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Uniform energy decay rates of hyperbolic equations with nonlinear boundary and interior dissipation*

by

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Abstract: We consider the problem of uniform stabilization of nonlinear hyperbolic equations, epitomized by the following three canonical dynamics: (1) the wave equation in the natural state space $L_2(\Omega) \times H^{-1}(\Omega)$, under nonlinear (and non-local) boundary dissipation in the Dirichlet B.C., as well as nonlinear internal damping; (2) a corresponding Kirchhoff equation in the natural state space $[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, under nonlinear boundary dissipation in the 'moment' B.C. as well as nonlinear internal damping; (3) the system of dynamic elasticity corresponding to (1). All three dynamics possess a strong, hard-to-show 'boundary $\rightarrow$ boundary' regularity property, which was proved, also by invoking a micro-local argument, in Lasiecka and Triggiani (2004, 2008). This is by no means a general property of hyperbolic or hyperbolic-like dynamics (Lasiecka and Triggiani, 2003, 2008). The present paper, as a continuation of Lasiecka and Triggiani (2008), seeks to take advantage of this strong regularity property in the case of those PDE dynamics where it holds true. Thus, under the above boundary $\rightarrow$ boundary regularity, as well as exact controllability of the corresponding linear model, uniform stabilization of nonlinear models is obtained under minimal nonlinear assumptions, provided that a corresponding unique continuation property holds true.

The treatment of the present paper is cast in the abstract setting (Lasiecka, 1989, 2001; Lasiecka and Triggiani, 2000, Ch. 7, 2003, 2008), which is proper for these hyperbolic dynamics and recovers the results of Lasiecka and Triggiani (2003, 2008) in the absence of the nonlinear interior damping, in particular in the linear case.

Keywords: non-linear hyperbolic equations, uniform energy decay rates.

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1. Description of the problem. Literature

The present paper is a direct successor of Lasiecka and Triggiani (2008), which, in turn, followed from Lasiecka and Triggiani (2003, 2004). These works dealt with the problem of uniform stabilization of certain classes of hyperbolic and Petrowski-type partial differential equations, which are defined on a bounded multi-dimensional domain and which are subject to linear (Lasiecka and Triggiani, 2003, 2004), or nonlinear (Lasiecka and Triggiani, 2003, 2008) boundary dissipation. The problem of nonlinear boundary dissipation for these evolution equations has, of course, been the subject of intensive *direct* studies over the past many years, with *direct* analysis which was based on the prior treatment—in results as well as techniques—of the corresponding linear dissipative models. A general *direct* approach, yielding sharp energy decay results under minimal assumptions on the nonlinear dissipative terms, was proposed in Lasiecka and Tataru (1993) in connection with a wave equation with nonlinear dissipation in the *Neumann B.C.* It was later exported and pursued in the study of nonlinear boundary, uniform stabilization of other hyperbolic-like dynamics, including shells (Lasiecka and Triggiani, 2002), Schrödinger equations (Lasiecka and Triggiani, 2006); and, moreover, also of uniform stabilization of wave equations with homogeneous Neumann B.C. but with dissipation localized on a selected layer or neighborhood of the boundary (Lasiecka and Toundykov, 2006).

In contrast, paper Lasiecka and Triggiani (2008)—on the basis of the linear results in Lasiecka and Triggiani (2003, 2004)—sought to revisit the nonlinear boundary uniform stabilization problem of several selected classes of hyperbolic or Petrowski-type evolution equations, by marrying the sharp energy-based approach in Lasiecka and Tataru (1993) with two additional properties required of the dynamics under study. One is the exact controllability of the corresponding linear version: This is a natural property, which in fact has been known for several decades (Russell, 1978) to be a necessary condition for uniform stabilization of linear hyperbolic-type PDE models. Moreover, the availability of this property of exact controllability for the hyperbolic/Petrowski-type classes has been established over the past two decades, to an advanced level of generality and checkability (Bardos, Lebeau and Rauch, 1992; Triggiani and Yao, 2002; Gulliver et al., 2003; Lasiecka, Triggiani, and Yao, 1999; Lasiecka, Triggiani and Zhang, 2000).

The second property taken as an assumption in the nonlinear approach to uniform stabilization followed in Lasiecka and Triggiani (2003, 2008) is an 'abstract boundary–boundary' regularity property of the corresponding boundary control models (boundedness of the operator B^*L, assumption (A.2) below). Unlike the first, this second assumption is less natural and, indeed, is too strong for stabilization purposes. In fact, Lasiecka and Triggiani (2003, 2004)—prompted by Guo and Luo (2002) and Ammari (2002)—show that while this property does hold true for certain explicit classes of hyperbolic/Petrowski-type PDEs, it also fails to be true for some other relevant classes. Moreover, even for the classes where it fails, the desirable result of uniform stabilization has been known to hold true via a *direct* analysis, for several decades! Thus, even though the aforementioned 'abstract boundary-boundary' regularity property (of B^*L) is only a (strong) sufficient condition for stabilization, but by no means a necessary one, nevertheless—having ascertained it in Lasiecka and Triggiani (2003, 2004) for several explicit physically relevant classes—one may as well take advantage of this finding and analyze its consequences, also as they pertain to nonlinear problems. This was the goal of the paper Lasiecka and Triggiani (2008). Indeed, by relying on both exact controllability and the 'abstract boundary-boundary' regularity property of the corresponding linear model, Lasiecka and Triggiani (2003) and further Lasiecka and Triggiani (2008) provided an amenable abstract proof for uniform stabilization, with optimal decay rates—due to the infusion of the energy-based approach in Lasiecka and Tataru (1993)—which applied to, and was motivated by, boundary dissipation in concrete (selected) classes of hyperbolic/Petrowski-type PDEs. As a consequence, decay rates given in Lasiecka and Triggiani (2008) *were truly uniform*; that is, *with constant independent of the radius of the sphere containing the initial condition.*

Canonical cases, where this nonlinear approach and theory apply, refer to the uniform stabilization of two *new* nonlinear boundary dissipative models: (i) the wave equation, and (ii) the Schrödinger equation, both with (necessarily, nonlocal) *boundary dissipation in the Dirichlet B.C.* (unlike Lasiecka and Tataru, 1993, which treated the wave equation with dissipation in the Neumann B.C.—the linear versions of which were first given in Lasiecka and Triggiani,1987 and 1992, respectively). For these two classes—as well as for the corresponding Kirchhoff plate equation—paper Lasiecka and Triggiani (2008) gives *new sharp results of uniform, nonlinear, boundary stabilization.*

The present paper continues the nonlinear uniform stabilization analysis of Lasiecka and Triggiani (2008)—along the same approach which combines the energy-based sharp analysis of Lasiecka and Tataru (1993) with the exact controllability and the 'abstract boundary-boundary' regularity property (of B^*L). However, it restricts the scope (due to space limitations) to three PDE-dynamics: the wave equation, as well as its natural generalizaiton, the system of dynamic elasticity, both with Dirichlet nonlinear *boundary* dissipation; and the corresponding Kirchhoff-plate problem—*while adding a nonlinear dissipative term also in the interior.* This latter feature was hinted at in Lasiecka and Triggiani (2008, Remark 4.4.1), but left out from the analysis given there. Moreover (again due to space limitations), we shall more conveniently restrict to models with constant coefficients, while Lasiecka and Triggiani (2008) also included explicitly the models with variable coefficients (in space) in the principal part of the dynamic operator.

Our presentation is first carried out at the appropriate abstract level—given in Lasiecka (1989, 1999), Lasiecka and Triggiani (2000 – Chapter 7, 2003, 2008)—and leads to abstract nonlinear *uniform* stabilization results where uniformity is with respect to all initial conditions in a given ball of the state space. Subsequently, the abstract theory is specialized to two concrete PDE cases: (i) the wave equation with Dirichlet nonlinear dissipation (Section 4); and (ii) the Kirchhoff plate equation with nonlinear dissipation in the 'moment' B.C. (Section 5). In contrast, some results on the lack of uniform stabilization/exact controllability for linear systems are given in Triggiani (1989/1990, 1990, 1991).

2. Abstract models and results

We begin with the abstract setting—given in Lasiecka (1989, 2001), Lasiecka and Triggiani (2000 – Chapter 7, 2003, 2008) – which appropriately captures the dynamical properties of the class of hyperbolic/Petrowski-type nonlinear models, which we intend to cover.

Abstract model. Let $\mathcal{H}$ and U be two Hilbert spaces. The present paper studies the following abstract dynamical system in closed-loop feedback form:

$$w_{tt} + \mathcal{A}w + \mathcal{B}g(\mathcal{B}^* w_t) + \mathcal{F}(w) = 0 \text{ in } [\mathcal{D}(\mathcal{A})]', \tag{2.1}$$

on the state space

$$H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H}, \tag{2.2}$$

for $\{w, w_t\}$, subject to the following sets of assumptions, each targeted to an appropriate conclusion, as explicitly noted below.

Preliminary dynamical assumptions for local well-posedness

(i) $\mathcal{A} : \mathcal{H} \supset \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is a positive, self-adjoint operator with compact resolvent;

(ii) $\mathcal{B} : U \to [\mathcal{D}(\mathcal{A})]'$ is a linear operator; here, $[\mathcal{D}(\mathcal{A})]'$ is the dual space of $\mathcal{D}(\mathcal{A})$, with respect to $\mathcal{H}$ as a pivot space; moreover, there exists a space $\tilde{U}$, with $\tilde{U} \subset U \subset \tilde{U}'$ and: (ii$_1$) $\mathcal{B} \in \mathcal{L}(\tilde{U}'; [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$; (ii$_2$) the subsequent adjoint $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \tilde{U})$ is surjective.

(iii) the continuous function g is the Gateaux gradient of a convex function $\Phi : U \to \mathbb{R}$ and satisfies $g(0) = 0$. Hence, g is maximal monotone on U (Barbu, 1976). In particular,

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle_U \geq 0, \quad \forall\, u_1, u_2 \in U; \tag{2.3}$$

(iv) the operator $\mathcal{F}$, with $\mathcal{F}(0)=0$, is assumed to be locally Lipschitz $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to \mathcal{H}$:

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{\mathcal{H}} \leq c_\rho \|v_1 - v_2\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})}, \quad \forall\, v_1, v_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), \tag{2.4}$$

with $\|v_i\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \leq \rho,\ i = 1, 2.$

Many examples of this abstract model are given under one cover in Lasiecka (2001), Lasiecka and Triggiani (2008, 2000 – Chapter 7).

Abstract first-order model. As usual, the second-order evolution model (2.1) can be recast as a first-order abstract equation in the variable $y = \{w, w_t\}$, as follows, Lasiecka and Triggiani (2000 – Chapter 7):

$$y_t = Ay - Bg(B^*y) + F(y), \text{ in } [\mathcal{D}(A^*)]', \; y = [w, w_t]; \tag{2.5}$$

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}, \; Bu = \begin{bmatrix} 0 \\ \mathcal{B}u \end{bmatrix}, \; B^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{B}^* v_2; \; F(y) = \begin{bmatrix} 0 \\ \mathcal{F}(w) \end{bmatrix}, \tag{2.6}$$

$F(0) = 0$. The skew-adjoint operator $A = -A^*$,

$$A : H \supset \mathcal{D}(A) \equiv \mathcal{D}(A^*) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to H, \tag{2.7}$$

generates a strongly continuous (C_0-) unitary group e^{At} on the space H in (2.2).

Local well-posedness. It is known, Lasiecka (1989), that, under the above assumptions (i), (ii), (iii), (iv), Eqn. (2.1) [or else (2.5)] *admits a local (in time) unique semigroup solution defined on the state space H in (2.2)*,

$$\{w, w_t\} \in C([0, T]; \; H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H}); \; T > 0 \text{ sufficiently small.} \tag{2.8}$$

We next introduce additional sets of assumptions for the purpose of obtaining *uniform stabilization* of (2.1) [or (2.5)].

