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Reachability, controllability to zero and observability of the positive discrete-time Lyapunov systems*

by

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Abstract: The new necessary and sufficient conditions for the reachability, controllability to zero and observability of the positive discrete-time Lyapunov systems are established. The notion of the dual positive Lyapunov system is introduced and the relationship between the reachability and observability are given. The considerations are illustrated with numerical examples.

Keywords: reachability, controllability to zero, observability, Lyapunov systems.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems in more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs of Farina and Rinaldi (2000) and Kaczorek (2001). The realization problem for positive linear systems without and with time delays has been considered in Benvenuti and Farina (2004), Kaczorek (2001, 2004, 2006a,b, 2007b).

The reachability, controllability to zero and observability of dynamical systems have been considered in Klamka (1991). The reachability and minimum energy control of positive linear discrete-time systems have been considered in

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Busłowicz and Kaczorek (2004). The controllability and observability of Lyapunov systems have been investigated by Murty and Apparao (2005). The positive discrete-time and continuous-time Lyapunov systems have been considered in Kaczorek (2007b), Kaczorek and Przyborowski (2007a). The positive linear time-varying Lyapunov systems have been investigated in Kaczorek and Przyborowski (2007c). The Lyapunov cone systems have been considered in Kaczorek and Przyborowski (2007b).

In this paper, the new necessary and sufficient conditions for the reachability, controllability to zero and observability of the positive discrete-time Lyapunov systems will be established, the notion of the dual positive Lyapunov system will be introduced and the relationship between the reachability and observability will be given. The considerations will be illustrated with numerical examples.

2. Preliminaries

Let $R^{n\times m}$ be the set of real $n\times m$ matrices with and $R^n = R^{n\times 1}$. The set of real $n\times m$ matrices with nonnegative entries will be denoted by $R_+^{n\times m}$. The set of nonnegative integers will be denoted by Z_+.

Consider the discrete-time linear Lyapunov system (Kaczorek, 2007b) described by the equations:

$$X_{i+1} = A_0X_i + X_iA_1 + BU_i \tag{1a}$$

$$Y_i = CX_i + DU_i \tag{1b}$$

where $X_i \in R^{n\times n}$ is the state-space matrix, $U_i \in R^{m\times n}$ is the input matrix, $Y_i \in R^{p\times n}$ is the output matrix, $A_0, A_1 \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, $i \in Z_+$.

The solution of the equation (1a) satisfying the initial condition X_0 is given by (Kaczorek, 2007b):

$$X_i = \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} A_0^k X_0 A_1^{i-k} + \sum_{j=0}^{i-1}\sum_{k=0}^{j} \frac{j!}{k!(j-k)!} A_0^k B U_{i-j-1} A_1^{j-k},\ i \in Z_+. \tag{2}$$

Definition 1 *The Kronecker product $A\otimes B$ of the matrices $A = [a_{ij}] \in R^{m\times n}$ and $B \in R^{p\times q}$ is the block matrix (Kaczorek, 1998):*

$$A\otimes B = [a_{ij}B]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in R^{mp\times nq}\ . \tag{3}$$

Lemma 1 *Let us consider the equation:*

$$AXB = C \tag{4}$$

where: $A \in R^{m\times n}$, $B \in R^{q\times p}$, $C \in R^{m\times p}$, $X \in R^{n\times q}$.

Equation 4 is equivalent to the following one:

$$(A \otimes B^T)x = c \tag{5}$$

where: $x := \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$, $c := \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^T$, *and* x_i *and* c_i *are the* i*-th rows of the matrices* X *and* C*, respectively.*

Proof. See Kaczorek (1998). ∎

LEMMA 2 *The Lyapunov system (1) can be transformed to the equivalent standard discrete-time,* nm*-inputs and* pn*-outputs, linear system in the form:*

$$\bar{x}_{i+1} = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i \tag{6a}$$
$$\bar{y}_i = \bar{C}\bar{x}_i + \bar{D}\bar{u}_i \tag{6b}$$

where $\bar{x}_i \in R^{n^2}$ *is the state-space vector,* $\bar{u}_i \in R^{(nm)}$ *is the input vector,* $\bar{y}_i \in R^{(pn)}$ *is the output vector,* $\bar{A} \in R^{n^2 x n^2}$, $\bar{B} \in R^{n^2 \times (nm)}$, $\bar{C} \in R^{(pn) \times n^2}$, $\bar{D} \in R^{(pn) \times (nm)}$.

