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Representations of hypersurfaces and minimal smoothness of the midsurface in the theory of shells* †

by

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Abstract: Many hypersurfaces ω in $\mathbf{R}^N$ can be viewed as a subset of the boundary Γ of an open subset Ω of $\mathbf{R}^N$. In such cases, the gradient and Hessian matrix of the associated *oriented distance function* b_Ω to the underlying set Ω completely describe the normal and the N fundamental forms of ω, and a fairly complete intrinsic theory of Sobolev spaces on $C^{1,1}$-hypersurfaces is available in Delfour (2000). In the theory of thin shells, the asymptotic model only depends on the choice of the *constitutive law*, the *midsurface*, and the space of solutions that properly handles the loading applied to the shell and the boundary conditions. A central issue is the minimal smoothness of the midsurface to still make sense of asymptotic *membrane shell* and *bending equations* without ad hoc mechanical or mathematical assumptions. This is possible for a $C^{1,1}$-midsurface with or without boundary and without local maps, local bases, and Christoffel symbols via the purely intrinsic methods developed by Delfour and Zolésio (1995a) in 1992.

Anicic, LeDret and Raoult (2004) introduced in 2004 a family of surfaces ω that are the image of a connected bounded open Lipschitzian domain in $\mathbf{R}^2$ by a bi-Lipschitzian mapping with the assumption that the normal field is globally Lipschizian. >From this, they construct a *tubular neighborhood* of thickness $2h$ around the surface and show that for sufficiently small h the associated *tubular neighborhood mapping* is bi-Lipschitzian. We prove that such surfaces are $C^{1,1}$-surfaces with a bounded measurable second fundamental form. We show that the tubular neighborhood can be completely described by the algebraic distance function to ω and that it is generally not a Lipschitzian domain in $\mathbf{R}^3$ by providing

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the example of a plate around a flat surface ω verifying all their assumptions. Therefore, the G_1-join of K-regular patches in the sense of Le Dret (2004) generates a new K-regular patch that is a $C^{1,1}$-surface and the join is $C^{1,1}$. Finally, we generalize everything to hypersurfaces generated by a bi-Lipschitzian mapping defined on a domain with facets (e.g. for sphere, torus). We also give conditions for the decomposition of a $C^{1,1}$-hypersurface into $C^{1,1}$-patches.

Keywords: thin shell, asymptotic shell, midsurface, smoothness, representation of a surface, oriented distance function, bi-Lipschitz mapping, tubular neighborhood.

1. Introduction

Many hypersurfaces ω in $\mathbf{R}^N$ can be viewed as the boundary or a subset of the boundary Γ of an open subset Ω of $\mathbf{R}^N$. In such cases the associated *oriented distance function* b_Ω to the underlying set Ω completely describes the surface ω: its (outward) normal is the gradient ∇b_Ω, its first, second, third, ..., and N-th fundamental forms are $\nabla b_\Omega \otimes \nabla b_\Omega$, its Hessian $D^2 b_\Omega$, $(D^2 b_\Omega)^2$, ... and $(D^2 b_\Omega)^{N-1}$ restricted to the boundary Γ (Delfour and Zolésio, 1994, 2001, Chapter 8, § 5). In addition, a fairly complete intrinsic theory of Sobolev spaces on $C^{1,1}$-surfaces is available in Delfour (2000).

In the theory of thin shells, the asymptotic model, when it exists, only depends on the choice of the *constitutive law*, the *midsurface*, and the space of solutions that properly handles the loading applied to the shell and the boundary conditions. A central issue is how rough this midsurface can be to still make sense of asymptotic *membrane shell* and *bending equations* without ad hoc mechanical or mathematical assumptions. This is possible for a general $C^{1,1}$-midsurface with or without boundary such as a sphere, a torus, or a closed reservoir. Moreover, it can be done without local maps, local bases, and Christoffel symbols via the purely intrinsic methods developed by Delfour and Zolésio starting in 1992 with Delfour and Zolésio (1995a) and in a number of subsequent papers, Delfour and Zolésio (1995b, 1996, 1997), Delfour (1998, 1999a,b, 2002), Delfour and Bernadou (2002), Bernadou and Delfour (2000). Results and a brief review are given in Section 2.