Further sets of assumptions for uniform stabilization. We shall divide these into three subsets: (a) structural assumptions on the nonlinearities g and $\mathcal{F}$; (b) 'abstract boundary → boundary' regularity and exact controllability of the corresponding linear controlled problem $z_t = Az + Bu$; in short, the pair $\{A, B\}$; (c) abstract unique continuation.

Structural assumptions on g, $\mathcal{F}$.

(A.1): (a) For the continuous function $g : U \to U$ with $g(0) = 0$, in (iii) above, there exists a (real-valued) continuous, concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$, strictly increasing, with $h(0) = 0$, such that

$$\|g(u)\|_U^2 + \|u\|_U^2 \le h(\langle g(u), u \rangle_U), \quad \forall \; u \in U. \tag{2.9}$$

In the case where g is a Nemytski operator of substitution, (2.9) is a *property*, not an assumption, Lasiecka and Triggiani, 2008, Section 3, Lemma 3.2.

(b) The locally Lipschitz operator $\mathcal{F}$, with $\mathcal{F}(0) = 0$, is Frechet differentiable $\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}) \to \mathcal{H}$ and satisfies the further property

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{\mathcal{H}} \le C_\rho \|v_1 - v_2\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})}, \tag{2.10}$$

for all $\|v_i\|_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})} \le \rho, \; i = 1, 2,$ for some arbitrarily small $\epsilon > 0$.

(c) There exists a potential function $\Pi : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to \mathbb{R}^+$, such that Π is differentiable, $\Pi(w) \geq 0$ for $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, and the following identity holds for $w, z \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$:

$$(\mathcal{F}(w), z)_{\mathcal{H}} = (\Pi'(w), z)_{\mathcal{H}}, \text{ where } (\Pi'(w), w)_{\mathcal{H}} \geq 0. \tag{2.11}$$

Here, $\Pi'(w)$ is the Frechet derivative at the point $w \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. Accordingly, we define the energy function of problem (2.1) [or (2.5)], by

$$E(t) \equiv \|w_t(t)\|^2_{\mathcal{H}} + \|\mathcal{A}^{\frac{1}{2}} w(t)\|^2_{\mathcal{H}} + 2\Pi(w(t)) \geq 0. \tag{2.12}$$

Global well-posedness. It is well known that under assumption (A.1)(a)-(b)-(c), *there exists a unique global finite energy solution*: that is, in (2.8), T can be an arbitrarily finite positive number.

This conclusion follows from the general theory of maximal monotone operators (Barbu, 1976), as applied to Eqn. (2.5): see Lasiecka (1989, Theorem 2.1). In fact, the evolution defined by Eqn. (2.5) can be represented as a locally Lipschitz perturbation of a maximal monotone operator. Surjectivity of $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}^{\frac{1}{2}}); \tilde{U})$ is used here to guarantee that the operator $A - Bg(B^* \cdot) + F \cdot)$ of (2.5) is maximal monotone (Lasiecka, 1989, Theorem 2.1). This gives *local (in time) existence.* Furthermore, *global existence* of solutions is then guaranteed by *a-priori* bounds that arise from the positivity of the energy function in (2.12) and from the energy identity posted in (3.1) below, in Section 3.

The boundary input-solution operator L of the linear model $z_t = Az + Bu$. Next, we introduce the boundary input-solution operator of the corresponding open-loop linear part of (2.5),

$$(Lu)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.13a}$$

$$: \text{ continuous } L_2(0, T; U) \to C([0, T]; H), \tag{2.13b}$$

whose regularity indicated in (2.13b) (needs to be assumed, but, in fact) follows as a consequence of the following 'abstract boundary $\to$ boundary' regularity assumption, which we now introduce.

The 'abstract boundary-boundary' regularity: The operator B^*L is bounded on $L_2(0, T; U)$. With reference to (2.13), we assume that

(A.2):

$$\text{The operator } B^*L \text{ is bounded, } L_2(0, T; U) \to L_2(0, T; U); \tag{2.14a}$$

$$(B^*Lu)(t) = \int_0^t B^* e^{A(t-\tau)} Bu(\tau) d\tau. \tag{2.14b}$$

Remark 2.1 The stated implication: (2.14) $\Rightarrow$ (2.13b) in the above (linear) setting was proved (in two ways) in Lasiecka and Triggiani (2004, 2008).

Remark 2.2 It is well known (Flandoli, Lasiecka and Triggiani, 1988; Lasiecka and Triggiani, 1991, 2000), that the regularity property (2.13b) is *equivalent* (via duality) to the following (regularity) inequality

$$B^* e^{A^* t} : \text{ continuous } \mathcal{H} \to L_2(0,T;U); \text{ or } \int_0^T \|B^* e^{A^* t} h\|_U^2 dt \le C_T \|h\|_{\mathcal{H}}^2, \; h \in \mathcal{H}. \tag{2.15}$$

The interior input-solution operator K of the linear model $z_t = Az + h$. We likewise introduce the interior input-solution operator of the corresponding open-loop model $z_t = Az + h$:

$$(Kh)(t) = \int_0^t e^{A(t-\tau)} h(\tau) d\tau \tag{2.16a}$$

$$: \text{ continuous } L_1(0,T;\mathcal{H}) \to C([0,T];\mathcal{H}), \tag{2.16b}$$

so that

$$(B^* Kh)(t) = \int_0^t B^* e^{A(t-\tau)} h(\tau) d\tau \tag{2.17a}$$

$$: \text{ continuous } L_2(0,T;\mathcal{H}) \to L_2([0,T];U). \tag{2.17b}$$

The regularity (2.17b) follows *a-fortiori* from (2.15). [Take the inner product on $L_2(0,T;U)$ of $B^* Kh$ with a function $\psi \in L_2(0,T;U)$; change the order of integration, and invoke (2.15).] Thus, *a-fortiori*, from assumption (A.2) = (2.14), which implies (2.13b) (by Remark 2.1), which in turn is equivalent to (2.15) (by Remark 2.2).

The nonlinear semigroup $S_F(t)$ describing (2.5); the variation of parameter formula. As noted before, under assumptions (A.1)(a),(b),(c), it follows, Lasiecka (1989 – Theorem 2.1, 1999), that the operator

$$A_F \equiv A - Bg(B^*) + F \tag{2.18}$$

is maximal monotone and generates a nonlinear semigroup of contractions on H, which we shall call $S_F(t)$. Accordingly, we may represent the solution of the first-order equation (2.5) by means of the following variation of parameter formula, via L in (2.13a) and K in (2.16a):

$$\begin{aligned} y(t; y_0) &= S_F(t) y_0 \\ &= e^{At} y_0 - \{Lg(B^* S_F(\cdot) y_0)\}(t) + \{KFS_F(\cdot) y_0\}(t), \end{aligned} \tag{2.19}$$

$y(t) = \{w_t), w_t(t)\}$, to be later invoked.

The exact controllability assumption of the open-loop linear problem $\dot{z} = Az + Bu$ (or $w_{tt} + \mathcal{A} w + \mathcal{B} u = 0$). We further assume that:

(A.3):

$$\begin{cases}\text{The linear open-loop dynamics } w_{tt} + \mathcal{A}w + \mathcal{B}u \equiv 0 \text{ is exactly} \\ \text{controllable in } \{w, w_t\} \text{ on the state space } H \text{ in (2.2), within} \\ \text{the class of } L_2(0,T;U)\text{-controls, for a sufficiently large } T > 0.\end{cases} \tag{2.20}$$

By the well-known duality between exact controllability and continuous observability (Russell, 1978), assumption (A.3) is equivalent to the following Continuous Observability Inequality (Flandoli, Lasiecka and Triggiani, 1988, Lasiecka and Triggiani, 1991a, 2003, 2008; Triggiani, 1988; Gulliver et al., 2003)

(A.3′):

$$\|x\|_H^2 \le C_T \int_0^T \|B^* e^{A^* t} x\|_U^2 dt, \quad \forall\, x \in H, \tag{2.20′}$$

which in turn, since $A = -A^*$ is skew-adjoint, is equivalent to the inequality (Lasiecka and Triggiani, 2003):

(A.3″):

$$\|x\|_H^2 \le C_T \int_0^T \|B^* e^{At} x\|_U^2 dt, \quad \forall\, x \in H. \tag{2.20″}$$

The fourth and last abstract assumption needed in the study of the asymptotic behavior of the evolution equation (2.1) [or (2.5)] is an 'abstract unique continuation property,' to be invoked in Section 3 to absorb lower-order terms.

Abstract unique continuation assumption

(A.4): Consider the evolution problem with interior feedback term

$$w_{tt} + \mathcal{A}w + \mathcal{F}(w) \equiv 0, \quad 0 < t \le T, \tag{2.21a}$$

whose time derivative version, under assumption (A.1)(b), is

$$(w_t)_{tt} + \mathcal{A}(w_t) + \mathcal{F}'(w)w_t \equiv 0, \quad 0 < t \le T, \tag{2.21b}$$

along with the following over-determined boundary condition (zero observation):

$$\mathcal{B}^* w_t \equiv 0, \qquad 0 \le t \le T. \tag{2.22}$$

We assume that, for T sufficiently large (at least as large as the time of exact controllability in (A.3)):

$$\text{(2.21a–b), (2.22)} \Rightarrow w_t \equiv 0, \text{ hence } w \equiv \text{const}, \; 0 < t \le T. \tag{2.23}$$

Implication of assumptions (A.4) and (A.1)(c): $w \equiv 0$. Assumption (A.4) yields $w_t \equiv 0$ for $0 < t \le T$. Accordingly, Eqn. (2.21a) becomes then $\mathcal{A}w + \mathcal{F}(w) \equiv 0$ and hence yields $(\mathcal{A}w, w)_{\mathcal{H}} + (\mathcal{F}(w), w)_{\mathcal{H}} \equiv 0$. By assumption (A.1)(c) = (2.11) and $\mathcal{A}$ positive, self-adjoint, we have $(\mathcal{F}(w), w)_{\mathcal{H}} = (\Pi'(w), w)_{\mathcal{H}} \ge 0$, and then $(\mathcal{A}w, w)_{\mathcal{H}} \equiv 0$, hence, $w \equiv 0$, $0 < t \le T$.

Main result. The main result of the present paper is given next. It generalizes the abstract result of Lasiecka and Triggiani (2008, Theorem 3.1) in the case of the purely 'boundary case,' to which it reduces when $\mathcal{F} \equiv 0$ (except that, in Lasiecka and Triggiani, 2008, uniformity of the decay is with respect to all initial conditions in H; here, instead, uniformity is with respect to all initial conditions within a given ball of H, centered at the origin). To state it, we need to introduce some additional quantities, following Lasiecka and Triggiani (2008) (ultimately, Lasiecka and Tataru, 1993), Lasiecka and Triggiani (2000, 2006), Lasiecka and Toundykov (2006). With reference to the concave function h noted in assumption (A.1)(a) = (2.9), we next introduce the following three functions, which are defined in succession (Lasiecka and Tataru, 1993): first the function $H(\cdot)$ on $\mathbb{R}^+$:

$$H(x) = \frac{C_{T,E(0)}}{T} h\left(\frac{x}{2T}\right)$$
$$: \text{positive for } x > 0, \text{ continuous, strictly increasing}, H(0) = 0; \quad (2.24)$$

where $C_{T,E(0)}$ is the constant depending on T and the initial energy $E(0)$, occurring in (3.37). [In PDE applications, one has $h\left(\frac{x}{2T \times \text{meas}(\Gamma)}\right)$ with a scaling factor inside the argument of $h(\cdot)$ which depends on the geometry of the domain Ω, $\partial\Omega = \Gamma$, on which the PDE is defined. We can absorb this scaling factor with h.] Next, the function $p(x)$, the inverse of $H(x)$:

$$p(x) = H^{-1}(x)$$
$$: \text{positive for } x > 0, \text{ continuous, strictly increasing}, p(0) = 0; \quad (2.25)$$

finally, the function

$$q(x) = x - (I + p)^{-1}(x) = p(I + p)^{-1}(x) = (I + p)^{-1} p(x)$$
$$: \text{positive for } x > 0, \text{ continuous, strictly increasing, } q(0) = 0. \quad (2.26)$$

Thus, H, p, q do depend on the initial energy $E(0)$. We can now state the main uniform stabilization result of the present paper. Subsequent sections will provide a few PDE illustrations.

