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Fuzzy regions: interpretations of surface area and distance*

by

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Abstract: This contribution concerns the modelling of fuzzy information in geographic databases. For this purpose, fuzzy regions and fuzzy points have been defined in the past, along with a number of suitable operations. For numerical information - such as the surface area of fuzzy regions, or the distance between fuzzy regions and/or fuzzy points - the computation depends on the interpretation given to the fuzzy regions or points. Consequently, it is important to differentiate between the interpretation in order to obtain correct results. This article explains in detail the impact of the interpretation in terms of both the surface area and the distance.

Keywords: fuzzy topology, fuzzy EIS, fuzzy surface, fuzzy distance.

1. Fuzzy regions

1.1. Introduction

Geographic information systems (GIS for short) are complex pieces of software, in which a multitude of information from various sources is combined for viewing or analysis (Rigaux, Scholl and Voisard, 2002; Shekhar, Chawla, 2003). The information can consist of data obtained through various sources, from measurements in the field, over satellite or aerial images, to data derived from such images. Most of these sources inherently introduce imperfection (e.g. missing measurements, failure to extract data, etc.) and/or loss of accuracy (e.g. due to conversions between coordinate systems, incorrect derivation of data). In practice, however, most data is represented as crisp and certain, effectively losing the inherent imperfection of the data in the subsequent computations. As these systems are used to reflect reality, the currently used crisp data models are sometimes inadequate (Morris, 2001); consider for example the soil composition: where is the boundary between one type of soil and the neighboring type of soil?

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Several concepts have been developed to improve on the traditional model, by allowing regions with undetermined boundaries to be modelled (Clementini and Di Felice, 1994; Cohn and Gotts, 1994; Schneider, 1996). To better deal with such information and perform richer analysis, we have introduced the concept of fuzzy regions.

1.2. Fuzzy regions

A theoretical concept for dealing with fuzzy spatial information has been introduced by a number of authors. Examples are, for instance, the works of Burrough and Frank (Burrough, 1996) and Zahn (1998); most authors consider aspects and properties of fuzzy regions at a theoretical level, without having an effective model to represent fuzzy regions. We introduced our concept for fuzzy regions in Verstraete et al. (2005); it was particularly developed as a first step toward a solid theoretical foundation on which a usable fuzzy GIS can be constructed, and so, implementation limitations and practical considerations have been integrated from the start. Based on the developed model, various aspects and behaviours have been constructed. In traditional geographic information systems, geographic data are often represented by means of basic geographic structures: points are used to indicate locations, lines to indicate roads and rivers and polygons are used to represent regions; this results in an outline being used as a representation of a region. Many extensions to generalize this concept have been developed. Examples are the egg-yolk model (Cohn and Gotts, 1994) and the broad boundary model (Clementini and Di Felice, 1994). However, when considering a region, one can also consider it to be a subset of the two dimensional space, containing all the points located inside the polygon that represents the region. Using this approach, it becomes possible to extend regions to fuzzy regions, quite similar to how sets have been extended in the past to fuzzy sets (Zadeh, 1971). The definition of a fuzzy region, then, resembles the definition of a fuzzy set, specified over the two dimensional numeric domain $\mathbb{R}^2$. This is illustrated in Fig. 1

Definition 1 *Let $A \subseteq U$ be the set of all the points that belong to the region (this is a crisp set). The crisp set (or region) A is then generalized to a fuzzy set (or region) $\tilde{A}$, defined as:*

$$\tilde{A} = \{(p, \mu_{\tilde{A}}(p))|p \in U, \mu_(\tilde{A})(p) > 0\}$$

where

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow\ ]0,1] \\ p &\mapsto \mu_{\tilde{A}}(p). \end{aligned}$$

Here, U is the universe of all locations p; the membership grade $\mu_{\tilde{A}}(p)$ expresses the membership grade of the region.

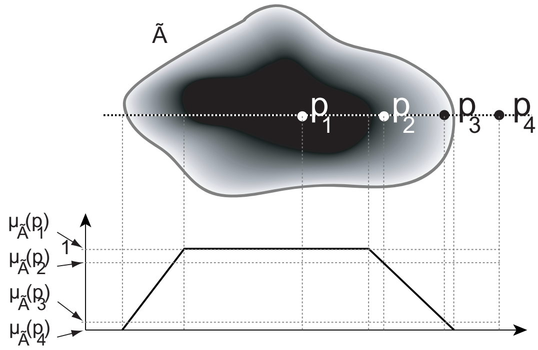


Figure 1. Illustration of the concept of fuzzy regions. Greyscales are used to indicate the membership grade (the outline is drawn for clarity), the shown membership function can be seen as an example of the concept.

