big)(t,u,\dot u)\Big] dt < -\rho^1 \int_a^b d^2(t,\ x,\ u) dt \\
& b_1(x,u) \int_a^b \eta(t,x,u)^t \Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot u) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2 \dot x}(t,u,\dot u)\Big] dt \\
& \quad \leqq -\rho^2 \int_a^b d^2(t,\ x,\ u) dt.
\end{aligned}
$$

Since $b_1(x,u)$ is positive, $\lambda \geq 0$ and $b_0(x,u) \geqq 0$, we get (4.10) again contradicting (4.1).

If (c) holds, then from (4.2) and (4.5), we get

$$
\begin{aligned}
& b_0(x,u) \int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 x}(t,u,\dot u) \\
& \quad - \frac{d}{dt}\Big(f_{\dot x}(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 \dot x}\Big)(t,u,\dot u)\Big] dt < -\rho^1 \int_a^b d^2(t,\ x,\ u) dt \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
& b_1(x,u) \int_a^b \eta(t,x,u)^t \Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot u) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2 \dot x}(t,u,\dot u)\Big] dt \\
& \quad < -\rho^2 \int_a^b d^2(t,\ x,\ u) dt. \qquad (4.12)
\end{aligned}
$$

From (4.11) and (4.12), we have $b_0(x,u) \neq 0$ and $b_1(x,u) \neq 0$, which give

$$
\begin{aligned}
& \int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 x}(t,u,\dot u) \\
& \quad - \frac{d}{dt}\Big(f_{\dot x}(t,u,\dot u) + e y_{J_1}(t)^t g_{J_1 \dot x}(t,u,\dot u)\Big)\Big] dt \\
& \quad < -\frac{\rho^1}{b_0(x,u)} \int_a^b d^2(t,\ x,\ u) dt \qquad (4.13)
\end{aligned}
$$

$$\int_a^b \eta(t,x,u)^t\Big[y(t)^t g_{J_{2x}}(t,u,\dot{u}) - \frac{d}{dt}y(t)^t g_{J_{2\dot{x}}}(t,u,\dot{u})\Big]dt$$
$$< -\frac{\rho^2}{b_1(x,u)}\int_a^b d^2(t,\ x,\ u)dt. \tag{4.14}$$

Because $\lambda \geq 0$, (4.13) gives

$$\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_{1x}}(t,u,\dot{u})$$
$$-\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_{1\dot{x}}}(t,u,\dot{u})\Big)\Big]dt$$
$$\leqq -\frac{\lambda^t\rho^1}{b_0(x,u)}\int_a^b d^2(t,\ x,\ x^*)dt. \tag{4.15}$$

Adding (4.14) and (4.15), we obtain

$$\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y(t)^t g_x(t,u,\dot{u})$$
$$-\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y(t)^t g_{\dot{x}}(t,u,\dot{u})\Big)\Big]dt$$
$$< -(\frac{\lambda^t\rho^1}{b_0(x,u)} + \frac{\rho^2}{b_1(x,u)})\int_a^b d^2(t,\ x,\ u)dt \leqq 0$$

which contradicts again (4.1). ■

COROLLARY 4.1 *(Aghezzaf and Khazafi; 2004) Let* (u^*, λ^*, y^*) *be a feasible solution for* $(XMOP)$. *Assume that* $y^*_{J_1}(t)^t g_{J_1}(t, u^*, \dot{u}^*) = 0$ *and assume that* u^* *is feasible for* (MOP). *If weak duality Theorem* 4.1 *holds between* (MOP) *and* $(XMOP)$, *then,* u^* *is an efficient solution for* (MOP) *and* (u^*, λ^*, y^*) *is an efficient solution for* $(XMOP)$.

THEOREM 4.2 *(Strong Duality) (Aghezzaf and Khazafi; 2004) Let* x^* *be an efficient solution for* (MOP) *at which the Kuhn-Tucker qualification constraint is satisfied, then there exists* $\lambda^* \in I\!R^p, \lambda^* \geq 0, \lambda^{*t}e = 1$ *and a piecewise smooth function* $y^* : I \longrightarrow I\!R^m$ *such that* (x^*, λ^*, y^*) *is feasible for* $(XMOP)$ *with* $y^*_{J_1}(t)^t g_{J_1}(t, u^*, \dot{u}^*) = 0$.

If also weak duality Theorem 4.1 *holds between* (MOP) *and* $(XMOP)$, *then* (x^*, λ^*, y^*) *is an efficient solution for* $(XMOP)$.

5. Some related problems

The classes of functions introduced in this paper can be easily adapted to establish various incomplete vector-valued Lagrange saddle point optimality theorems for multiobjective variational programming problems, Aghezzaf and Khazafi

(2005). They can also be easily adapted to derive various sufficient optimality conditions and duality results for other classes of variational programming problems (see Bhatia and Mehra, 1999): Natural Boundary Value Problem, Fractional Programming Problem and Minimax Programming Problem. Analogous results can easily be obtained for the class of nonsmooth constrained fractional variational problems.

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