In the classical theory of shells (see, for instance, Ciarlet, 2000), the *midsurface* ω is defined as the image of a flat smooth bounded connected domain U in $\mathbf{R}^2$ via a C^2-immersion $\varphi : U \to \mathbf{R}^3$. When U is sufficiently smooth and the thickness sufficiently small, the associated *tubular neighborhood* $\mathbb{S}_h(\omega)$ *of thickness* $2h$ is a Lipschizian domain that is identified with a *thin shell of thickness* $2h$ *around* ω. Anicic, LeDret, and Raoult 2004 relaxed the classical assumptions by introducing a family of surfaces ω that are the image of a connected bounded open Lipschitzian domain U in $\mathbf{R}^2$ by a bi-Lipschitzian mapping φ with the assumption that the normal field only defined almost everywhere is globally

Lipschizian. Such surfaces are called *K-regular patches* by LeDret (2004). From this, they construct a *tubular neighborhood* $\mathbb{S}_h(\omega)$ of thickness $2h$ around the surface and show that for sufficiently small h the *tubular neighborhood mapping* is bi-Lipschitzian.

In Section 3, we prove that the surfaces of Anicic, Le Dret and Raoult (2004) (or K-regular patches) are $C^{1,1}$-surfaces with a bounded measurable second fundamental form. It was already known that $C^{1,1}$-surfaces have a globally Lipschitzian normal field, but it was not, a priori, clear whether midsurfaces generated in the parametrized set-up of Anicic, Le Dret and Raoult (2004) would be strictly rougher than $C^{1,1}$ or not. Moreover, since a K-regular patch does not see the singularities of the underlying bi-Lipschitzian parametrization, the G_1-join of K-regular patches along a join developed in Le Dret (2004) generates a new K-regular patch that is a $C^{1,1}$ surface and the join is in fact $C^{1,1}$. We first generalize everything to hypersurfaces in $\mathbf{R}^N$, $N \geq 2$, since the proofs are independent of the dimension. Secondly, we show that such tubular neighborhoods can be completely specified by the *algebraic distance* to ω and that they are generally not Lipschitzian domains in $\mathbf{R}^3$ since their tangential smoothness is not effectively controlled by the assumptions of Anicic, Le Dret and Raoult (2004) as illustrated by our Example 3.1 of a bi-Lipschitzian parametrization of the plane (see Remarks 3.1 and 3.2) that does not transform a Lipschizian domain into a Lipschizian domain. This means that classical results from three-dimensional linear elasticity over Lipschitzian domains cannot be directly applied to the class of thin shells considered in Anicic, Le Dret and Raoult (2004). At best, they must be recovered by direct methods. Therefore, $C^{1,1}$ is still the currently available minimum smoothness to make sense of asymptotic *membrane shell* and *bending equations*.

In Section 4 we extend the results of Section 3 to hypersurfaces defined on a connected domain with facets. For instance, such domains make it possible to parametrize surfaces such as a sphere or a torus. We show that under the same assumptions as in Section 3 the resulting hypersurface is $C^{1,1}$ and that the tubular neighborhood mapping theorem still holds.

In Section 5, we generalize the work of Le Dret (2004) on G_1-joins of K-regular patches to the G_1-joins of $C^{1,1}$-patches defined on a domain with facets.

Finally, in Section 6, we introduce natural assumptions to decompose a $C^{1,1}$-hypersurface into $C^{1,1}$-patches defined on a domain with facets. This construction seems to be of special interest as a basis of finite element methods for thin shells.

2. Intrinsic representation of hypersurfaces

2.1. $C^{1,1}$-hypersurfaces via the oriented distance function

We first recall the main underlying constructions. For an integer $N \geq 1$ the inner product and the norm in $\mathbf{R}^{\mathrm{N}}$ will be written $x \cdot y$ and $|x|$. The transpose of a matrix A will be denoted A^* and its image Im A. The *complement* $\{x \in \mathbf{R}^N : x \notin \Omega\}$ and the boundary $\overline{\Omega} \cap \overline{\complement\Omega}$ of a subset Ω of $\mathbf{R}^N$ will be respectively denoted by $\complement\Omega$ or $\mathbf{R}^N \backslash \Omega$ and by $\partial\Omega$ or Γ. Given $\Omega \subset \mathbf{R}^N$ and $x \in \overline{\Omega}$, denote by $T_x\Omega$ the *Bouligand's contingent cone* to Ω in x,

$$T_x\Omega \stackrel{\text{def}}{=} \left\{v \in \mathbf{R}^N : \exists \{x_n\} \subset \Omega \text{ and } \varepsilon_n \searrow 0 \text{ such that } (x_n - x)/\varepsilon_n \to v\right\}, \quad (2.1)$$

and by $(T_x\Omega)^*$ its *dual cone* $(T_x\Omega)^* \stackrel{\text{def}}{=} \left\{y \in \mathbf{R}^N : \forall v \in T_x\Omega, \quad y \cdot v \geq 0\right\}$. The *distance* and the *oriented distance function* from a point x to Ω are defined as

$$d_\Omega(x) \stackrel{\text{def}}{=} \inf_{y \in \Omega} |y - x|, \quad b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\complement\Omega}(x). \quad (2.2)$$