1.3. Interpretation of fuzzy regions

1.3.1. Interpretation of fuzzy sets

Membership grades can have a number of interpretations (Dubois and Prade, 2000). They can have a *veristic* interpretation, a *possibilistic* interpretation, or an interpretation as degrees of truth. Other interpretations for membership grades are possible, but it was shown in Dubois and Prade (2006) that these are equivalent to one of these three interpretations.

In a veristic interpretation, each membership grade indicates a degree of belonging to the set for the element it is associated with. In other words: all elements with a membership grade greater than 0 belong to the set, but some more than others; elements with membership grade 1 are said to *fully* belong to the set. The higher the membership grade, the more the element belongs to the set. Such an interpretation is also called *conjunctive*, and is often illustrated with the example of "the languages a person speaks"; a GIS example can concern the vegetation in a given region: there can be different types of vegetation, but some types more prominently present than others. The set $\{(\text{grass}, 1), (\text{bushes}, 0.8), (\text{trees}, 0.2)\}$ indicates that there is a lot of grass, less bushes, and only few trees.

In a possibilistic interpretation, there is uncertainty about which elements belong to the fuzzy set. The membership grade indicates this possibility: a value 0 indicates that the element does not belong to the set, a value 1 indicates that the element certainly belongs to the set. The higher the membership grade, the higher the possibility the element belongs to a given set; all elements with a value greater than 0 can belong to the set. This is also called *disjunctive*.

The interpretation as degrees of truth can have meaning for fuzzy regions as well, but to indicate degrees of truth, the concept of possibilistic truth values

(De Tré, 2002) is used. As this extends beyond the scope of this article, we refer to Verstraete, Hallez and De Tré (2006) for more details.

1.3.2. Different interpretations of fuzzy regions

In fuzzy regions, both the veristic interpretation as well as the possibilistic interpretation can be applied.

The veristic interpretation will be used in the context of fuzzy regions to represent regions. Every point belongs to some extent to the area, an example is the region considered as "near" Ghent. This concept yields the region that holds all locations more or less in the vicinity of Ghent; the fuzziness indicates that some of these locations are more "near" than others.

The possibilistic interpretation can be used in the context of fuzzy regions to represent fuzzy positions. An example would be the modelling of the position of a person that is located "near" Ghent. The possible locations of this person can be represented by a single fuzzy region, but the membership grade of a position in this region now needs to be interpreted as an indication how possible a position this is for the location of the person: a possiblistic interpretation. As a person can only be in one place at one time, only one position is the valid position; the fuzziness indicates that this one position is not known exactly or with certainty.

These different interpretations provide for the concept of fuzzy regions to be used both as regions and as points. This will not only impact on the meaning of the regions, but also on the operations. For set operations, the difference is minimal: the union, intersection or difference of two fuzzy regions yields a result that carries the same interpretation. For operations such as surface area and distance, matters are more complex. The surface area can be defined in different ways, depending both on the desired result and the interpretation of the fuzzy regions. The definition of the distance between fuzzy regions will depend on the whether they carry a veristic or a possibilistic interpretation.

1.4. Preliminaries

1.4.1. α-cut

When working with fuzzy sets, it can be necessary to extract crisp sets from the fuzzy information, which can for instance be used to process the data with non-fuzzy techniques. This is often called "defuzzifying" a fuzzy set, for which the α-cut operation is commonly used. The α-cut retains the elements for which the membership grades are greater than a given value α (the α-level). As a fuzzy region is fundamentally a fuzzy set, the α-cut is easily defined, along with its special cases. These definitions are very straightforward, and are only repeated here for completeness.

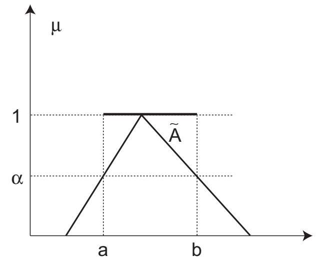


Figure 2. Schematic illustration of the α-cut of fuzzy regions. For the illustrated value α, only the points that satisfy the criterion are kept.

Definition 2 (weak α-cut)

$$\tilde{A}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Definition 3 (strong α-cut)

$$\tilde{A}_{\overline{\alpha}} = \{x \mid \mu_{\tilde{A}}(x) > \alpha\}.$$

1.4.2. Fuzzy numbers

A particular application of fuzzy set theory is the use of fuzzy numbers. Fuzzy numbers are in essence nothing more than fuzzy sets defined over a numerical domain. Using the Zadeh's extension principle (Zadeh, 1965), various operations on numbers are extended, making arithmetic operations on fuzzy numbers possible (Dubois and Prade, 2000; Klir and Yuan, 1995). Fuzzy arithmetic allows working with numbers that are imprecise or uncertain; each number is represented by a membership function that associates membership grades in the range $[0, 1]$ to crisp numbers. This membership function is a model for the uncertainty or imprecision of the number: higher membership grades (up to 1) mean a high certainty or precision, lower membership grades (down to 0) indicate a low certainty or precision. When determining numerical properties of fuzzy regions, the concept of fuzzy numbers will also be applied.