Proof. The transformation is based on Lemma 1. The matrices X, U, Y are transformed to the vectors:

$$\tilde{x} = \begin{bmatrix} X_1 & X_2 & \dots & X_n \end{bmatrix}^T, \tilde{u} = \begin{bmatrix} U_1 & U_2 & \dots & U_m \end{bmatrix}^T,$$
$$\tilde{y} = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_p \end{bmatrix}^T$$

where X_i, U_i, Y_i denote the i-th rows of the matrices X, U, Y, respectively.

The matrices of (6) are:

$$\begin{array}{l} \bar{A} = (A_0 \otimes I_n + I_n \otimes A_1^T)\ \bar{B} = B \otimes I_n \\ \bar{C} = C \otimes I_n \ , \quad \bar{D} = D \otimes I_n \, . \end{array} \tag{7}$$

DEFINITION 2 *The Lyapunov system 1 is called* (internally) positive *if and only if* $X_i \in R_+^{n \times n}$ *and* $Y_i \in R_+^{p \times n}$ *for any* $X_0 \in R_+^{n \times n}$ *and for every input* $U_i \in R_+^{m \times n}$, $i \in Z_+$.

LEMMA 3 *The Lyapunov system (1) is positive if and only if:*

$$A_0, A_1 \in R_+^{n \times n}, B \in R_+^{n \times m}, C \in R_+^{p \times n}, D \in R_+^{p \times m}. \tag{8}$$

Proof. See Kaczorek (2007b). ∎

3. Reachability

DEFINITION 3 *The positive Lyapunov system (1) is called* reachable *if for any given* $X_f \in R_+^{n \times n}$ *there exists* $q \in Z_+$, $q > 0$ *and an input sequence* $U_i \in R_+^{n \times m}$, $i = 0, 1, \dots, q-1$ *that steers the state of the system from* $X_0 = 0$ *to* X_f*, i.e.* $X_q = X_f$.

THEOREM 1 *The positive system (1) is reachable:*
a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n$, $a \in R$, if and only if the matrix

$$R_n = \left[B\ \bar{A}_0B\ \ldots\ \bar{A}_0^{n-1}B\right] \tag{9}$$

contains n linearly independent monomial columns, $\bar{A}_0 = A_0 + A_1$.
b) For $A_1 \neq aI_n$, $a \in R$, if and only if the matrix B contains n linearly independent monomial columns.

Proof. From Kaczorek (2007b) it follows that positive Lyapunov system is reachable if and only if the matrix:

$$R_{n^2} = [B\otimes I_n,\ (A_0\otimes I_n+I_n\otimes A_1^T)(B\otimes I_n)\ ,\ldots,\ (A_0\otimes I_n+I_n\otimes A_1^T)^{n^2-1}(B\otimes I_n)] \tag{10}$$

contains n^2 linearly independent monomial columns.