Theorem 2.1 *With reference to the nonlinear closed-loop feedback problem (2.1) [or (2.5)], assume the standing hypotheses (i), (ii), (iii), (iv) posted below (2.2), as well as assumptions (A.1), (A.2), (A.3), (A.4). Then, the semigroup solution $S_F(t)$ with generator A_F in (2.18) describing the solution of the closed-loop, dissipative, nonlinear problem (2.1) [or (2.5)] (as guaranteed by Lasiecka, 1989, 2001), decays to zero on the space H as $T \to +\infty$, uniformly with respect to initial data within a given ball of H. More precisely, its decay rate is described by the following nonlinear ODE in the scalar function $s(t)$ (nonlinear contraction),*

$$\frac{d}{dt} s(t) + q(s(t)) = 0, \qquad s(0) = E(0), \quad (2.27)$$

where q is the function defined in (2.26), which depends on the initial energy $E(0)$. The decay rate is given by the solution $s(t) \searrow 0$:

$$E(t) \leq s(t)(E(0)) \searrow 0 \qquad \text{as } t \nearrow +\infty, \tag{2.28}$$

where the notation means that the solution $s(t)$ depends on the initial energy $E(0)$ (since q does in (2.26), via (2.24), (2.25)). Thus, uniformity of the decay is with respect to all initial conditions in H, which are within a given ball of H centered at the origin.

REMARK 2.3 It is shown in Lasiecka and Toundykov (2006), Lasiecka and Triggiani (2006) that the asymptotic rate of energy provided by Theorem 2.1 can be approximated by solving the ODE (2.27) with the function q replaced by the function h^{-1}; that is, by solving

$$\frac{d}{dt} s(t) + h^{-1}(s(t)) = 0, \quad s(0) = E(0). \tag{2.29}$$

3. Proof of Theorem 2.1

Step 1. The energy identity (Lasiecka and Triggiani, 2008, Eqn. 3.3).

LEMMA 3.1 *Assume (A.1a-b-c) (as well as (i), (ii), (iii), (iv) of Section 2). Then, the evolution equation (2.1) [or (2.5)], whose solution is given by the variation of parameter formula (2.19), obeys the following energy identity*

$$E(t) + 2\int_s^t \langle g(\mathcal{B}^* w_t(t)), \mathcal{B}^* w_t(t)\rangle_U dt = E(s), \;\; 0 \leq s \leq t, \tag{3.1}$$

where the integrand is non-negative by (2.3) or (2.9), so that the energy is decreasing: $E(t) \leq E(s)$, for $s \leq t$. Hence, the problem is dissipative.

Proof. The proof of this lemma follows from standard energy methods, as applied to Lipschitz perturbations of monotone problems (Barbu, 1976; Lasiecka, 1989). ∎

Step 2. The first part of the next lemma pertains to the continuous observability inequality.

LEMMA 3.2 *Assume (A.1), (A.2), (A.3) (and (i)–(iv) of Section 2).*

(a) Let $T > 0$ be sufficiently large as in assumption (A.3). Then, the following inequality holds true for the energy $E(\cdot)$ in (2.12)

$$E(T) + \int_0^T E(t)dt \leq TC_T C_{E(0)} \int_0^T \left[\|\mathcal{B}^* w_t(t)\|_U^2 + \|g(\mathcal{B}^* w_t(t))\|_U^2\right] dt + TC_T C_{E(0)} \int_0^T \|\mathcal{F}(w(t))\|_{\mathcal{H}}^2 dt \tag{3.2}$$

(b) Eqn. (3.2) leads to the following estimate:

$$\int_0^T E(t)dt \leq C_{E(0)}C_T h\left(\frac{1}{T}\int_0^T \langle g(\mathcal{B}^* w_t(t)), \mathcal{B}^* w_t(t)\rangle_U dt\right) + C_{E(0)}C_T \int_0^T \|w(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})} dt, \tag{3.3}$$

where $\epsilon > 0$ is the one occurring in (2.10).

Proof. We work with the first-order dynamics (2.5), where we recall that $y = \{w, w_t\}$ with reference to the original second-order problem (2.1); in particular, with its variation of parameter formula (2.19). We proceed as in Lasiecka and Triggiani (2006, Theorem 3.1; 2003, Section 3.1).

Part (a). Step (i). We apply the operator B^* on both sides of the solution formula (2.19) and obtain

$$B^* e^{At} y_0 = B^* S_F(t) y_0 + \{[B^* L] g(B^* S_F(\,\cdot\,) y_0)\}(t) + \{[B^* K] F S_F(\,\cdot\,) y_0\}(t). \tag{3.4}$$

We next invoke assumption (A.2) = (2.14): the operator B^*L is bounded in $L_2(0,T;U)$ and thus, *a-fortiori*, the operator B^*K is bounded from $L_2(0,T;\mathcal{H})$ to $L_2(0,T;U)$, by (2.17b). This way, identity (3.4) yields

$$\|B^* e^{At} y_0\|^2_{L_2(0,T;U)} \leq \text{const}_T \Big\{ \|B^* S_F(\,\cdot\,) y_0\|^2_{L_2(0,T;U)} + \|g(B^* S_F(\,\cdot\,) y_0)\|^2_{L_2(0,T;U)} + \|F S_F(\,\cdot\,) y_0\|^2_{L_2(0,T;\mathcal{H})} \Big\}. \tag{3.5}$$

Step (ii). We next invoke the exact controllability assumption (A.3) for $T > 0$ sufficiently large, of the pair $\{A, B\}$, which for $A = -A^*$ skew-adjoint is equivalent to its version (A.3″) = (2.20″). We thus obtain, by combining (A.3″)=(2.20″) with (3.5) (where T is as in assumption (A.3)):

$$\|y_0\|^2_H \leq C_T \int_0^T \|B^* e^{At} y_0\|^2_U dt \tag{3.6}$$

$$\text{(by (3.5))} \quad \leq \text{const}_T \Big\{ \int_0^T [\|B^* S_F(t) y_0\|^2_U + \|g(B^* S_F(t) y_0)\|^2_U] dt + \int_0^T \|F S_F(t) y_0\|^2_{\mathcal{H}} dt \Big\}. \tag{3.7}$$

Step (iii). We return to the energy $E(t)$ in (2.12) and evaluate it at $t = 0$. We thus estimate by assumption (A.1)(c) on $\Pi : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \Rightarrow \mathbb{R}^+$:

$$E(0) = \|w_1\|^2_{\mathcal{H}} + \|\mathcal{A}^{\frac{1}{2}} w_0\|^2_{\mathcal{H}} + 2\Pi(w(0)) \tag{3.8}$$

$$\leq \|w_1\|^2_{\mathcal{H}} + \|\mathcal{A}^{\frac{1}{2}} w_0\|^2_{\mathcal{H}} + C_{E(0)} \|w(0)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} \tag{3.9}$$

$$\leq C_{E(0)} \|[w_0, w_1]\|^2_H = C_{E(0)} \|y_0\|^2_H. \tag{3.10}$$

Next, by Lemma 3.1, $E(t)$ is decreasing, hence by (3.7), (3.10):

$$\int_0^T E(t)dt \leq TE(0) \leq TC_{E(0)} \|y_0\|^2_H \tag{3.11}$$

$$\leq TC_T C_{E(0)} \int_0^T \left[\|B^* S_F(t) y_0\|^2_U + \|g(B^* S_F(t) y_0)\|^2_U \| \right] dt$$
$$+ TC_T C_{E(0)} \int_0^T \|F S_F(t) y_0\|^2_{\mathcal{H}} dt. \tag{3.12}$$

Then, (3.12) and $E(T) \leq E(0)$ prove the desired inequality (3.2) via (2.6) and $y(t) = S_F(t) y_0 = \{w(t), w_t(t)\}$, $\mathcal{B}^* S_F(t) y_0 = \mathcal{B}^* w_t(t)$, $F S_F(t) y_0 = [0, \mathcal{F}(w(t))]$ [(3.12) is a first-order version, (3.2) is a second-order version].

Part (b). On the RHS of (3.12), we first recall assumption (A.1)(a) = (2.9) and next invoke the Jensen inequality (Lieb and Loss, 1996, p. 138). We thus obtain

$$\int_0^T E(t)dt \leq TC_T C_{E(0)} \int_0^T h\left(\langle g(B^* S_F(t) y_0), B^* S_F(t) y_0 \rangle_U\right) dt$$
$$+ TC_T C_{E(0)} \int_0^T \|F S_F(t) y_0)\|^2_{\mathcal{H}} dt$$
$$\text{(by Jensen in.)} \leq TC_T C_{E(0)} h\left(\frac{1}{T} \int_0^T \langle g(B^* S_F(t) y_0), B^* S_F(t) y_0 \rangle_U dt \right)$$
$$+ TC_T C_{E(0)} \int_0^T \|F S_F(t) y_0\|^2_{\mathcal{H}} dt. \tag{3.13}$$

Inequality (3.13) is the first-order version which yields the claimed second-order version inequality (3.3). Indeed, (3.13) yields estimate (3.3), as desired, by invoking again (2.6) and $y(t) = S_F(t) y_0 = \{w(t), w_t(t)\}$, $\mathcal{B}^* S_F(t) y_0 = \mathcal{B}^* w_t(t)$, $F S_F(t) y_0 = [0, \mathcal{F}(w(t))]$, as well as the following estimate obtained from assumption (A.1)(c) = (2.10) on $\mathcal{F}$:

$$\int_0^T \|F S_F(t) y_0\|^2_H dt = \int_0^T \|F y(t)\|^2_H dt$$
$$= \int_0^T \|\mathcal{F}(w)\|^2_{\mathcal{H}} dt \leq C_{T,E(0)} \int_0^T \|w(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})} dt. \tag{3.14}$$

Step 3. Absorption of the last $\ell.o.t.$ in estimate (3.3). In this step, we absorb the last term in inequality (3.3), which is due to the presence of the interior terms $\mathcal{F}$ in model (2.1) [or F in its version (2.5)]. [This step did not occur in Lasiecka and Triggiani, 2008, Section 3, which dealt only with abstract boundary damping.] This step is carried out by a usual compactness-uniqueness argument, albeit in a nonlinear setting, as in Lasiecka and Tataru (1993, Section 5).

LEMMA 3.3 *Consider the setting of Lemma 3.2(b), Eqn. (3.3). Assume, furthermore, assumption (A.4) = (2.23) for a time $T > 0$ sufficiently large. Then, there is a positive constant $C_{T,E(0)}$, depending on T and the initial energy $E(0)$, such that the following inequality holds true:*

$$\int_0^T \|w(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})} dt \leq C_{T,E(0)} \int_0^T [\|g(B^* S_F(t) y_0)\|_U^2 + \|B^* S_F(t) y_0\|_U^2] dt. \tag{3.15}$$

Proof (Orientation). As mentioned, the proof is based on a, by now, familiar compactness-uniqueness argument, albeit in the present nonlinear setting, as in the concrete case of Lasiecka and Tataru (1993, Section 5). Compactness results from the embedding $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})$, $\epsilon > 0$, since $\mathcal{A}^{-1}$ is assumed compact (see (i) below (2.2)), while uniqueness is guaranteed by both the controllability assumption (A.3) of the linear version and assumption (A.4) of uniqueness in the nonlinear setting. The size of T depends on the speed of propagation of the underlying dynamics. A sketch will follow, patterned after the concrete case in Lasiecka and Tataru (1993, Section 5).