2. Surface area

2.1. Introduction

The surface area of a crisp region is commonly calculated and defined using the surface area of the geometric object (usually a polygon) that approximates the region. For a region consisting of several disconnected regions, or a region containing holes, appropriate solutions are available (Rigaux, Scholl and Voisard, 2002; Shekhar and Chawla, 2003). For fuzzy regions, calculating the surface area implies performing computations on fuzzily defined structures. Depending on how the fuzziness is dealt with, different definitions are possible: the first definition will yield a fuzzy number, whereas the second will yield a crisp number.

2.2. Fuzzy number as result

The fuzzy surface area $\tilde{S}^f$ of a fuzzy region $\tilde{A}$ in the first interpretation will result in a fuzzy number (indicated by f). This fuzzy number essentially represents all the valid surface areas. The interpretation is meaningful in a system capable of working with fuzzy numbers and fuzzy arithmetic. Current systems do not have this functionality, but this interpretation is just one part of our model for fuzzy regions leading towards a GIS with support for fuzzy information. Using a fuzzy number to represent the area of a fuzzy region makes sense: any imprecision or uncertainty in the region should be reflected in the number representing this surface area.

To obtain the fuzzy result, first all possible surface areas for the given region must be considered; these are obtained from all the possible α-cuts of the fuzzy region. Both the the strong and the weak α-cuts are required, (2), (3). The weak and the strong α-cut of a fuzzy region $\tilde{A}$ both yield a crisp region, denoted $\tilde{A}_\alpha$ respectively $\tilde{A}_{\overline{\alpha}}$. For these crisp regions and for every $\alpha \in]0,1]$:

$$S(\tilde{A}_{\overline{\alpha}}) \leq S(\tilde{A}_\alpha)$$

where S is the notation for the calculation of the surface area of a crisp region. The equality only occurs if $S(\tilde{A}_\alpha - \tilde{A}_{\overline{\alpha}}) = 0$; this happens if the points p for which $\mu_{\tilde{A}}(p) = \alpha$ form a one dimensional object. This allows us to define the surface area as follows.

Definition 4 (surface area) $\quad \tilde{S}^f(\tilde{A}) = \{(x, \mu_{\tilde{S}^f(\tilde{A})}(x)), \forall \alpha \in]0,1]\}$
where

$$\begin{array}{rcl} \mu_{\tilde{S}^f(\tilde{A})}(x) : \mathbb{R} & \rightarrow & [0,1] \\ x & \mapsto & \begin{cases} 1 & \textit{if } \ x = S(\tilde{A}_1) \\ \sup\{\alpha | S(\tilde{A}_{\overline{\alpha}}) < x \leq S(\tilde{A}_\alpha)\} & \\ 0 & \textit{elsewhere} \end{cases} \end{array}.$$

Each surface area x between $S(\tilde{A}_{\overline{\alpha}})$ and $S(\tilde{A}_\alpha)$ is considered and assigned an appropriate membership grade. This grade is the largest α for which the $x \in [S(\tilde{A}_{\overline{\alpha}}), \leq S(\tilde{A}_\alpha)]$. The reason for this is that this would result in the fuzzy surface area being a fuzzy number, which allows for fuzzy arithmetic to be applied in further computations. To verify that the result is a fuzzy number, different properties must be fulfilled (Klir, Yuan, 1995).

- The result is always normalized.

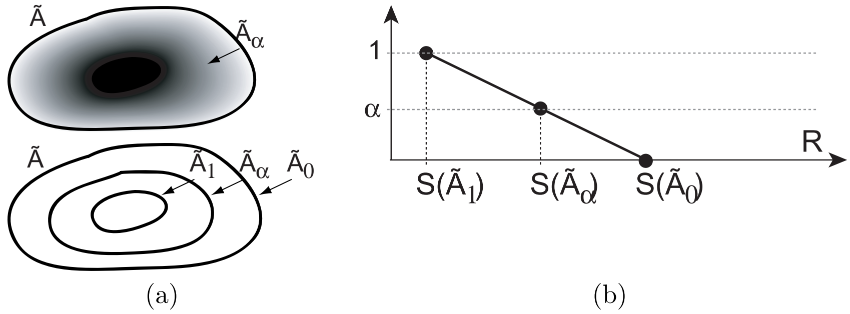


Figure 3. Illustration of the fuzzy surface area yielding a fuzzy number: (a) fuzzy region $\tilde{A}$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy surface area of $\tilde{A}$.

- The support is bounded.
- Every α-cut for $\alpha \in]0,1]$ yields a closed interval.