The following three cases will be considered:

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n$, $a \in R$, matrix (10) has the form:

$$\begin{aligned}R_{n^2} &= [B \otimes I_n,\ (A_0 \otimes I_n)(B \otimes I_n),\ \ldots,\ (\bar{A}_0 \otimes I_n)^{n^2-1}(B \otimes I_n)]\\ &= [B \otimes I_n,\ (\bar{A}_0B \otimes I_n),\ \ldots,\ (\bar{A}_0^{n^2-1}B \otimes I_n)]\\ &= [B,\ \bar{A}_0B,\ \ldots,\ \bar{A}_0^{n^2-1}B] \otimes I_n.\end{aligned}$$

This matrix contains n^2 linearly independent monomial columns if and only if the matrix:

$$R_n = \left[B,\ \bar{A}_0B,\ \ldots,\ \bar{A}_0^{n-1}B\right]$$

contains n linearly independent monomial columns.

b) For $A_0 = aI_n$, $A_1 \neq bI_n$, $a, b \in R$ the matrix 10 has the form:

$$\begin{aligned}R_{n^2} &= [B \otimes I_n,\ (I_n \otimes A_1^T)(B \otimes I_n),\ \ldots,\ (I_n \otimes A_1^T)^{n^2-1}(B \otimes I_n)]\\ &= [B \otimes I_n,\ (B \otimes A_1^T),\ \ldots,\ (B \otimes (A_1^T)^{n^2-1})].\end{aligned}$$

If B contains n linearly independent monomial columns then the matrix R_{n^2} contains n^2 linearly independent monomial columns, and the system is reachable. If B contains $r < n$ linearly independent monomial columns, then $(B \otimes I_n)$ contains $r \cdot n$ linearly independent monomial columns and each of matrices $(B \otimes A_1^T), \ldots, (B \otimes (A_1^T)^{n^2-1})$ contains not more than $r \cdot n$ linearly independent monomial columns, but they are linearly dependent with monomial columns of the matrix $(B \otimes I_n)$. Therefore, the matrix R_{n^2} contains less than

n^2 linearly independent monomial columns, and the system is not reachable. In this case the system is reachable if and only if the matrix B contains n linearly independent monomial columns.

c) For $A_0 \neq aI_n$, $A_1 \neq bI_n$, $a, b \in R$ the matrix 10 has the form:

$$\begin{aligned} R_{n^2} = [&B \otimes I_n,\ (A_0 B \otimes I_n + B \otimes A_1^T), \\ &(A_0^2 B \otimes I_n + 2A_0 B \otimes A_1^T + B \otimes (A_1^T)^2),\ \dots \\ &\dots,\ (A_0 \otimes I_n + I_n \otimes A_1^T)^{n^2-1}(B \otimes I_n)]. \end{aligned}$$

The block elements of the matrix R_{n^2} are equal to:

$$R_{n^2}(1,k) = \sum_{i=0}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} A_0^i B \otimes (A_1^T)^{k-1-i},\ \text{for}\, k = 1, \dots, n^2.$$

If B contains n linearly independent monomial columns then matrix R_{n^2} contains n^2 linearly independent monomial columns, and the system is reachable. If B contains $r < n$ linearly independent monomial columns then $(B \otimes I_n)$ contains $r \cdot n$ linearly independent monomial columns and each of matrices $R_{n^2}(1,k)$, for $k = 2, \dots, n^2$ contains not more than $r \cdot n$ linearly independent monomial columns, but they are linearly dependent with monomial columns of the matrix $(B \otimes I_n)$. Therefore, the matrix R_{n^2} contains less than n^2 linearly independent monomial columns, and the system is not reachable. Thus, in this case the system is reachable if and only if the matrix B contains n linearly independent monomial columns. ∎

Example 1 Consider the positive Lyapunov system (1) with:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

In this case $n = 2$ and the system is reachable, since B contains two linearly independent monomial columns.

The matrices of the equivalent standard system have the form

$$\bar{A} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and the reachability matrix

$$R_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 10 & 0 & 8 & 0 & 38 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 16 & 0 & 18 & 0 & 64 & 0 & 122 \\ 0 & 0 & 2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 18 & 0 & 0 & 0 & 54 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 50 & 0 & 0 & 0 & 250 \end{bmatrix}$$

contains $4 = \bar{n} = n^2$ linearly independent monomial columns. Therefore the system is reachable.