Step 1. Let the initial condition $\{w_0, w_1\}$ satisfy $E(0) \leq M$. By contradiction, assume that inequality (3.15) is false. Then, there exists a sequence of solutions $\{w^n, w_t^n\}$ of problem (2.1), hence with regularity $\{w^n, w_t^n\} \in L_\infty(0, T; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H})$ as in (2.8), and satisfying the observability-type inequality (3.2) and also (3.3) [where $y(t) = S_F(t) y_0$], such that

$$\begin{cases} w^n \to \text{ some } w, \text{ weak}^* \text{ in } L_\infty(0, T; \mathcal{D}(\mathcal{A}^{\frac{1}{2}})); & (3.16a) \\ w_t^n \to w_t, \text{ weak}^* \text{ in } L_\infty(0, T; \mathcal{H}), & (3.16b) \end{cases}$$

as $n \to \infty$, while, with $y^n = \{w^n, w_t^n\}$:

$$\frac{\|w^n\|^2_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}))}}{\int_0^T [\|g(B^* y^n)\|_U^2 + \|B^* y^n\|_U^2] dt} \to \infty, \tag{3.17}$$

as $n \to \infty$. As the numerator of (3.17) is uniformly bounded in n by (3.16a), it follows via (2.6) for B^* that

$$\int_0^T [\|g(\mathcal{B}^* w_t^n)\|_U^2 + \|\mathcal{B}^* w_t^n\|_U^2] dt \to 0 \text{ for } n \to \infty. \tag{3.18}$$

Step 2. On the other hand, (3.16a) and (3.16b), along with the compactness of $\mathcal{A}^{-1}$ [assumed in (i) below (2.2)], and hence compactness of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})$ in the space variable yields (Simon, 1987; Aubin, 1963) the strong convergence

$$w^n \to w \text{ strongly in } L_\infty(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})). \tag{3.19}$$

Furthermore, by assumption (A.1)(b), in particular (2.10) on $\mathcal{F}$ (which is *a-fortiori* continuous $\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}) \to \mathcal{H}$, we also obtain as a consequence of (3.19), that

$$\mathcal{F}(w^n) \to \mathcal{F}(w) \text{ strongly in } L_\infty(0,T;\mathcal{H}). \tag{3.20}$$

Step 3. Let us first assume that $w \neq 0$ in (3.19). Passing with the weak*-limit on the original equation (2.1)

$$w^n_{tt} + \mathcal{A}w^n + \mathcal{B}g(\mathcal{B}^* w^n_t) + \mathcal{F}(w^n) = 0, \tag{3.21}$$

satisfied by $\{w^n, w^n_t\}$ and invoking (3.18) and (3.20) yields

$$w_{tt} + \mathcal{A}w + \mathcal{F}(w) = 0, \quad \mathcal{B}^* w_t \equiv 0,\ 0 < t \leq T. \tag{3.22}$$

The abstract uniqueness assumption (A.4), as in (2.23), plus the implication noted below (2.23) yields then that

$$w \equiv 0, \quad 0 < t \leq T, \tag{3.23}$$

a contradiction with the original assumption $w \not\equiv 0$ of the present case.

Step 4. Let us assume next that $w \equiv 0$ in (3.19). Normalize the solution w^n by setting

$$\hat{w}^n \equiv \frac{w^n}{c_n},\ c_n = \|w^n\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}))}, \tag{3.24}$$

so that

$$c_n \to 0 \text{ by (3.19) with } w \equiv 0; \text{ and } \|\hat{w}^n\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}))} \equiv 1. \tag{3.25}$$

Next, divide numerator and denominator of the blowing-up fraction in (3.17) by c_n^2, use the normalization condition in (3.25) and obtain via (2.6)

$$\frac{1}{c_n^2}\int_0^T [\|g(\mathcal{B}^* w^n_t)\|^2_U + \|\mathcal{B}^* w^n_t\|^2_U] dt \to 0 \text{ as } n \to 0. \tag{3.26}$$

As noted before, the solution $\{w^n, w^n_t\}$ satisfies the observability-type inequality (3.2) of Lemma 3.2(b). We divide this inequality (3.2) for $\{w^n, w^n_t\}$ by c_n^2, invoke (3.26), as well as

$$\frac{1}{c_n^2}\int_0^T \|\mathcal{F}(w^n(t))\|^2_{\mathcal{H}} dt \leq C_{T,E(0)} \int_0^T \|\hat{w}^n(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})} dt \equiv C_{T,E(0)} \tag{3.27}$$

—which is obtained from (3.14) and (3.24), (3.25): in this way we obtain

$$\frac{1}{c_n^2}\int_0^T [\|\mathcal{A}^{\frac{1}{2}} w^n(t)\|^2_{\mathcal{H}} + \|w^n_t(t)\|^2_{\mathcal{H}}]dt \le \text{const}_{T,E(0)}, \tag{3.28a}$$

dropping positive terms $E^n(T)/c_n^2$ and $\Pi(w^n(t)) \ge 0$, see (2.12) for $E^n(t)$. Then, by (3.24), we rewrite (3.28a) as

$$\|\hat{E}^n\|_{L_2(0,T;H)} \le C_{T,E(0)},\ 0 \le t \le T, \text{ where } \hat{E}^n(t) \equiv \|\hat{w}^n(t)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})} + \|\hat{w}^n_t(t)\|^2_{\mathcal{H}}. \tag{3.28b}$$

Then, by (3.28) we deduce that

$$\begin{cases} \hat{w}^n \to \hat{w} \ \text{ weakly in } L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}})); & (3.29a)\\ \hat{w}^n_t \to \hat{w}_t \ \text{ weakly in } L_2(0,T;\mathcal{H}); & (3.29b)\\ \hat{w}^n \to \hat{w} \ \text{ strongly in } L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})), \\ \qquad\quad \text{by Aubin (1963), Simon (1987),} & (3.29c) \end{cases}$$

again, since $\mathcal{A}^{-1}$ is compact. Dividing Eqn. (3.21) for w^n by c_n and passing to the weak limit yields by virtue of (3.29), (3.26),

$$\hat{w}_{tt} + \mathcal{A}\hat{w} + \lim_n \frac{\mathcal{F}(w^n)}{c_n} = 0 \text{ as well as } \mathcal{B}^*\hat{w}_t \equiv 0,\ 0 \le t \le T. \tag{3.30}$$

As a last step of our present argument, we shall show that the differentiability of $\mathcal{F} : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to \mathcal{H}$, due to assumption (A.1)(b), will imply

$$\lim_n \frac{\mathcal{F}(w^n)}{c_n} = \mathcal{F}'(0)\hat{w},\ \text{weakly in } L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})). \tag{3.31}$$

Proof of (3.31). The differentiability of $\mathcal{F}$ in (A.1)(b), Eqn. (2.10), gives, recalling c_n from (3.24)

$$\mathcal{F}(w^n) - \mathcal{F}'(0)w^n = o\left(\|w^n\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}))}\right) = o(c_n), \tag{3.32}$$

where, by definition of o we get (3.31) from (3.32) via (3.24) on $\hat{w}^n$ and (3.29a).

By use of (3.31) in (3.30), we then obtain

$$\hat{w}_{tt} + \mathcal{A}\hat{w} + \mathcal{F}'(0)\hat{w} = 0, \quad \mathcal{B}^*\hat{w}_t \equiv 0,\ 0 < t \le T. \tag{3.33}$$

Invoking the uniqueness assumption (A.4) on (3.33) then yields $\hat{w} \equiv 0$. Indeed, Eqn. (3.33) coincides with Eqn. (2.21b) in the solution now being $\hat{w}$ and with respect to the reference point now being 0. But $\hat{w} \equiv 0$ is a contradiction with

$$1 \equiv \lim_{n\to\infty} \|\hat{w}^n\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}))} \equiv \|\hat{w}\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})}, \tag{3.34}$$

which follows by (3.25) and (3.29c).

The proof of Lemma 3.3 is complete. ∎

Step 4: Final Estimate. As a corollary of estimate (3.15) being used on the RHS of estimate (3.3), we obtain via assumption (A.1)(a) = (2.9) and Jensen inequality as in (3.13):

PROPOSITION 1 *Assume (A.1)–(A.4) (as well as (i)–(iv) of Section 2). Let $T > 0$ be sufficiently large, as required by assumption (A.4) = (2.23) and (A.3). Then, the following estimate holds true for solutions of problem (2.1) [or (2.5)]:*

$$\int_0^T E(t)dt \le C_{E(0)} C_T h \left(\frac{1}{T} \int_0^T \langle g(B^* S_F(t) y_0, B^* S_F(t) y_0 \rangle_U dt \right). \quad (3.35)$$

Step 5. Completion of the proof of Theorem 2.1. In view of the energy identity (3.1) in Lemma 3.1, which implies that $E(t)$ is decreasing, we obtain by (3.35)

$$TE(T) \le \int_0^T E(t)dt \le C_{T,E(0)} h \left(\frac{1}{2T} (E(0) - E(T)) \right). \quad (3.36)$$

Dividing (3.36) by T and recalling the definition of the function $H(\,\cdot\,)$ in (2.24) yields

$$E(T) \le \frac{C_{T,E(0)}}{T} h \left(\frac{E(0) - E(T)}{2T} \right) \equiv H(E(0) - E(T)). \quad (3.37)$$

Since $H(\,\cdot\,)$ is strictly increasing, see (2.24), we apply $H^{-1}(\,\cdot\,)$ across (3.37), recall $p = H^{-1}$ by (2.25), and obtain

$$H^{-1}(E(T)) \le E(0) - E(T); \text{ or } E(T) + p(E(T)) \le E(0). \quad (3.38)$$

The above inequality now leads (Lasiecka and Tataru, 1993, Lemma 5.1) to the desired conclusion of Theorem 2.1, in particular, to Eqn. (2.27), with q as in (2.26). ∎

4. Wave equation with nonlinear dissipation both in the Dirichlet B.C. and in the interior

4.1. Model and results

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$, with sufficiently smooth boundary Γ. On it, we consider the following wave equation problem, with both boundary dissipation in the Dirichlet B.C. and in the interior:

$$w_{tt} = \Delta w - \mathcal{F}(w) \text{ in } (0, \infty] \times \Omega \equiv Q; \quad (4.1.1a)$$

$$w(0, \cdot) = w_0, \; w_t(0, \cdot) = w_1 \text{ in } \Omega; \quad (4.1.1b)$$

$$w|_\Sigma \equiv g \left(\left[\frac{\partial (\mathcal{A}^{-1} w_t)}{\partial \nu} \right]_\Gamma \right) \text{ in } (0, \infty] \times \Gamma \equiv \Sigma. \quad (4.1.1c)$$

Here $\nu =$ outward unit normal vector defined on Γ. With reference to problem (4.1.1), we introduce two positive self-adjoint operators and a corresponding functional setting, in line with the notation of Sections 2 and 3. First, we define the

(i) positive, self-adjoint operator $\mathcal{A}_0$ in $L_2(\Omega)$:

$$\mathcal{A}_0 h = -\Delta h, \; L_2(\Omega) \supset \mathcal{D}(\mathcal{A}_0) = H^2(\Omega) \cap H_0^1(\Omega) \to L_2(\Omega); \tag{4.1.2a}$$

$$\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) = H_0^1(\Omega), \quad [\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})]' \equiv H^{-1}(\Omega) \tag{4.1.2b}$$

(duality with respect to $L_2(\Omega)$ as a pivot space). Next,

(ii) the positive, self-adjoint operator $\mathcal{A}$ on $H^{-1}(\Omega)$:

$$\mathcal{A} \equiv \text{realization of } \mathcal{A}_0 \text{ considered on the underlying space } H^{-1}(\Omega); \tag{4.1.3a}$$

$$: \; \mathcal{H} \equiv H^{-1}(\Omega) \supset \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \equiv H_0^1(\Omega) \to H^{-1}(\Omega); \tag{4.1.3b}$$

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv L_2(\Omega), \; H \equiv L_2(\Omega) \times H^{-1}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H}; \; U = L_2(\Gamma). \tag{4.1.4}$$

Linear case: $F \equiv 0$**,** $g(u) = u$. This boundary feedback system was first introduced in Lasiecka and Triggiani (1987), where uniform stabilization was shown in the state space $L_2(\Omega) \times H^{-1}(\Omega)$ of optimal regularity with $w|_\Sigma = -\left.\frac{\partial(\mathcal{A}^{-1} w_t)}{\partial \nu}\right|_\Sigma \in L_2(0, \infty; L_2(\Gamma))$ (Lasiecka and Triggiani, 1981, 1983; Lasiecka, Lions and Triggiani, 1986), at least when Ω is convex (or the set-theoretic difference of two convex sets). Via Russell (1978), this result gave for the first time exact controllability of the corresponding open loop problem on the state space $L_2(\Omega) \times H^{-1}(\Omega)$ of optimal regularity with controls $w|_\Sigma = u \in L_2(0, T; L_2(\Gamma))$. Later, geometric conditions were much relaxed, Lasiecka and Triggiani (1992).