The first property is always fulfilled: if there are no points p for which $\mu_{\tilde{A}}(p) = 1$, then $\mu_{S^f(\tilde{A})}(0) = 1$. The other two properties are automatically fulfilled if the fuzzy region satisfies the property that every α-cut of the region is contained within every α-cut with smaller α-values:

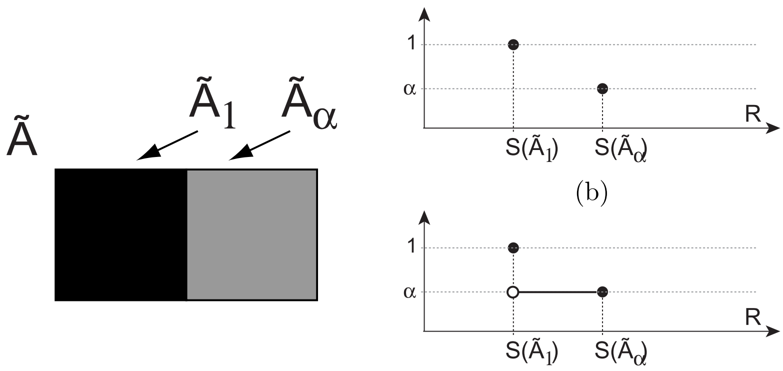
$$\forall \alpha_1, \alpha_2 \in]0,1] : \alpha_1 < \alpha_2 \Rightarrow \tilde{A}_{\alpha_2} \subseteq \tilde{A}_{\alpha_1}.$$

To clarify the origin and requirement of the supremum in the definition, consider the examples in Fig. 4. The region represented by the image in Fig. 4a is a fuzzy region that consists of two portions: both portions are squares, but all points in the left portion have membership grade 1 whereas points in the right portion have membership grade 0.5. This is a region in accordance with Definition 1; it can be called *discontinuous* as there is no continuity in membership grades of the elements of the region (some points have membership grade 0.5, others have membership grade 1, but no points have values in between).

If one were to simply associate the (crisp) surface area of points with a given membership grade as a contributing element of the resulting fuzzy surface area, the obtained surface area would be the fuzzy set $\{(s,1),(s,0.5)\}$, as illustrated in Fig. 4b. This fuzzy set, however, is not a fuzzy number, as it does not satisfy the aforementioned requirements (the support is not bounded). Consequently, this result is not suited for future computations, even though this fuzzy set could be a possible valid representation of a fuzzy surface area.

Consider now an application of Definition 4. An application of this definition on the current example yields

$$\tilde{S}^f(\tilde{A}) = \{(x, \mu_{\tilde{S}^f(\tilde{A})}(x)), \forall \alpha \in]0,1]\}.$$



(b)

(a) (c)

Figure 4. Illustration of the surface of a discontinuous fuzzy region: (a) the fuzzy region, (b) the surface area as would be obtained without use of the weak α-cut, (c) the surface area as obtained from the definition.

To determine the membership function, it is necessary to find out the membership grade that will be associated with each possible surface area. The surface area of the portion with membership grade 1 is assigned membership grade 1 (thus the element s contributes to the fuzzy surface area with degree 1. The other occurring surface areas are all in the range $]s, 2s]$. For every x in this interval, an α level must be determined such that

$$S(\tilde{A}_{\overline{\alpha}}) < x \leq S(\tilde{A}_\alpha).$$

In this simple example, just one α level suffices for all x; the level 0.5. As a result, all x are assigned the membership grade given by the equation

$$\sup\{\alpha | S(\tilde{A}_{\overline{\alpha}}) < x \leq S(\tilde{A}_\alpha)\} = 0.5.$$

Combining these results yields the fuzzy number for the example:

$$\tilde{S}^f(\tilde{A}) = \{(s, 1)\} \cup \{(x, 0.5) | x \in]s, 2s]\}$$

with s the surface area of each of the squares.

This membership function is illustrated in Fig. 4c. The main difference with the result illustrated in Fig. 4b is that now surface areas in between $S(\tilde{A}_{\overline{\alpha}})$ and $S(\tilde{A}_\alpha)$ are also given membership grade α. By assigning the value α to each x for which $S(\tilde{A}_{\overline{\alpha}}) \leq x \leq S(\tilde{A}_\alpha)$ holds, this result becomes a fuzzy number. The support is now bounded and every α cut yields a closed interval: for $\alpha \in]0, 0.5[$, the strong α-cut results in the interval $[s, 2s]$; for $\alpha \in [0.5, 1]$, the strong α-cut yields the (degenerate) interval $[1, 1]$.

Apart from the benefits of having a fuzzy number result, there is a valid intuitive ground to accept this definition: the region in the example only has possible surface areas in this range, and membership grades of elements of a fuzzy set are independent of one another: the fact that two points in the fuzzy region have a possibility 0.5 does not imply that these points are both in the region or both not in the region, just that they have an equal possibility.