EXAMPLE 2 Consider the positive Lyapunov system (1) with:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this case $n = 2$ and the system is not reachable, since B contains only $1 < n$ linearly independent monomial column.

The matrices of the equivalent standard system have the form

$$\bar{A} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the reachability matrix

$$R_4 = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 \\ 0 & 1 & 0 & 4 & 0 & 16 & 0 & 64 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

contains $2 < \bar{n} = n^2$ linearly independent monomial columns. Therefore the system is not reachable.

4. Controllability to zero

DEFINITION 4 *The positive Lyapunov system* (1) *is called* controllable to zero *if for any given nonzero $X_0 \in R_+^{n\times n}$ there exists $q \in Z_+$, $q > 0$ and an input sequence $U_i \in R_+^{n\times m}$, $i = 0, 1, \ldots, q-1$ that steers the state of the system from X_0 to $X_f = 0$, i.e. $X_q = 0$.*

THEOREM 2 *The positive Lyapunov system (1) is controllable to zero:*
a) in a finite number of steps (not greater than n^2) if and only if the matrix $\bar{A}$ is nilpotent, i.e. has all zero eigenvalues.
b) in an infinite number of steps if and only if the system is asymptotically stable.

Proof. From the equivalence of the systems (1) and (6) it follows that the positive Lyapunov system (1) is controllable to zero if and only if the pair $(\bar{A}, \bar{B})$ (defined in (7)) is controllable to zero.

For the equivalent system described by the equation (6a) we have:

$$\bar{A}^{n^2}\bar{x}_0 = -\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \ldots & \bar{A}^{n^2-1}\bar{B} \end{bmatrix} \begin{bmatrix} \bar{u}_{n^2-1} \\ \bar{u}_{n^2-2} \\ \vdots \\ \bar{u}_0 \end{bmatrix}. \tag{11}$$

For $\bar{u}_i = 0$, $i = 0, 1, \dots, n^2 - 1$ the equation (11) is satisfied if and only if the matrix $\bar{A}$ is nilpotent. Thus, the positive Lyapunov system (1) is controllable to zero in a finite number of steps if and only if the matrix $\bar{A}$ is nilpotent.

In case (b) the equation (11) may be satisfied for $\bar{u}_i = 0$, $i \in Z_+$ only if $\lim \bar{A}^i \bar{x}_0 = 0$ for every $\bar{x}_0 \in R_+^{n^2}$, that is—when the system is asymptotically stable. ∎

From the theorem we have the following important corollary.

COROLLARY 1 *The positive Lyapunov system* (1) *is controllable to zero only if it is asymptotically stable.*

EXAMPLE 3 Consider the positive Lyapunov system (1) with:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{12}$$

In this case $n = 2$ and the matrix

$$\bar{A} = (A_0 \otimes I_n + I_n \otimes A_1^T) = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent, since $\bar{A}^3 = 0$. Therefore, the system is controllable to zero in three steps from every $X_0 \neq 0$ with zero inputs.

LEMMA 4 *If the matrices A_0 and A_1 are nilpotent then the matrix $\bar{A}$ is also nilpotent with the nilpotency index $v_{\bar{A}} \leq 2n$.*

Proof. By induction it is easy to show that:

$$\begin{aligned} \bar{A}^{2n} &= (A_0 \otimes I_n + I_n \otimes A_1^T)^{2n} \\ &= \sum_{i=1}^{2n-1} \frac{(2n)!}{i!(2n-i)!} A_0^{2n-i} \otimes A_1^i + A_0^{2n} \otimes I_n + I_n \otimes (A_1^T)^{2n}. \end{aligned} \tag{13}$$

Taking into account the fact that if A_0 and A_1 are nilpotent then $A_0^n = 0$ and $(A_1^T)^n = 0$ and from (13) we have $\bar{A}^{2n} = 0$. ∎

THEOREM 3 *If the matrices A_0 and A_1 are nilpotent then the positive system* (1) *is controllable to zero in number of steps equal to $v_{\bar{A}} \leq 2n$.*