Nonlinear case: $\mathcal{F} \equiv 0$, g as in assumptions (iii), (A.1a). This case was explicitly studied in Lasiecka and Triggiani (2008, Section 4), where uniform stabilization of problem (4.1.1a–c) with $\mathcal{F} \equiv 0$ was obtained with optimal rates of energy decay and with constant independent on the norm of the initial condition, by following the approach of the present paper, which is based on assumptions (A.2) and (A.3). As noted above, assumption (A.3) had been known to hold true since Lasiecka and Triggiani (1987), see also Triggiani (1988), Lions (1988), Ho (1986). Verification of assumption (A.2) is a delicate issue and was established in Lasiecka and Triggiani (2004, 2006), by combining trace regularity results of Lasiecka, Lions and Triggiani (1986) with a micro-local analysis argument.

Nonlinear case: $\mathcal{F}$, g to be specified here as to fall into the abstract setting of Sections 2 and 3. Before doing this, we recall the abstract setting for the control operator from Triggiani (1978), Lasiecka and Triggiani (1983, 1987, 1991a, 2000), and for problem (4.1). We set:

$$v = Du \iff \{\Delta v = 0 \text{ in } \Omega; \; v|_\Gamma = u\}$$

$$D = \text{harmonic extension of the Dirichlet datum}; \tag{4.1.5}$$

$$\mathcal{B}g = \mathcal{A}Dg \text{ in } [\mathcal{D}(\mathcal{A})]'; \quad (\mathcal{B}g, v)_{\mathcal{H}} = \langle g, \mathcal{B}^* v\rangle_U; \tag{4.1.6a}$$

$$\mathcal{B}^*\varphi = -\left.\frac{\partial \mathcal{A}^{-1}\varphi}{\partial \nu}\right|_{\Gamma}, \text{ first on } \mathcal{D}(\mathcal{A}) \text{ and next extended } L_2(\Omega) \to L_2(\Gamma). \tag{4.1.6b}$$

$\tilde{U} = H^{\frac{1}{2}}(\Gamma)$, so that $\mathcal{B}^* = -\frac{\partial}{\partial\nu}\mathcal{A}^{-1}$ is surjective

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv L_2(\Omega) \text{ onto } \tilde{U} = H^{\frac{1}{2}}(\Gamma), \tag{4.1.6c}$$

by trace theory, as required by the preliminary assumption (ii) just below (2.2). In view of (4.1.3), (4.1.5), (4.1.6), the feedback problem (4.1.1a–c) can be rewritten abstractly as

$$w_{tt} + \mathcal{A}w + \mathcal{B}g(\mathcal{B}^* w_t) + \mathcal{F}(w) = 0 \text{ in } [\mathcal{D}(\mathcal{A})]', \tag{4.1.7}$$

$\{w_0, w_1\} \in H$, as desired, in line with (2.1).

The nonlinear terms f and g. In what follows, we shall consider the term $\mathcal{F}(w)$ as defined by

$$\mathcal{F}(w) = f(\mathcal{A}^{-1}w), \tag{4.1.8}$$

where f is a Nemytski operator of substitution $(f(v))(x) = f(v(x))$, generated by a scalar-valued function $f \in C^1(\mathbb{R})$. The following hypotheses are placed on f and g:

(H_f): The scalar function $f(s)$ is differentiable,

$$f(0) = 0 \text{ and } f(s)s \geq 0, \text{ so that } \hat{f}(x) \geq 0, \ \forall\, x \in \mathbb{R}, \ \hat{f}'(x) = f(s). \tag{4.1.9}$$

[In fact, taking *w.l.o.g.* $\hat{f}(0) = 0$, we have $\hat{f}(x) = \int_0^x f(s)ds \geq 0$ with $x > 0$, so that $f(s) \geq 0$, $ds \geq 0$; as well as $x < 0$, so that $f(s) \leq 0$, $ds \leq 0$.]

(H_g): The operator $g : U \to U$, $U = L_2(\Gamma)$ is a Nemytski operator of substitution $(g(u))(x) = g(u(x))$ defined by the scalar function $g(s)$ which is continuous, monotone increasing, $g(0) = 0$, and satisfies the following growth condition at infinity:

$$ms^2 \leq g(s)s \leq Ms^2, \quad |s| \geq 1, \ 0 < m < M. \tag{4.1.10}$$

We then recall from Lasiecka and Tataru (1993) (and then Lasiecka and Triggiani, 2006; Lasiecka and Toundykov, 2006) that it is always possible to select a continuous, concave, strictly increasing function $h_0 : \mathbb{R}^+ \to \mathbb{R}^+$, with $h_0(0) = 0$, such that

$$s^2 + g(s)s \leq h_0(sg(s)), \quad |s| \leq 1. \tag{4.1.11}$$

The main result of the present section on problem (4.1.1a–c) is the following:

Theorem 4.1.1 *Assume hypotheses* $(H_f) = (4.1.9)$ *and* $(H_g) = (4.1.10)$. *Then, problem (4.1.1a–c) is well posed in the state space* $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{H}$ $(\equiv L_2(\Omega) \times H^{-1}(\Omega))$ *in the sense that it generates a nonlinear semigroup there. Moreover, the energy of the solution* $E_w(t) \equiv \|\{w(t), w_t(t)\}\|_H^2$ *decays uniformly to zero with rate specified by Theorem 2.1, if, in addition, the uniqueness property (4.2.10), (4.2.11)* $\Rightarrow$ *(4.2.12) holds true. Here the function* h *(in (2.24) leading to the function* q *in (2.26)) is now replaced by the function* h_0 *in (4.1.11), and uniformity of the decay rate is with respect to all initial conditions contained in a given ball of the space* H *in (2.2).*

Remark 4.1 Theorem 4.1.1 is a specialization of the abstract stabilization Theorem 2.1, as applied to the wave/Dirichlet problem (4.1.1a–c). As is the case in the energy-based approach of Lasiecka and Tataru (1993)—which is incorporated in the proof of Theorem 2.1—the decay rates of Theorem 4.1.1 depend entirely on the behavior of the nonlinear function $g(s)$ near the origin (see also the examples of Lasiecka and Tataru, 1993; Lasiecka and Triggiani, 2006; Lasiecka and Toundykov, 2006; Triggiani, 2007). Indeed, as noted in Remark 2.3, the decay rate is driven by the equation

$$\frac{d}{dt}s(t) + h_0^{-1}(s(t)) = 0, \quad s(0) = E(0), \tag{4.1.12}$$

where h_0 is the function in (4.1.11) (more specifically, the dominant behavior of h, for small frequencies, depends on h_0). We illustrate this algorithm with several examples (all in line with the aforementioned references).

Case 1: Linear growth of $g(\cdot)$ *near the origin.* Let $ms \le g(s)s \le Ms^2$ near the origin, $0 < m < M$. Then from (4.1.11) we can take $h_0(s) \sim as$ near the origin, for some nonzero constant a. In this case, as expected, the decay of $s(t)$ provided by Eqn. (2.27) or (4.1.12) is *exponential.*

Case 2: Polynomial growth of $g(s)$ *near the origin.* Let now $g(s) \sim s^p$, $p > 1$, near the origin. Then, from (4.1.11), we can take $h(s) = h_0(s) = s^{\frac{2}{p}}$, and the corresponding ODE (4.1.12) to be solved, takes the form

$$s_t + s^{\frac{p}{2}} = 0, \quad s(0) = E(0). \tag{4.1.13}$$

The above leads to the algebraic decay rates $\frac{1}{t^{\frac{2}{p-2}}}$.

Case 3: Exponential growth. We consider $g(s) = e^{-\frac{1}{s^2}}s$. In this case we can take $h(s) = \frac{1}{\ln s}$, so that $h^{-1}(s) = e^{-\frac{1}{s}}$ and the resulting ODE (4.1.12) to be solved becomes

$$s_t + e^{-\frac{1}{s}} = 0. \tag{4.1.14}$$

The decay rates are then logarithmic.

Case 4: We consider the sublinear growth $g(s) = s^{\frac{1}{3}}$. Then $h(s) = \sqrt{s}$, $s > 0$. The ODE (4.1.12) to be solved becomes $s_t + s^2 = 0$, whose solution has asymptotic behavior $s(t) \sim \frac{1}{t}$.

4.2. Proof of Theorem 4.1.1

Theorem 4.1.1 will follow as a specialization of the abstract Theorem 2.1 to the actual setting in Section 4.1, once we verify the four abstract assumptions (A.1) through (A.4) in the present case.

Verification of assumption (A.3). In the present case, assumption (A.3) is the well-known result of exact controllability of the wave equation with Dirichlet control in the state space $H \equiv L_2(\Omega) \times H^{-1}(\Omega)$, which was first established in Lasiecka and Triggiani (1987) (as a consequence of uniform stabilization via Russell, 1978), under some geometric conditions. Direct proofs were later given in Triggiani (1988), Ho (1986), Lions (1988), with more general results given in Bardos, Lebeau and Rauch (1992), Lasiecka, Triggiani and Yao (1999), Lasiecka, Triggiani and Zhang (2000), Triggiani and Yao (2002). All these works actually established—in various degrees of generality—the equivalent Continuous Observability Inequality (A.3′) = (2.20′), which in the present case reads: There exists a constant $C_T > 0$, such that

$$\|\{y_0, y_1\}\|^2_{H^1_0(\Omega)\times L_2(\Omega)} \le C_T \int_\Sigma \left|\frac{\partial y}{\partial \nu}\right|^2 d\Sigma, \tag{4.2.0}$$

for the homogeneous dual problem

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q; & (4.2.1a)\\ y(T) = y_0,\ y_t(T) = y_1 & \text{in } \Omega; & (4.2.1b)\\ y|_\Sigma \equiv 0, & & (4.2.1c) \end{cases}$$

for $T > T_0 =$ sufficiently large > 0. Thus, assumption (A.3) holds true in the present case of model (4.1.1).

Verification of assumption (A.2). In the present case of model (4.1.1), it was shown in Lasiecka and Triggiani (2004, 2008) that assumption (A.2) is equivalent to the following inequality:

$$\int_\Sigma \left|\frac{\partial \mathcal{A}^{-1} v_t}{\partial \nu}\right|^2 d\Sigma \le C_T \int_\Sigma |u|^2 d\Sigma, \tag{4.2.2}$$

where v solves the following problem

$$\begin{cases} v_{tt} = \Delta v & \text{in } (0,T]\times\Omega = Q; & (4.2.3a)\\ v(0,\,\cdot\,) = 0,\ v_t(0,\,\cdot\,) = 0 & \text{in } \Omega; & (4.2.3b)\\ v|_\Sigma = u \in L_2(\Sigma) & \text{in } (0,T]\times\Gamma \equiv \Sigma. & (4.2.3c) \end{cases}$$

[This regularity was first stated in Ammari, 2002, but the key part of the proof sketched in this reference is referred to the author's Ph.D. thesis, which is inaccessible to us. The complete proof in Lasiecka and Triggiani, 2004, 2008, is very different.]

Regularity (4.2.2) for the v-problem (4.2.3a–c) is very delicate and challenging for dim $\Omega \geq 2$ (while it is simple for dim $\Omega = 1$, Lasiecka and Triggiani, 2003). It was proved in Lasiecka and Triggiani (2004) (see also Lasiecka and Triggiani, 2008, Section 4) by combining trace regularity results from Lasiecka, Lions and Triggiani (1986) with a micro-local analysis argument. (Lasiecka and Triggiani, 2003, provided an incorrect counter-example in the half-space, due to a spurious appearance of the symbol 'Real part' in Lasiecka and Triggiani, 2003, Eqn. (5.2.18). Once the symbol Re is omitted, as it should have been, the same analysis and computation provide a positive conclusion for the half-space version of problems (4.2.2), (4.2.3) in dimension greater or equal to two.)