2.3. Crisp result

In the second interpretation, the fuzziness is used to indicate the intrinsic vagueness of a region. The surface area, therefore, becomes a crisp number. This number takes all points into consideration, where the membership grade for each point determines how much it will contribute: a point with a membership grade 0.5 will only contribute half of what a point with membership grade 1 will contribute. In a discrete set, this number resembles the fuzzy cardinality, but for infinite sets this needs to be extended.

Definition 5 (fuzzy surface area $\tilde{S}^c$)

$$\tilde{S}^c(\tilde{A}) = \int_{(x,y)\in U} \mu_{\tilde{A}}(p(x,y))d(x,y).$$

Definition 5 is easily illustrated using the example in Fig. 4a, where the computation yields:

$$\begin{aligned}\tilde{S}^c(\tilde{A}) &= \int_{(x,y)\in U} \mu_{\tilde{A}}(p(x,y)))d(x,y)\\ &= 1S(\tilde{A}_1) + 0.5S(\tilde{A}_{0.5})\\ &= 1s + 0.5s\\ &= 1.5s.\end{aligned}$$

The example is a single region, which consists of two square sub-regions, one with membership grade 1 and a second with membership grade 0.5. Following the definition, the surface area of the square with membership grade 0.5 only counts half the area of the square (as it is multiplied with the membership grade). This interpretation always yields a crisp result, which does take into account some aspects of the fuzziness, and can be used when the results need to be processed by a non-fuzzy system.

2.4. Interpretation of the results

Different scenarios are now possible. First, consider a fuzzy region $\tilde{A}$ with a veristic interpretation; this means that all the points p with membership $\mu_{\tilde{A}}(p)$ belong to some extent to the region. The surface area of this region can be computed as a fuzzy number (Definition 4). However, as there is certainty about all the elements of the fuzzy region, there is certainty about its surface area: the resulting fuzzy number also needs to be interpreted in a veristic way.

As there is certainty about the region (all points that belong to the region are certain, as is the extent to which they belong to the region), the surface area can also be computed as a crisp number (Definition 5). The choice regarding which definition to use will depend on both the application (can it deal with fuzzy numbers? is the crisp number sufficient?), but may also depend on the origin of the fuzziness. If the fuzziness is an inherent part of the region (implying that the fuzziness will not change), the crisp number might be more appropriate; if the fuzziness is introduced (and thus able to change when more or better data becomes available), the fuzzy number might be more suitable.

Now, consider a fuzzy region $\tilde{B}$ with a possibilistic interpretation: for each of the points p it is known how possible it is ($\mu_{\tilde{B}}(p)$) that it belongs to the region. As there is no certainty regarding which points belong to the region, there can be no certainty about the surface area. This uncertainty in surface area is reflected when using Definition 4 to obtain a fuzzy number. However, contrary to the previous example, the fuzzy number now needs to carry a possibilistic interpretation, as the membership grades now represent the possibility of each surface area. The need to express uncertainty about the surface area itself, makes the use of Definition 5 less appropriate.

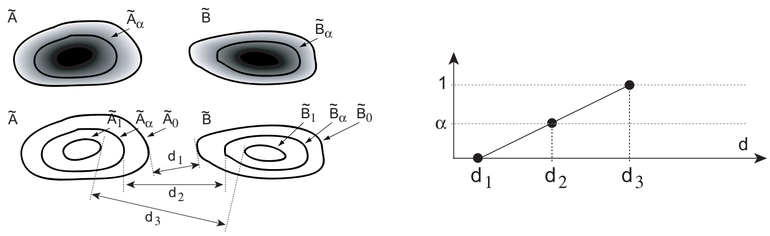
3. Distance

3.1. Introduction

The distance to a fuzzy region is a complicated concept. Consider for instance the distance from Spain to the United Kingdom. Gibraltar is part of the United Kingdom, but this distance might not always be desired: for a criminal evading the Spanish police, Gibraltar might be the easiest way out, but most people will refer to the distance to the main island of the UK. We propose two possible distance measurements that are related to the crisp concept of the shortest distance between regions. Without having further information on how the membership grades of the fuzzy region are interpreted, it is difficult to pinpoint the most appropriate definition.

3.2. α-level approach

The first definition is based on α-levels. With this definition, the distance to a fuzzy region is represented by a fuzzy number, which models all the possible distances between the corresponding α-levels. To define the distance calculation, consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$. The distance between crisp regions is defined as the shortest distance between them (Rigaux, Scholl and Voisard, 2002; Shekhar and Chawla, 2003). The fuzzy distance $\tilde{d}(\tilde{A}, \tilde{B})$ is a fuzzy number representing the possible distances between the α-cuts of A and B.