Proof. The proof follows immediately from Theorem 1 and Lemma 4. For the matrices (12) and $v_{\bar{A}} = 3$ the hypothesis of Theorem 2 holds. ∎

5. Observability

DEFINITION 5 *The positive Lyapunov system* (1) *is called* observable in q-steps, *if X_0 can be uniquely determined from the knowledge of the output in the q following time instants: $Y_0, Y_1, \ldots, Y_{q-1}$, generated by initial instant $X_0 \in R_+^{n\times n}$ and $U_i = 0$, $i \in Z_+$.*

DEFINITION 6 *The positive Lyapunov system* (1) *is called observable if there exists a natural number $q \geq 1$, such that the system is observable in q-steps.*

THEOREM 4 *The positive system* (1) *is observable if and only if:*

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n$, $a \in R$, if and only if the matrix

$$O_n = \begin{bmatrix} C \\ C\bar{A}_0 \\ \vdots \\ C\bar{A}_0^{n-1} \end{bmatrix} \tag{14}$$

contains n linearly independent monomial rows, $\bar{A}_0 = A_0 + A_1$.

b) For $A_1 \neq aI_n$, $a \in R$, if and only if the matrix C contains n linearly independent monomial rows.

Proof. The positive Lyapunov system is observable if and only if the equivalent positive standard system (6) is observable, or equivalently the matrix:

$$O_{n^2} = \begin{bmatrix} C \otimes I_n \\ (C \otimes I_n)(A_0 \otimes I_n + I_n \otimes A_1^T) \\ \vdots \\ (C \otimes I_n)(A_0 \otimes I_n + I_n \otimes A_1^T)^{n^2-1} \end{bmatrix} \tag{15}$$

contains n^2 linearly independent monomial rows.

The following three cases will be considered:

a) For A_1 satisfying the condition $XA_1 = A_1X$, i.e. $A_1 = aI_n$, $a \in R$, the matrix (15) has the form:

$$O_{n^2} = \begin{bmatrix} C \otimes I_n \\ (C \otimes I_n)(\bar{A}_0 \otimes I_n) \\ \vdots \\ (C \otimes I_n)(\bar{A}_0 \otimes I_n)^{n^2-1} \end{bmatrix} = \begin{bmatrix} C \otimes I_n \\ (C\bar{A}_0 \otimes I_n) \\ \vdots \\ (C\bar{A}_0^{n^2-1} \otimes I_n) \end{bmatrix} = \begin{bmatrix} C \\ C\bar{A}_0 \\ \vdots \\ C\bar{A}_0^{n-1} \end{bmatrix} \otimes I_n.$$

This matrix contains n^2 linearly independent monomial rows if and only if the matrix:

$$O_n = \begin{bmatrix} C \\ C\bar{A}_0 \\ \vdots \\ C\bar{A}_0^{n-1} \end{bmatrix}$$

contains n linearly independent monomial rows, $\bar{A}_0 = A_0 + A_1$.
b) For $A_0 = aI_n$, $A_1 \neq bI_n$, $a, b \in R$ the matrix (15) has the form:

$$O_{n^2} = \begin{bmatrix} C \otimes I_n \\ (C \otimes I_n)(I_n \otimes A_1^T) \\ \vdots \\ (C \otimes I_n)(I_n \otimes A_1^T)^{n^2-1} \end{bmatrix} = \begin{bmatrix} C \otimes I_n \\ (C \otimes A_1^T) \\ \vdots \\ (C \otimes (A_1^T)^{n^2-1}) \end{bmatrix}.$$