Thus, in conclusion, assumption (A.2) holds true in the present case of model (4.1.1).

Verification of assumption (A.1)(a) for g. It is established in Lasiecka and Triggiani (2008, Section 3, Eqns. (3.24)–(3.34)) that a Nemytski operator $g : U \to U$ (operator of substitution $(g(u))(x) = g(u(x))$) satisfies assumption (A.1)(a), if the scalar function g satisfies assumption (H_g), as presently postulated. Indeed, we review the analysis.

For small s, monotonicity of g implies the existence of a concave function h_0 such that (Lasiecka and Tataru, 1993; Lasiecka and Triggiani, 2006; Lasiecka and Toundykov, 2006):

$$s^2 + g^2(s) \leq h_0(sg(s)), \qquad |s| \leq 1, \tag{4.2.4}$$

with h_0, moreover, continuous, increasing and $h_0(0) = 0$. For large s, the linear bounds (4.1.10) imposed on g give

$$s^2 + g^2(s) \leq \frac{g(s)s}{m} + Msg(s) = (m + M)sg(s), \quad |s| \geq 1. \tag{4.2.5}$$

This gives

$$h(s) = h_0(s) + (m + M)(s) \tag{4.2.6}$$

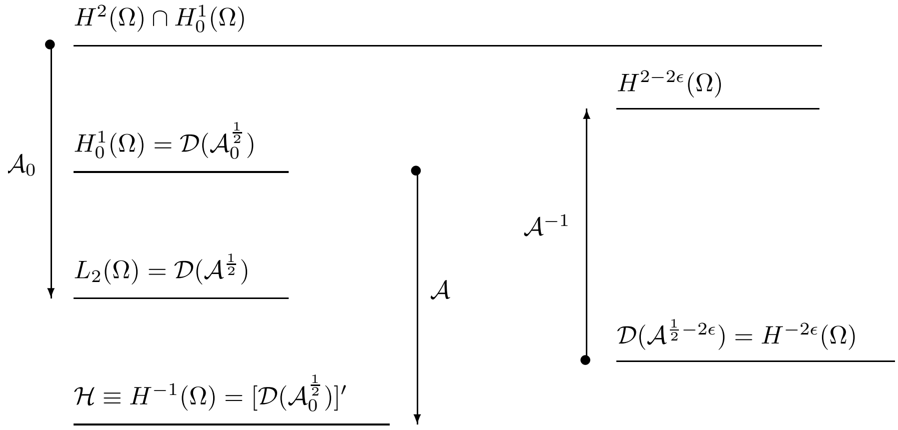
as the final choice for the concave function h which satisfies all the required conditions.

Verification of assumption (A.1)(b),(c) for $\mathcal{F}$. The operator $\mathcal{F}$ is defined in (4.1.8) with f satisfying assumption $(\mathrm{H}_f) = (4.1.9)$.

To verify (A.1)(b) = (2.10), we note that differentiability of $\mathcal{F}$ in (A.1)(b) follows from the postulated differentiability of $f(\,\cdot\,)$, as well as from Sobolev embedding $H^{2-2\epsilon}(\Omega) \subset C(\Omega)$ for dim $\Omega = 1, 2, 3$, and finally from the property that

$$\mathcal{A}^{-1} : \text{ bounded } H^{-2\epsilon}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon}) \to H^{2-2\epsilon}(\Omega), \tag{4.2.7}$$

with $\mathcal{A}$ defined in (4.1.3). [The claimed embedding follows from $W^{s,p} \subset W^{t,q}$, $0 \leq t \leq s < \infty$, $1 < p \leq q < \infty$ for $s - \frac{n}{p} \geq t - \frac{n}{q}$, with $s = 2 - 2t$, $p = 2$; $t = 0$, $q \to \infty$, which is valid for $n < 2(2 - 2\epsilon)$, or $n = 1, 2, 3$)



Next, to verify (A.1)(c) = (2.11), we recall (4.1.8) $\mathcal{F} = f(\mathcal{A}^{-1}w)$, define $\hat{f}' = f$, recall $\mathcal{H} = H^{-1}(\Omega) = [\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})]'$ from (4.1.3b), (4.1.2b), and compute

$$(\mathcal{F}(w), z)_{\mathcal{H}} = (f(\mathcal{A}^{-1}w), z)_{\mathcal{H}} = (\hat{f}'(\mathcal{A}^{-1}w), z)_{\mathcal{H}} = (\hat{f}'(\mathcal{A}^{-1}w), \mathcal{A}^{-1}z)_{L_2(\Omega)}; \tag{4.2.8}$$

$$(\mathcal{F}(w), w)_{\mathcal{H}} = (\hat{f}'(\mathcal{A}^{-1}w), w)_{\mathcal{H}} = (f(\mathcal{A}^{-1}w), \mathcal{A}^{-1}w)_{L_2(\Omega)} = (\Pi'(w), w)_{\mathcal{H}} \geq 0, \tag{4.2.9}$$

since $f(s)s \geq 0$, where $\Pi(w) = \int_\Omega \hat{f}(\mathcal{A}^{-1}w)d\Omega \geq 0$, as required, by (4.1.9) (i.e., $\hat{f}(x) \geq 0$, $\forall\, x \in \mathbb{R}$).

Verification of assumption (A.4). Property (2.23) in the setting of (2.21a–b), (2.22) can be restated in the present case as the following unique continuation principle: Consider the dynamics (corresponding to (2.21b) in our present case)

$$y_{tt} - \Delta y + f'(\mathcal{A}^{-1}w)\mathcal{A}^{-1}y = 0, \tag{4.2.10}$$

obtained from differentiating (4.1.1a) in t and setting $w_t = y$, along with the over-determined B.C.,

$$y = \frac{\partial}{\partial \nu}\mathcal{A}^{-1}y \equiv 0 \text{ on } \Sigma = (0, T] \times \Gamma. \tag{4.2.11}$$

Here, the first B.C. is incorporated in the operator $\mathcal{A}$ in (2.21a) (via (4.1.2a)), while the second B.C. corresponds to (2.22) with $\mathcal{B}^*$ given by (4.1.6b) (and $y = w_t$). We then seek to conclude that, in fact, (4.2.10), (4.2.11) imply

$$y \equiv 0 \text{ on } 0 < t \leq T \tag{4.2.12}$$

(i.e., (2.23)). Due to the non-local character of the nonlinear term in (4.2.10), one may need to assume additional conditions, such as small internal damping, to obtain $y \equiv 0$ in (4.2.12).

Alternatively, we may convert the y-problem (4.2.10), (4.2.11) into the following v-problem

$$\begin{cases} -\Delta v_{tt} + \Delta^2 v + f'(\mathcal{A}^{-1}w)v \equiv 0 \quad \text{in } Q; & \text{(4.2.13a)} \\ v|_\Gamma = \left.\dfrac{\partial v}{\partial \nu}\right|_\Gamma = \Delta v|_\Gamma \equiv 0 \quad \text{in } \Sigma, & \text{(4.2.13b)} \end{cases}$$

obtained by setting $v = \mathcal{A}^{-1}y$ or $\mathcal{A}v = -\Delta v$ with three homogeneous B.C. Whether the v-problem implies $v \equiv 0$, perhaps under some geometrical (star-shaped) conditions, remains to be studied.

5. Kirchhoff plate with nonlinear dissipation both in the 'moment' B.C. and in the interior

5.1. Model and results

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n = 1, 2$, with sufficiently smooth boundary Γ for $n = 2$. On it, we consider the following Kirchhoff problem with constant $\gamma > 0$, with both boundary dissipation in the 'moment' B.C. and in the interior

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + f(w) = 0 \quad \text{in } (0,\infty)\times\Omega \equiv Q; & \text{(5.1.1a)} \\ w(0,\cdot\,) = w_0,\ w_t(0,\cdot\,) = w_1 \quad \text{in } \Omega; & \text{(5.1.1b)} \\ w|_\Sigma \equiv 0,\ \Delta w|_\Sigma = -g\left(\left.\dfrac{\partial w_t}{\partial \nu}\right|_\Gamma\right) \quad \text{on } (0,\infty)\times\Gamma \equiv \Sigma. & \text{(5.1.1c)} \end{cases}$$

Let $\mathcal{A}_0$, D be the operators defined in (4.1.2a), (4.1.5), respectively. Then, the abstract model of problem (5.1.1) is (Lasiecka and Triggiani, 1991b; Horn and Lasiecka, 1994):

$$Mw_t + \mathcal{A}_0^2 w + \mathcal{A}_0 Dg\left(\left.\frac{\partial w_t}{\partial \nu}\right|_\Gamma\right) + f(w) = 0, \tag{5.1.2}$$

where the 'mass' operator $M = M_\gamma = (I + \gamma\mathcal{A}_0)$, $\gamma > 0$, is positive, self-adjoint on $L_2(\Omega)$. Applying M^{-1} to (5.1.2) yields the final model

$$w_{tt} + M^{-1}\mathcal{A}_0^2 w + M^{-1}\mathcal{A}_0 Dg\left(\left.\frac{\partial w_t}{\partial \nu}\right|_\Gamma\right) + M^{-1}f(w) = 0 \tag{5.1.3}$$

of the same form as (2.1), where (Lasiecka and Triggiani, 1991b, 2000, 2002):

(a) $\mathcal{A} \equiv M^{-1}\mathcal{A}_0^2$ is a positive, self-adjoint operator on the space

$$\mathcal{H} = \mathcal{D}(M^{\frac{1}{2}}) = \mathcal{D}((I+\gamma\mathcal{A}_0)^{\frac{1}{2}}) \equiv H_0^1(\Omega), \ (f_1, f_2)_{\mathcal{H}} = ((I+\gamma\mathcal{A}_0)f_1, f_2)_{L_2(\Omega)}. \tag{5.1.4}$$

(b) The state space H in (2.2) for the pair $\{w, w_t\}$ is, as $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_0)$:

$$H = \mathcal{D}(\mathcal{A}_0) \times \mathcal{D}((I+\gamma\mathcal{A}_0)^{\frac{1}{2}}) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega). \tag{5.1.5}$$

(c) The operator $\mathcal{B} \equiv M^{-1}\mathcal{A}_0 D$ has $\mathcal{H}$-adjoint $\mathcal{B}^*$ given by

$$(\mathcal{B}g, x)_{\mathcal{H}} = ((I+\gamma\mathcal{A}_0)(I+\gamma\mathcal{A}_0)^{-1}\mathcal{A}_0 Dg, x)_{L_2(\Omega)} = (g, D^*\mathcal{A}_0 x)_{L_2(\Gamma)}; \tag{5.1.6}$$

$$\mathcal{B}^* x = D^*\mathcal{A}_0 x = -\frac{\partial x}{\partial \nu} \quad \text{(Lasiecka and Triggiani, 1983, 1991a, 2000).} \tag{5.1.7a}$$

$\tilde{U} = H^{\frac{1}{2}}(\Gamma)$, so that $\mathcal{B}^* = -\frac{\partial}{\partial\nu}$ is surjective

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \mathcal{D}(\mathcal{A}_0) = H^2(\Omega) \cap H_0^1(\Omega) \text{ onto } \tilde{U} = H^{\frac{1}{2}}(\Gamma), \tag{5.1.7b}$$

by trace theory, as required by the preliminary assumption (i) just below (2.2). Thus, the operator B and its H-adjoint B^* are

$$B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{B}x_2 \end{bmatrix}, \ B^*\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathcal{B}^* x_2 = -\frac{\partial x_2}{\partial \nu}, \tag{5.1.8}$$

hence

$$B^*\begin{bmatrix} w \\ w_t \end{bmatrix} = -\frac{\partial w_t}{\partial \nu}. \tag{5.1.9}$$

(d) The operator $\mathcal{F}$ in model (2.1) is given by $\mathcal{F}(w) = M^{-1}f(w)$ according to (5.1.3).