(a) (b)

Figure 5. Illustration of the fuzzy distance yielding a fuzzy number: (a) fuzzy regions $\tilde{A}$ and $\tilde{B}$ (illustrated using grey scales, and using some contourlines), (b) the membership function representing the fuzzy distance between $\tilde{A}$ and $\tilde{B}$.

3.2.1. In a veristic interpretation

In a veristic interpretation, all the different α-levels can be considered to be crisp regions, thus the distance between crisp regions needs to be considered. Consequently, only the shortest distance between any two points of the regions need to be considered, as the distance between two fuzzy regions cannot exceed the core of the regions.

Definition 6 (distance between fuzzy regions) *(α-level approach)*

$$\tilde{d}(\tilde{A},\tilde{B}) = \{(x, \mu_{\tilde{d}(\tilde{A},\tilde{B})}(x)) \mid x \in \mathbb{R}\}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup\{\alpha \mid d(\tilde{A}_\alpha, \tilde{B}_\alpha) \le x \le d(\tilde{A}_{\overline{\alpha}}, \tilde{B}_{\overline{\alpha}})\}. \end{aligned}$$

In the case of fuzzy regions with discontinuous membership functions, the definition provides for a result similar to the result obtained by the definition of the fuzzy surface (Definition 4). The definition of the distance is illustrated on Fig. 5. Two fuzzy regions are shown on Fig. 5a, as well as the distances between some of their α levels. These α levels are combined (following the above definition) to yield the single membership function, shown in Fig. 5b, to represent the fuzzy distance.

In this definition, the distance to points with a membership grade 1 is the only distance which obtains membership grade 1. Intuitively, one might wonder about this: points with a membership grade greater than 0.5 already are considered to belong more to the region than to be outside of it.

3.2.2. In a possibilistic interpretation

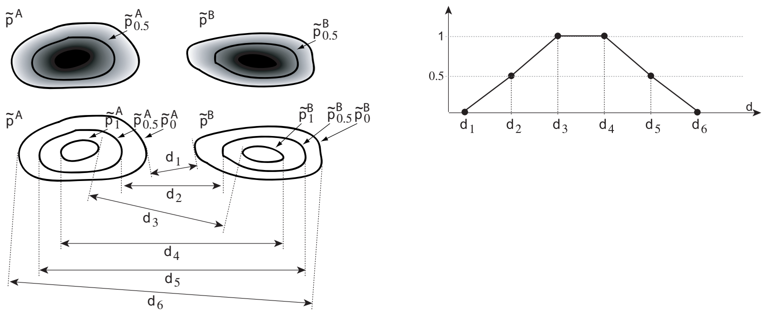
In a possibilistic interpretation, each individual α-level has to be considered as a region of candidate locations. As the location itself is unknown, it can be anywhere in these regions. Consequently, all distances between any two points of the regions need to be considered.

Definition 7 (distance $\tilde{d}(\tilde{p}^A, \tilde{p}^B)$ between fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$)

$$\tilde{d}(\tilde{p}^A, \tilde{p}^B) = \{(x, \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)}(x))\}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup_{\alpha \in]0,1]} \{\alpha \mid p_1 \in \tilde{p}^A_\alpha, p_2 \in \tilde{p}^B_\alpha \wedge d(p_1, p_2) = x\}. \end{aligned}$$



(a) (b)

Figure 6. Illustration of the fuzzy distance between fuzzy points: (a) fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy distance between $\tilde{p}^A$ and $\tilde{p}^B$.

3.3. Topological approach

3.3.1. Fuzzy topology

The second approach for defining distance makes use of topological aspects of fuzzy regions. Topology deals with relative positions of regions, for crisp regions, these aspects are defined using the concepts of *interior*, *exterior* and *boundary*. For crisp regions, the definitions are as follows:

$$\partial A = \overline{A} - A^\circ$$

where $\overline{A}$ is the closure, and A° is the interior of A in U. Simply put, the *boundary* contains all points that form the outline of the region.

The *interior* A° of a region A is defined as the set of points $p \in A$ such that A° contains a neighbourhood of p:

$$A^\circ = \{p \in A \mid \exists B \subset A, B \text{ is a neighbourhood for } p\}.$$

In practice, the interior of a region holds all the points that are inside the region.

$$A^- = \mathbb{R}^2 - \partial A$$

where "−" is the notation for set-minus. In practice, the *exterior* of a region encompasses all the points that are not part of the region.

Using these concepts for two different crisp regions, a case study can be performed to classify the relative positions based on the presence, or lack thereof, of the intersection of the interior of the first region with the interior of the second region, the boundary of the first with the interior of the second, ..., nine different cases in total. These nine cases are grouped in a matrix, the nine-intersection, by means of which a full case study can be performed.

For fuzzy regions, a similar case study using extended concepts for the interior, exterior and boundary is in preparation. In this case, as the regions themselves are defined in a fuzzy way, it stands to reason that these three concepts will also yield fuzzy regions.