If C contains n linearly independent monomial rows, then matrix O_{n^2} contains n^2 linearly independent monomial rows, and the system is observable. If C contains $r < n$ linearly independent monomial rows, then $(C \otimes I_n)$ contains $r \cdot n$ linearly independent monomial rows and each of matrices $(C \otimes A_1^T), \dots, (C \otimes (A_1^T)^{n^2-1})$ contains not more than $r \cdot n$ linearly independent monomial rows, but they are linearly dependent with monomial rows of the matrix $(C \otimes I_n)$. Therefore, matrix O_{n^2} contains less than n^2 linearly independent monomial rows and the system is not observable. Hence, in this case, the system is observable if and only if the matrix C contains n linearly independent monomial rows.

c) For $A_0 \neq aI_n$, $A_1 \neq bI_n$, $a, b \in R$ the matrix (15) has the form:

$$O_{n^2} = \begin{bmatrix} C \otimes I_n \\ (CA_0 \otimes I_n + C \otimes A_1^T) \\ (CA_0^2 \otimes I_n + 2CA_0 \otimes A_1^T + C \otimes (A_1^T)^2) \\ \vdots \\ (C \otimes I_n)(A_0 \otimes I_n + I_n \otimes A_1^T)^{n^2-1} \end{bmatrix}.$$

The block elements of the matrix O_{n^2} are equal:

$$O_{n^2}(k,1) = \sum_{i=0}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} CA_0^i \otimes (A_1^T)^{k-1-i}, \ \text{for}\, k = 1, \dots, n^2.$$

If C contains n linearly independent monomial rows, then matrix O_{n^2} contains n^2 linearly independent monomial rows, and the system is observable. If C contains $r < n$ linearly independent monomial rows , then $(C \otimes I_n)$ contains $r \cdot n$ linearly independent monomial columns and each of matrices $O_{n^2}(1, k)$, for $k = 2, \dots, n^2$ contains not more than $r \cdot n$ linearly independent monomial rows, but they are linearly dependent with monomial rows of the matrix $(C \otimes I_n)$. Therefore, matrix O_{n^2} contains less than n^2 linearly independent monomial rows and the system is not observable. Hence, in this case, the system is observable if and only if the matrix C contains n linearly independent monomial rows. ■

EXAMPLE 4 Consider the positive Lyapunov system (1) with:

$$A_0 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case $n = 2$, and the system is observable, since C contains two linearly independent monomial rows.

The matrices of the equivalent standard system have the form

$$\bar{A} = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the observability matrix

$$O_4 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 18 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \\ 108 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 \\ 11 & 0 & 25 & 0 \\ 0 & 5 & 0 & 4 \\ 648 & 0 & 0 & 0 \\ 0 & 81 & 0 & 0 \\ 91 & 0 & 125 & 0 \\ 0 & 19 & 0 & 8 \end{bmatrix}$$

contains $4 = \bar{n} = n^2$ linearly independent monomial rows. Therefore the system is observable.

EXAMPLE 5 Consider the positive Lyapunov system (1) with:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In this case $n = 2$, and the system is not observable, since C contains only $1 < n$ linearly independent monomial row.

The matrices of the equivalent standard system have the form

$$\bar{A} = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the observability matrix

$$O_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \\ 11 & 0 & 25 & 0 \\ 0 & 5 & 0 & 4 \\ 91 & 0 & 125 & 0 \\ 0 & 19 & 0 & 8 \end{bmatrix}$$

contains $2 < \bar{n} = n^2$ linearly independent monomial rows. Therefore the system is not observable.

6. Dual positive Lyapunov system

DEFINITION 7 *The positive Lyapunov system described by the equations:*

$$X_{i+1} = A_0^T X_i + X_i A_1^T + C^T U_i \tag{16a}$$

$$Y_i = B^T X_i + DU_i \tag{16b}$$

is called the dual system of the positive system (1), *with matrices* A_0, A_1, B, C, D, X_i, U_i, Y_i *the same as in the system* (1).