This way, model (5.1.3) is cast in the abstract setting (2.1). The natural energy associated with problem (5.1.1) is

$$E(t) = \|\Delta w(t)\|^2_{L_2(\Omega)} + \|M^{\frac{1}{2}} w_t(t)\|^2_{L_2(\Omega)} + \int_\Omega \hat{f}(w(t))d\Omega, \ \hat{f}' = f. \tag{5.1.10}$$

The main result of the present section on problem (5.1.1) is the following.

THEOREM 5.1.1 *Assume hypotheses* $(H_f) = (4.1.9)$ *and* $(H_g) = (4.1.10)$ *on f and g. Then, problem (5.1.1a–c) is well posed on the state space H defined by (5.1.5) in the sense that it generates a nonlinear semigroup here. Moreover, the energy E(t) of the system given by (5.1.10) decays uniformly to zero with rate specified by Theorem 2.1, provided that, in addition, the uniqueness property (5.1.13), (5.1.14)* $\Rightarrow$ *(5.1.15) holds true.*

5.2. Proof of Theorem 5.1.1

Theorem 5.1.1 will follow as a specialization of the abstract Theorem 2.1 to the actual setting of Section 5.1, once we verify the four abstract assumptions (A.1) through (A.4) in the present case.

Verification of assumption (A.3). In the present case, assumption (A.3) is the well-known result of exact controllability of the Kirchhoff equation with just one boundary control in the 'moment' B.C. $\Delta w|_\Sigma = g$ (Horn and Lasiecka, 1994; Lasiecka and Triggiani, 1991b). Thus, assumption (A.3) holds true in our case.

Verification of assumption (A.2). That the corresponding map B^*L is continuous on $L_2(0,T;L_2(\Gamma))$ was shown in Lasiecka and Triggiani (2004) on the basis of validity of the same assumption (A.2) of the wave equation case of Section 4 (see (4.2.2), (4.2.3)). Hence, by a partially micro-local analysis argument, the present Kirchhoff equation case can be reduced to the wave equation case. Reference Lasiecka and Triggiani (2004) corrects an erroneous conclusion of Lasiecka and Triggiani (2003), as noted below (4.2.3c). Thus, assumption (A.2) holds true in the present case.

Verification of assumption (A.1)(a) on g. This was noted in Section 4.2; from (4.2.4) to (4.2.7).

Verification of assumption (A.1)(b), (c) for $\mathcal{F}$. Let $\mathcal{F}(w) \equiv M^{-1}f(w)$ as in (d) of Section 5.1, with f satisfying assumption $(\mathrm{H}_f) = (4.1.9)$. Then, we compute recalling $\mathcal{H}$ in (5.1.4):

$$(\mathcal{F}(w), z)_{\mathcal{H}} \equiv (MM^{-1}f(w), z)_{L_2(\Omega)} = (f(w), z)_{L_2(\Omega)}; \tag{5.2.1}$$

$$\begin{aligned}(\mathcal{F}(w), w)_{\mathcal{H}} &= (f(w), w)_{L_2(\Omega)} = (\hat{f}'(w), w)_{L_2(\Omega)} \\ &= (\Pi'(w), w)_{\mathcal{H}} = (\Pi'(w), Mw)_{L_2(\Omega)} \geq 0,\end{aligned} \tag{5.2.2}$$

since $f(s)s \geq 0$, where $\Pi(w) = \int_\Omega \hat{f}(w)d\Omega \geq 0$, as required, by (4.1.9) (i.e., $\hat{f}(x) \geq 0$, $\forall\, x \in \mathbb{R}$).

Verification of assumption (A.4). Property (2.23) in the setting of (2.21a–b), (2.22) can be restated in the present case as the following unique continuation statement: Consider the dynamics

$$y_{tt} - \gamma\Delta y_{tt} + \Delta^2 y + f'(w)y = 0 \text{ in } Q, \tag{5.2.3}$$

obtained from differentiating (5.1.1a) in t and setting $y = w_t$, along with the over-determined B.C.,

$$y|_\Sigma \equiv \Delta y|_\Sigma \equiv \left.\frac{\partial y}{\partial \nu}\right|_\Sigma \equiv 0 \text{ on } \Sigma = (0,T] \times \Gamma. \tag{5.2.4}$$

Here, the first two B.C. are incorporated in the dynamic operator $\mathcal{A}$ in (5.1.4) (via (4.1.2a), while the third B.C. corresponds to (2.22) with $\mathcal{B}^*$ given by (5.1.9) (and $y \equiv w_t$). We then seek to conclude that, in fact, (5.2.3), (5.2.4) imply

$$y \equiv 0 \text{ on } 0 < t \leq T \tag{5.2.5}$$

(i.e., (2.23)).

6. The system of dynamic elasticity with nonlinear dissipation both in the Dirichlet B.C. and in the interior

Model. Let Ω be an open bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$, with sufficiently smooth boundary Γ. In this section, we present, briefly, the case of a system of dynamic elasticity with nonlinear feedback control in the Dirichlet B.C., as well as in the interior. Our brief discussion is jusified by two reasons: (a) such system is the perfect counterpart, *mutatis mutandis*, of the scalar wave equation of Section 4, modulo the new technicalities due to the more complex structure of the system of dynamic elasticity over the scalar wave equations, which in fact can be handled as in Horn (1998a,b); (b) space constraints. Thus, with $w = [w_1, w_2, w_n]$ [say, typically $n = 3$], the counterpart of the feedback problem (4.1.1a–c) is now

$$w_{tt} - \mu\Delta w - (\lambda + \mu)\nabla(\text{div } w) = -\mathcal{F}(w) \quad \text{in } (0, \infty] \times \Omega = Q; \tag{6.1a}$$

$$w(0, \cdot\,) = w_0, \; w_t(0, \cdot\,) = w_1 \quad \text{in } \Omega; \tag{6.1b}$$

$$w|_\Sigma = g\left(\left[\mu\frac{\partial \mathcal{A}^{-1} w_t}{\partial \nu} + (\lambda{+}\mu)\nu \text{ div}(\mathcal{A}^{-1} w_t)\right]_\Gamma\right) \text{ on } (0, \infty){\times}\Gamma \equiv \Sigma. \tag{6.1c}$$

For the corresponding linear open loop problem, to be given in (6.4) below, we refer, e.g., to Lions (1988, Chapter IV, Section 1).

The operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^*$; the abstract model. As in (4.1.2), we let now $\mathcal{A}_0$: $\mathcal{D}(\mathcal{A}_0) = [H^2(\Omega) \cap H_0^1(\Omega)]^n$ be the (free dynamic) operator, whose action is defined by the LHS of Eqn. (6.1a): $\mathcal{A}_0 = -[u\Delta + (\lambda + \mu)\nabla(\text{div } (\,\cdot\,))]$. Likewise, we let $\mathcal{A}$ be the realization of $\mathcal{A}_0$ considered on the underlying space $[H^{-1}(\Omega)]^n$ (counterpart of (4.1.3)).

As in (4.1.5), (4.1.6), and Lasiecka and Triggiani (2000), we now let $\mathcal{B}$ be defined by

$$\mathcal{B}y = \mathcal{A}Du \text{ in } [\mathcal{D}(\mathcal{A})]'; \; y = Du; \quad \begin{aligned} &\mu\Delta y + (\lambda + \mu)\nabla(\text{div } y) = 0 \quad \text{in } \Omega; && (6.2a) \\ &y = u \quad \text{on } \Gamma. && (6.2b) \end{aligned}$$

Then, the same computations as in Lasiecka and Triggiani (1987, p. 345) for the wave case yield

$$\mathcal{B}^* v = D^*\mathcal{A}v = -\left[\mu\,\frac{\partial \mathcal{A}^{-1} v}{\partial \nu} + (\lambda + \mu)\nu \text{ div}(\mathcal{A}^{-1} v)\right]_\Gamma = -\sigma(\mathcal{A}^{-1} v)\nu \tag{6.3}$$

[counterpart of (4.1.6b)], and we take $\tilde{U} = [H^{\frac{1}{2}}(\Gamma)]^n$, so that $\mathcal{B}^*$ is surjective: $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \equiv [L_2(\Omega)]^n$ onto $\tilde{U}$, as required by the preliminary assumption (ii) just below (2.2) [counterpart of (4.1.6c). Then, problem (6.1a–c) can be rewritten abstractly as in (4.1.7).

Linear case, open loop problem. Here we consider the problem in Lions (1988, Chapter IV, Section 1), where $v = [v_1, \ldots, v_n]$:

$$v_{tt} - \mu\Delta v - (\lambda + \mu)\nabla(\text{div } v) = f \quad \text{in } Q; \tag{6.4a}$$

$$v(0, \cdot) = v_0, \; v_t(0, \cdot) = v_1 \quad \text{in } \Omega; \quad v_{tt} = -\mathcal{A}v + \mathcal{A}Dg; \tag{6.4b}$$

$$v|_\Sigma = g \quad \text{in } \Sigma. \tag{6.4c}$$

Optimal interior/boundary regularity of problem (6.4). Belishev and Lasiecka (2002) provide an optimal interior/boundary regularity theory not only for the constant coefficient system of dynamic elasticity (6.4a-b-c), but also for the more general dynamical Lame system *with variable (in space) coefficients* of class C^3, where the model is

$$v_{tt} = \mathbb{A}v + f \quad \text{in } Q; \quad (\mathbb{A}v)_i = \rho^{-1} \sum_{j,k,\ell=1}^{3} \partial_j c_{ijk\ell} \partial_\ell v_k, \; i = 1, 2, 3; \tag{6.5a}$$

$$v(0, \cdot) = v_0, \; v_t(0, \cdot) = v_1 \quad \text{in } \Omega; \quad c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{6.5b}$$

$$v|_\Sigma = g \quad \text{in } \Sigma; \quad \mathcal{C}v = \mu(v + v^T) + \lambda(\text{tr } v)I, \quad \mathcal{C} = \{c_{ijk\ell}\}_1^3. \tag{6.5c}$$

Here $\mathcal{C}$ satisfies the symmetricity conditions $c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$, so that $\mathcal{C}$ is the elasticity tensor of the Lame model, where ρ, λ, μ are smooth C^3-functions depending on the spatial variable only. They satisfy the usual ellipticity condition: $\rho > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, which allows one to establish the positivity of $\mathcal{C} : \langle \mathcal{C}\alpha, \alpha \rangle \geq c_0\|\alpha\|^2$, $c_0 > 0$. Introducing the strain tensor

$$\epsilon(v) = \frac{1}{2}[\nabla v + (\nabla v)^T], \text{ then } \mathbb{A} = \rho^{-1} \text{ div } \mathcal{C}\epsilon(\cdot). \tag{6.6}$$

The aforementioned interior/boundary regularity theory in Belishev and Lasiecka (2002) for problem (6.4) with Dirichlet B.C. is the perfect counterpart of the optimal interior/boundary regularity theory for general scalar second-order hyperbolic equations with (time and space) variable coefficients and Dirichlet B.C. given in Lasiecka and Triggiani (1981, 1983), and in Lasiecka, Lions and Triggiani (1986), and invoked in Section 4. A result at the basic energy level for the constant coefficient case (6.4a–c) is given also in Lions (1988, Chapter IV, Section 1). All these are obtained by using the same multiplier (energy method) as in the canonical case, Lasiecka, Lions and Triggiani (1986) (with variable coefficients). In the next result, we report only the results needed below in the treatment of the present section.