Definition 8 (boundary $\Delta\tilde{A}$ of a fuzzy region $\tilde{A}$) *The boundary $\Delta\tilde{A}$ of a fuzzy region $\tilde{A}$ will be defined as a new fuzzy region such that points, for which $\mu_{\tilde{A}}(p) = 0.5$ are assigned the membership grade $\mu_{\Delta\tilde{A}}(p) = 1$, and membership grades in the boundary decrease in a linear way to 0 as the membership grades in the original region differ more from 0.5 and come closer to either 0 or 1. This results in the definition:*

$$\Delta\tilde{A} = \bigcup_{\alpha\in]0,1]} \{(p, 2(0.5 - |0.5 - \alpha|)) \mid p \in \partial\tilde{A}_\alpha\}$$

where $\partial\tilde{A}_\alpha$ represents the crisp boundary of the α-cut at level α.

Similarly, the interior will hold all the points for which $\mu_{\tilde{A}}(p) > 0.5$, with decreasing membership grades in the boundary, i.e. as the membership grade in the boundary of the original region nears 1.

Definition 9 (interior $\tilde{A}^\circ$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^\circ = \{(p, \mu_{\tilde{A}^\circ}(p)\}$$

Where

$$\begin{aligned} \mu_{\tilde{A}^\circ} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} 0 & \mu_{\tilde{A}}(p) \leq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(p) & \textit{elsewhere} \end{cases} \end{aligned}.$$

The exterior is defined similarly to the interior; only points p for which $\mu_{\tilde{A}}(p) < 0.5$ are now considered, with decreasing membership grades as the membership grades in the boundary of the original region near 1.

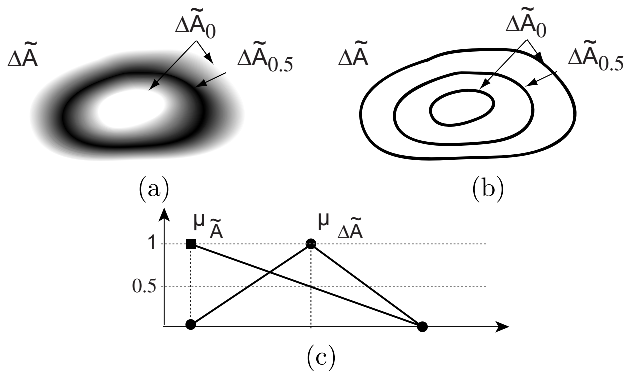


Figure 7. Illustration of the fuzzy boundary: (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\Delta\tilde{A}$; this can be considered as a cross section of the boundary of $\tilde{A}$, ranging from the inside to the outside.

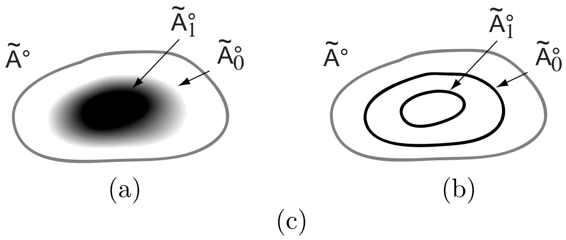


Figure 8. Illustration of the fuzzy interior (the outline of $\tilde{A}$ is shown): (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\tilde{A}^\circ$; this can be considered as a cross section of the boundary of $\tilde{A}$, ranging from the inside to the outside.

Definition 10 (exterior $\tilde{A}^-$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^- = \{(p, \mu_{\tilde{A}^-}(p)\}$$

Where

$$\begin{aligned} \mu_{\tilde{A}^-} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} 0 & \mu_{\tilde{A}}(p) \geq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(p) & \textit{elsewehere} \end{cases} \end{aligned}$$

A full case study regarding topology of fuzzy regions has been performed, but this is in preparation to be published. This contribution introduces and illustrates the concepts used.

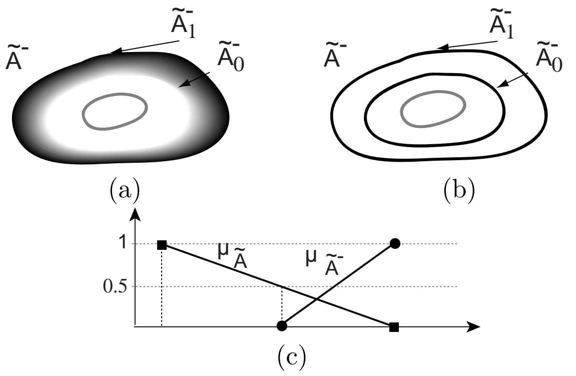


Figure 9. Illustration of the fuzzy exterior (the outline of the core of $\tilde{A}$ is also shown): (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\tilde{A}^-$; this can be considered as a cross section of the boundary of $\tilde{A}$, ranging from the inside to the outside. The region does not end at the boundary $\tilde{A}_1^-$, but extends infinitely.