THEOREM 5 *The positive Lyapunov system* (1) *is observable if and only if the dual system is reachable.*

Proof. The following two cases will be considered:

a) For $A_1 = 0$ the positive system (1) is observable if and only if the matrix O_n contains n linearly independent monomial rows. Upon transposing the matrix O_n we obtain:

$$O_n^T = \left[C^T \ C^T A_0^T \ \ldots \ C^T (A_0^T)^{n-1}\right]$$

that is, the reachability matrix R_n of the positive dual system (16). Therefore, by Theorem 1, the positive system (1) is observable if and only if the positive dual system is reachable.

b) For $A_1 \neq 0$ the positive system (1) is observable if and only if matrix C contains n linearly independent monomial rows; taking into account the fact that C^T is the matrix of the dual positive system, which contains n linearly independent monomial columns, by Theorem 1, the positive system (1) is observable if and only if the positive dual system is reachable. ■

7. Concluding remarks

New necessary and sufficient conditions for reachability (Theorem 1), controllability to zero (Theorem 2,3) and observability (Theorem 4) of the positive discrete-time Lyapunov systems have been established. The notion of the dual positive Lyapunov system has been introduced and the relationship between the reachability and observability has been given (Theorem 5). The considerations have been illustrated with numerical examples. Extension of these considerations for positive continuous-time Lyapunov systems is an open problem.

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References

Benvenuti, L. and Farina, L. (2004) A tutorial on positive realization problem. *IEEE Trans. Autom. Control* **49** (5), 651-664.

Busłowicz, M. and Kaczorek, T. (2004) Reachability and minimum energy control of positive linear discrete-time systems with one delay. *12th Mediterranean Conference on Control and Automation*, June 6-9, Kusadasi, Izmir, Turkey.

Farina, L. and Rinaldi, S. (2000) *Positive Linear Systems. Theory and Applications*, J. Wiley, New York.

Kaczorek, T. (1998) *Vectors and Matrices in Automation and Electrotechnics.* Wydawnictwo Naukowo-Techniczne, Warszawa (in Polish).

Kaczorek, T. (2001) *Positive 1D and 2D Systems*, Springer Verlag, London.

Kaczorek, T. (2004) Realization problem for positive discrete-time systems with delay. *System Science* **30** (4), 117-130.

Kaczorek, T. (2006a) Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs. *Int. J. Appl. Math. Comp. Sci.* **16** (2), 101-106.

Kaczorek, T. (2006b) A realization problem for positive continuous-time systems with reduced number of delays. *Int. J. Appl. Math. Comp. Sci.* **16** (3), 101-117.

Kaczorek, T. (2007a) New reachability and observability tests for positive linear discrete-time systems. *Bull. Pol. Acad. Sci. Techn.* **55** (1).

Kaczorek, T. (2007b) Positive discrete-time linear Lyapunov systems. Submitted to *The 15th Mediterranean Conference of Control and Automation – MED*, 27-29 June, Athens.

Kaczorek, T. and Busłowicz, M. (2004) Minimal realization problem for positive multivariable linear systems with delay. *Int. J. Appl. Math. Comp. Sci.* **14** (2), 181-187.

Kaczorek, T. and Przyborowski, P. (2007a) Positive continuous-time linear Lyapunov systems. In: *Proc. of the International Conference on Computer as a Tool EUROCON*, 9-12 September, Warsaw.

Kaczorek, T. and Przyborowski, P. (2007b) Continuous-time linear Lyapunov cone-systems. In: *Proc. of the 13th IEEE IFAC International Conference on Methods and Models in Automation and Robotics*, 27-30 August, Szczecin - Poland.

Kaczorek, T. and Przyborowski, P. (2007c) Positive continuous-time linear time-varying Lyapunov systems. In *Proc. of the XVI International Conference on Systems Science*, 4-6 September, Wroclaw - Poland.

Klamka, J. (1991) *Controllability of Dynamical Systems.* Kluwer Academic Publ., Dordrecht.

Murty, M.S.N. and Apparao, B.V. (2005) Controllability and observability of Lyapunov systems. *Ranchi University Mathematical Journal* **32**, 55-65.