THEOREM 6.1 *(Belishev and Lasiecka, 2002) With reference to the v-problem (6.5) [in particular, the constant coefficient problem, (6.4)], we have:*

(a) the map

$$\{v_0, v_1, f, g\} \in [L_2(\Omega) \times H^{-1}(\Omega)]^n \times L_1(0,T;[H^{-1}(\Omega)]^n) \times L_2(0,T;(L_2(\Gamma))^n) \Rightarrow \{v, v_t\} \in C([0,T];[L_2(\Omega)\times H^{-1}(\Omega)]^n) \quad (6.7)$$

is continuous. [In particular, so is the map L, corresponding to (2.13): $Lg = \{v, v_t\}$ *for* $v_0 = v_1 = 0$, $f = 0$.*]*

(b)

$$\mathcal{C}(v)\nu \in [H^{-1}(\Sigma)]^n, \quad (6.8a)$$

in particular,

$$\sigma(v)\nu \in [H^{-1}(\Sigma)]^n \text{ in the constant coefficient case (6.4)}, \quad (6.8b)$$

where $\mathbb{A}v = div\ \sigma(v)$, $\sigma(\omega) = \mu(\omega + \omega^T) + \lambda\ div\ \omega$, $div\ (\omega) = tr\ \epsilon(\omega)$, $\epsilon(\omega) = \nabla\omega + (\nabla\omega)^T$*, in the usual notation.*

For a justification of part (a) of the above Theorem 6.1, we refer to Belishev and Lasiecka (2002, Theorem 1, Section 2.3, p. 148; via usual duality on Lemma 1, p. 149, or Proposition 1, p. 149). Indeed, the proof in Belishev and Lasiecka (2002, pp. 150–152) is recognized as following the same strategy as that of Lasiecka, Lions and Triggiani (1986) for the basic energy level result in (6.7), with support of computations performed in Lasiecka (1999) for the von Karman system. Moreover, for part (b) of Theorem 6.1, we refer to Belishev and Lasiecka (2002, Comments in Section 2.5, p. 154). These state that, once Theorem 6.1(a) has been established (as well as its dual result in Belishev and Lasiecka, 2002, Lemma 1, Proposition 1, p. 149), one can prove higher- (or lower-) level optimal regularity theory of the solutions with respect to various levels of function spaces for the controls g; and that this can be accomplished in the same way as done in Lasiecka, Lions and Triggiani (1986) for scalar second-order hyperbolic equations. Then the boundary regularity (6.8) is the perfect counterpart of Lasiecka, Lions and Triggiani (1986, Eqn. (2.14), p. 153).

Exact controllability of problem (6.4). Surjectivity of the map in (6.7) [or say, of the operator L defined below (6.7)], say for $v_0 = v_1 = 0$, $f = 0$, and $T > 0$ sufficiently large is established in Lions (1988, Chapter IV, Section 1), Alabau and Komornik (1998).

This is precisely the required result of exact controllability on the natural state space of problem (6.4): it corresponds to the abstract assumption (2.20) in the present case.

Nonlinear case: $\mathcal{F}$ and g. These are assumed precisely as in Section 4 for the canonical case of the scalar wave equation.

The boundary-boundary regularity: The operator B^*L is bounded on $L_2(0,T;U)$, $U=(L_2(\Gamma))^n$. Assumption (A.2). It remains to justify, in the case of the system of dynamic elasticity, the counterpart of the boundary regularity property (4.2.2) asserted in the case of the wave equation (or general second-order hyperbolic equations), which is proved in Lasiecka and Triggiani (2004, 2008). A statement, with the key part of the proof referred to the author's Ph.D. thesis is also in Ammari (2002). The strategy is quite different from Lasiecka and Triggiani (2004, 2008). As in Lasiecka and Triggiani (2003, 2004, 2008), one first obtains that assumption (A.2) is reformulated as follows with $B=[0,\mathcal{A}D]^{\mathrm{tr}}$, with reference to the v-problem in (6.4a–c) with $v_0=v_1=0$, $f=0$: by (6.3)

$$B^*Lg = D^*v_t = D^*\mathcal{A}\mathcal{A}^{-1}v_t = \mathcal{B}^*(\mathcal{A}^{-1}v_t) \equiv -\sigma(\mathcal{A}^{-1}v_t)\nu \tag{6.9}$$

$$= -\sigma(z)\nu; \quad z=\mathcal{A}^{-1}v_t; \tag{6.10}$$

$$\text{(by (6.4b))} \qquad z_t=\mathcal{A}^{-1}v_{tt}=\mathcal{A}^{-1}[-\mathcal{A}v+\mathcal{A}Dg] = -v+Dg; \tag{6.11}$$

$$z_{tt}=\mathbb{A}z+Dg_t \text{ in } Q, \quad z|_\Sigma=0, \tag{6.12}$$

counterpart of Lasiecka and Triggiani (2004, Eqn. 1.12; 2008), where we have recalled (6.4b). Moreover, B^* is the adjoint w.r.t. the state space $[L_2(\Omega)\times H^{-1}(\Omega)]^n$, while D^* is the adjoint w.r.t. $[L_2(\Omega)]^n$.

Eqns. (6.9), (6.10) above are the perfect counterpart of Lasiecka and Triggiani (2004, (1.10), Section 4.2), while the z-problem in (6.12) above is the perfect counterpart of the z-problem $\{z_{tt}=\Delta z+Dg_t,\ z|_\Gamma=0\}$ in Lasiecka and Triggiani (2004, p. 627, Eqn. (4.1.12)). [Of course, "D" in Lasiecka and Triggiani (2004) and "D" now are two different, but perfect countparts of each other, Dirichlet maps, corresponding to the respective elliptic (static) problems.] Thus, in conclusion, the perfect counterpart of the boundary estimate (4.2.2) for scalar waves is the following

Theorem 6.2 *With reference to the v-problem (6.4a–c) (or even (6.5a–c)), with $v_0=v_1=0$, $f=0$, we have*

$$B^*L: \text{ continuous } L_2(0,T;U)\to L_2(0,T;U),\ U=[L_2(\Gamma))]^n, \tag{6.13a}$$

equivalently, by (6.9):

$$\int_\Sigma \|\sigma(\mathcal{A}^{-1}v_t)\nu\|^2 d\Sigma \le C_T \int_\Sigma \|g(t)\|^2 d\Sigma. \tag{6.13b}$$

The proof of Theorem 6.2 is the perfect counterpart of (4.2.2) given in Lasiecka and Triggiani (2004, 2008, Section 4.2). We can only give here a quick account to stress the parallelism.

Step 1. (hyperbolic sector) As in Lasiecka and Triggiani (2004, p. 628 and (2.11), p. 630; 2008, Section 4.2), we have

$$\sigma(z_t)\nu\in[H^{-1}(\Sigma)]^n, \text{ hence } (I-\mathcal{X})\sigma(z)\nu\in L_2(0,T;U) \text{ in the hyperbolic sector,} \tag{6.14}$$

once the problem has been transported to the half-space $\tilde{\Omega}$, by use of partition of unity procedure and local change of coordinates. The underlying reason behind (6.14) is the trace regularity $\sigma(v)\nu \in [H^{-1}(\Sigma)]^n$, asserted (in (6.8a)).

Step 2. In the elliptic sector, the proof parallels that of Lasiecka and Triggiani (2004, 2008, Section 4.2), with additional computations, due to the more complicated structure of the elastic operator $\mathbb{A}$, over the wave operator, for which the analysis carried out (via Hörmander's volumes) in Horn (1998a, Section 2) is needed: but this, in turn, is precisely the perfect parallel counterpart of that provided in the case of scalar second-order hyperbolic equations in the original reference Lasiecka and Triggiani (1992). More precisely, as in Lasiecka and Triggiani (2004; 2008, Section 4.2), one extends the v-solution of (6.4) with zero initial conditions $v_0 = v_1 = 0$ and $f = 0$ by zero for negative times. We then introduce the same time cut-off function $\varphi(t)$ as in Lasiecka and Triggiani (2004; 2008, Section 4.2). This transforms [on the half-space $\tilde{\Omega}$] the original v-problem into a new w-problem, $w = \varphi v$, with an additional RHS term $\ell ot(v)$, due to commutators, still zero I.C., no interior forcing term, and Dirichlet boundary datum g. The interior regularity of $\{w, w_t\}$ is the same as that of $\{v, v_t\}$ given by (6.7). Next, as in Lasiecka and Triggiani (2004; 2008, Section 4.2), we split the w-problem into the sum of two problems, $w = u + y$, where (a) u solves the problem with zero I.C. and Dirichlet boundary datum $g \in L_2(0,T;\tilde{U})$, $\tilde{U} = [L_2(\tilde{\Gamma})]^n$, $\tilde{\Gamma}$ the boundary of $\tilde{\Omega}$; (b) y solves the problem with zero I.C., zero Dirichlet boundary datum, but an RHS term $f = \ell ot(v)$. By the regularity (6.7), we have

$$\{u, u_t\} \in C([0,T]; [L_2(\tilde{\Omega}) \times H^{-1}(\tilde{\Omega})]^n) \text{ continuously in } g \in L_2(0,T;\tilde{U}), \quad (6.15)$$

precisely the counterpart of Lasiecka and Triggiani (2004, Eqn. (2.5b); 2008, Eqn. (4.2.5b)). Moreover, regarding the y-problem, the regularity of $f = \ell ot(v) \in C([0,T]; [H^{-1}(\tilde{\Omega})]^n)$ by (6.7), and hence

$$\{y, y_t\} \in C([0,T]; [L_2(\tilde{\Omega}) \times H^{-1}(\tilde{\Omega})]^n) \text{ continuously in } g \in L_2(0,T;\tilde{U}), \quad (6.16)$$

precisely as in Lasiecka and Triggiani (2004, Eqn. (2.7); 2008, Eqn. (4.2.7)). The analysis in Step 3 (Lasiecka and Triggiani, 2004, 2008, Section 4.2) (using Horn, 1998; Lasiecka and Triggiani, 1992) has a perfect counterpart now. In fact, we recall from (6.9) that we seek to establish

$$D^* v_t \in L_2(0,T; L_2(\tilde{\Gamma})) \text{ continuously in } g \in L_2(0,T; L_2(\tilde{\Gamma})). \quad (6.17)$$

Moreover, we recall that v in Ω is transferred into $w = u + y$ on the half-space $\tilde{\Omega}$ (locally). Thus, with reference to the above y-problem, what suffices to show for y is the following regularity property

$$f \to D^* y_t = D^* \mathcal{A}\mathcal{A}^{-1} y_t : \text{ continuous } L_2(0,T; [H^{-1}(\tilde{\Omega})]^n) \to L_2(0,T; L_2(\tilde{\Gamma})), \quad (6.18)$$

whereby $D^* y_t$ is, ultimately, continuous in $g \in L_2(0, T; L_2(\tilde{\Gamma}))$ [Eqn. (6.18) is the counterpart of Lasiecka and Triggiani (2004, Eqn. (2.8); 2008, Eqn. (4.2.8). However, the above property (2.8) is precisely the result of Lasiecka, Lions and Triggiani (1986, Theorem 3.11, p. 182) in the case of scalar second-order hyperbolic equations, of which—as noted in Belishev and Lasiecka (2002, Comments in Section 2.5, p. 154)—a perfect counterpart holds true for the system of dynamic elasticity, the y-problem. So (6.18) holds true.]

Finally, the analysis of Lasiecka and Triggiani (2004, Step 4, p. 630–1; 2008, Step 4, p. 223–4) in the elliptic sector has a perfect counterpart as well, in particular for Lasiecka and Triggiani (2004, Eqns. (2.14), (2.15); 2008, Eqns. (4.2.14)–(4.2.15)). Ultimately, the entire analysis leads to the conclusion that

$$\mathcal{X}\sigma(z)\nu,\ \sigma(\mathcal{X}z)\nu \in L_2(0, T; \tilde{U}) \text{ in the elliptic sector,} \tag{6.19}$$

as in Lasiecka and Triggiani (2004, Eqn. (2.17); 2008, (4.2.17)). Combining the regularity of $\sigma(z)\nu$ in the hyperbolic sector, Eqn. (6.14), and in the elliptic sector, Eqn. (6.19), one finally obtains

$$\sigma(z)\nu \in L_2(0, T; \tilde{U}),\ \text{continuously in } g \in L_2(0, T; \tilde{U}), \quad \tilde{U} = L_2(\tilde{\Gamma}) \tag{6.20}$$

(counterpart of Lasiecka and Triggiani, 2004, (2.18)), and then (6.13b) is established, since $z = \mathcal{A}^{-1} v_t$ by (6.10).

In conclusion: A perfect counterpart of Theorem 4.1.1 applies now in the case of the system of dynamic elasticity given by (6.1a–c), *mutatis mutandis.*

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