3.3.2. Topological distance

To illustrate the topological approach for distances, one might consider, given that it is boundary that matters most in determining distances, that points with a membership degree greater than 0.5 already belong more to the region than points with less than 0.5. This would imply that these points contribute less to the distance of the region. Similarly, one might consider that as the membership grade of points is smaller than 0.5, the points belong less to the region, thus contribute less to the distance. This can be accomplished by defining the distance using both the distance to its (fuzzy) boundary and the distance to its (fuzzy) interior.[1]

Definition 11 (distance between fuzzy regions) *(topological approach)*

$$\tilde{d}^{\Delta}(\tilde{A},\tilde{B}) = \{(x, \mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})}(x))\}$$

$$\begin{aligned} \mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})} : \mathbb{R} &\to [0,1] \\ x &\mapsto \begin{cases} \mu_{\tilde{d}(\Delta\tilde{A},\Delta\tilde{B})}(x) & \textit{if } \; x < d(\Delta\tilde{A}_{0.5}, \Delta\tilde{B}_{0.5}) \\ 1 - \mu_{\tilde{d}(\tilde{A}^\circ,\tilde{B}^\circ)}(x) & \textit{if } \; x \geq d(\Delta\tilde{A}_{0.5}, \Delta\tilde{B}_{0.5}). \end{cases} \end{aligned}$$

[1]This definition of distance should not be confused with the topological distance as defined in Egenhofer and Sharma (1993), which expresses the distance between two different topology cases in a conceptual neighbourhood graph.

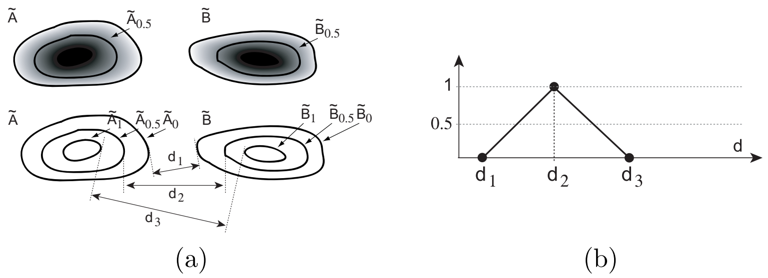


Figure 10. Illustration of the fuzzy distance yielding a fuzzy number: (a) fuzzy regions $\tilde{A}$ and $\tilde{B}$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy distance between $\tilde{A}$ and $\tilde{B}$.

Note that the membership grade for the distance decreases as the membership for the region increases from 0.5 to 1. This reflects the fact from the crisp case where the distance between a point that belongs to a region is 0; points with a membership grade greater than 0.5 are considered to be more inside the region than outside the region. The distance between such points is therefore assigned lower membership grade. The choice for 0.5 is in a sense arbitrary, but it stands to reason to consider the halfway-point of the membership grades.

Note that the membership grade decreases as the membership differs 0.5, so for the distance d_3 in Fig. 10 the membership grade is 0. This is because the distance is considered using the boundary as a reference, not the interior of the regions. The main argument is that points with a membership grade greater than 0.5 are more likely to belong to the region, so it is less likely that the distance needs to be considered up to these points. In a more elaborate case study, this also allows for a smooth transition from non-overlapping regions, over touching regions, to intersecting regions.

This definition yields a nice intuitive result for regions that are represented by normalized fuzzy sets (i.e. where the highest occurring membership grade is 1) and that are simple fuzzy regions (i.e. the membership grade decreases from the inside toward the outside), but also for regions that are a union of normalized, simple fuzzy regions. Note that this topological definition for distance is not appropriate for fuzzy points, as the topological concepts themselves can be considered dubious. The definitions presented in Section 3.2 can, however, be used for points.

3.4. Interpretation of the definitions

For the distance between regions with a veristic interpretation, there is a choice between the α-level approach (Definition 6) and the topological approach (Definition 11). Intuitively, we would suggest the choice based on the origin of fuzziness, and applying the topological approach if the fuzziness is inherent to the region, as the boundary is known with certainty and will not change, and consider the α-level approach for fuzziness that is introduced as the boundary might change when additional information might impact the region (and thus the boundary).

4. Conclusion

When extending a geographic system to work with fuzzy regions, it is interesting to have operations that yield a crisp number for crisp regions and a fuzzy number for fuzzy regions. In this paper, the intuitive definition for the surface area of a fuzzy region has been considered, and it was shown that this not necessarily yields a fuzzy number. Similar issues occur with other numeric properties, such as the distance to a fuzzy region. To overcome this for the surface area, an alternative definition has been presented. It was also illustrated that this alternative definition yields result which in all cases is a fuzzy number.

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