In particular, $d_\Omega = |b_\Omega|$. The *set of projections* of a point x onto $\overline{\Omega}$ will be denoted $\Pi_\Omega(x)$. When $\Pi_\Omega(x)$ is a singleton, the projection will be denoted $p_\Omega(x)$. The *h-neighborhood* of Ω is defined as

$$U_h(\Omega) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^N : d_\Omega(x) < h\right\}. \quad (2.3)$$

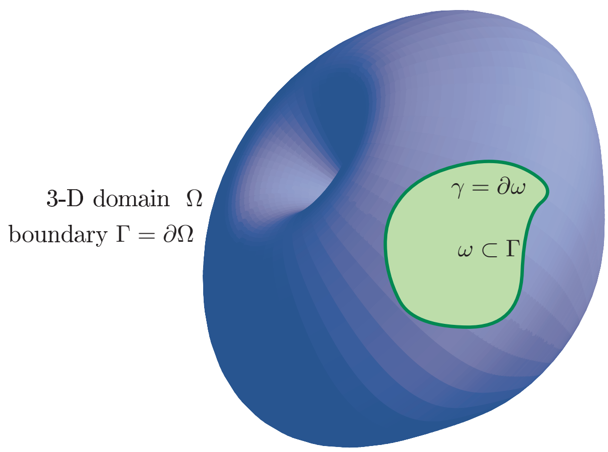


Figure 1. Domain ω with boundary γ

For a subset $\Omega \subset \mathbf{R}^N$ of class $C^{1,1}$ with a non-empty boundary $\Gamma \stackrel{\text{def}}{=} \partial\Omega$, Γ is a $C^{1,1}$-submanifold of codimension one, the normal coincides with ∇b_Ω and the first, second, third, ... , and N-th fundamental forms are $\nabla b_\Omega \otimes \nabla b_\Omega$, the Hessian $D^2 b_\Omega$, $(D^2 b_\Omega)^2$, ... and $(D^2 b_\Omega)^{N-1}$ restricted to the boundary Γ (Delfour and Zolésio, 1994, 2001, Chapter 8, § 5). In addition, a fairly complete tangential differential calculus and an intrinsic theory of Sobolev spaces on $C^{1,1}$-surfaces is available in Delfour (2000). We quote the following theorem from Delfour (2000) that is used to work in curvilinear coordinates in the neighborhood of Γ.

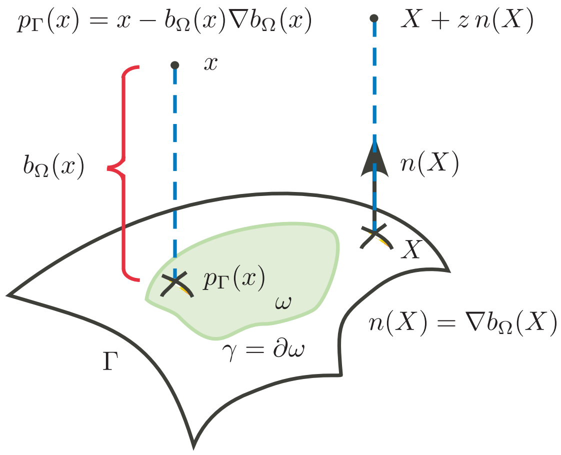


Figure 2. Bijective bi-Lipschitzian mapping T

THEOREM 2.1 *Let $\Omega \subset \mathbf{R}^N$ be a set of class $C^{1,1}$ such that its boundary $\Gamma \overset{\text{def}}{=} \partial\Omega \neq \varnothing$ be bounded. Then there exists $h > 0$ such that $b_\Omega \in C^{1,1}(\overline{U_h(\Gamma)})$,*

$$X, z \mapsto T(X, z) \overset{\text{def}}{=} X + z\nabla b_\Omega(X) : \Gamma\times\,]-h, h[\to U_h(\Gamma) \tag{2.4}$$

is a bi-Lipschitzian bijection, and

$$T^{-1}(x) = (p_\Gamma(x), b_\Omega(x)). \tag{2.5}$$

Assume for the moment that the assumptions of Theorem 2.1 are verified and let $h > 0$ be such that $b_\Omega \in C^{1,1}(\overline{U_h(\Gamma)})$. Given a relatively open subset ω of Γ, define the *tubular neighborhood* of thickness k, $0 < k \le h$, around ω

$$S_k(\omega) \overset{\text{def}}{=} \{x \in \mathbf{R}^N : |b_\Omega(x)| < k \text{ and } p_\Gamma(x) \in \omega\}. \tag{2.6}$$

By definition, $S_k(\Gamma) = U_k(\Gamma)$. But when $\omega \subsetneq \Gamma$, $U_k(\omega)$ is larger than or equal to $S_k(\omega)$.

COROLLARY 2.1.1 *Let $\Omega \subset \mathbf{R}^N$ be a set of class $C^{1,1}$ such that its boundary $\Gamma \neq \varnothing$ be bounded. Let ω be a relatively open subset of Γ. Then there exists $h > 0$ such that*

$$X, z \mapsto T(X, z) \overset{\text{def}}{=} X + z\nabla b_\Omega(X) : \omega\times\,]-h, h[\to S_h(\omega) \tag{2.7}$$

is a bi-Lipschitzian bijection and

$$T^{-1}(x) = (p_\Gamma(x), b_\Omega(x)). \tag{2.8}$$

Let γ be the relative boundary of ω in Γ. In view of Corollary 2.1.1, the boundary $\partial S_k(\omega)$ of $S_k(\omega)$ is made up of three parts: the *bottom* and *top boundaries*

$$T(\omega, -k) \text{ and } T(\omega, k) \tag{2.9}$$

and the *lateral boundary*

$$\Sigma_k(\gamma) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| \leq k \text{ and } p_\Gamma(x) \in \gamma\}. \tag{2.10}$$

The top and bottom boundaries $T(\omega, k)$ and $T(\omega, -k)$ are $C^{1,1}$ surfaces with respective normal ∇b_Ω and $-\nabla b_\Omega$ since the sets $\{x \in \mathbf{R}^N : b_\Omega(x) < k\}$ and $\{x \in \mathbf{R}^N : b_\Omega(x) > -k\}$ are still sets of class $C^{1,1}$. $\Sigma_k(\gamma)$ is *normal* to Γ, $T(\omega, k)$, and $T(\omega, -k)$. It is natural to characterize its smoothness in the tangent plane to Γ by specifying the smoothness of $\Sigma_k(\gamma)$ near γ.

Definition 2.1 (Delfour, 2000, §4.5) *Let ω be a bounded relatively open subset of Γ that satisfies the assumptions of Theorem* 2.1.

(i) Given an integer $k \geq 1$ and a real $0 \leq \lambda \leq 1$, γ is $C^{k,\lambda}$ if there exist $h > 0$ and $0 < h' \leq h$ such that the piece $\Sigma_{h'}(\gamma)$ of the lateral boundary of $S_h(\omega)$ is $C^{k,\lambda}$.

(ii) γ is Lipschitzian if there exist h', $0 < h' \leq h$, such that $\Sigma_{h'}(\gamma)$ is Lipschitzian.

(iii) ω is connected if there exists h', $0 < h' < h$, such that $S_{h'}(\omega)$ is connected.

The definitions correspond to the usual ones in $\mathbf{R}^{\mathrm{N}}$. For instance condition (i) is equivalent to saying that the oriented distance function $b_{S_h(\omega)}$ associated with the set $S_h(\omega)$ has the required smoothness in a neighborhood of $\Sigma_{h'}(\gamma)$.

Under the assumptions of Theorem 2.1, $S_h(\Gamma)$ is $C^{1,1}$ since Γ has no boundary; for a bounded relatively open subset ω of Γ with Lipschitzian relative boundary γ, $S_h(\omega)$ is Lipschitzian. In both cases, two versions of Korn's inequality are given in Delfour (2000, Thms 5.1 and 5.2) and the theory of linear elasticity over a Lipschitzian domain is readily available.

The basic idea in the application to the theory of thin shells is that a vector function $V : S_h(\omega) \to \mathbf{R}^N$ can be expressed in the tangential-normal coordinate system as

$$(X, z) \mapsto v(X, z) \stackrel{\text{def}}{=} V(T(X, z)) : \omega \times]-h, h[\to \mathbf{R}^N .$$

Once in that coordinate system, the vector function V can be approximated by a polynomial function in the z-variable

$$v_h(X, z) \stackrel{\text{def}}{=} v_0(X) + z\, v_1(X) + z^2\, v_2(X),$$

but now this approximation can be transported back into the Euclidean space as $v_h \circ T^{-1}$ and

$$V_h \stackrel{\text{def}}{=} v_h \circ T^{-1} = v_0 \circ p_\Gamma + b_\Omega\, v_1 \circ p_\Gamma + b_\Omega^2\, v_2 \circ p_\Gamma.$$

The big advantage is that now V_h is the approximation of V in the equations of linear or nonlinear elasticity and all the computations are done in the Euclidean neighborhood of Γ. The derivatives of b_Ω is the normal $\nabla b_\Omega \circ p_\Gamma$ evaluated at the projection and the Jacobian matrix of p_Γ is $I - \nabla b_\Omega \otimes \nabla b_\Omega + b_\Omega D^2 b_\Omega$. The Jacobian matrix of $v_i \circ p_\Gamma$ on Γ is the tangential derivative $D_\Gamma v_i$ of v_i.

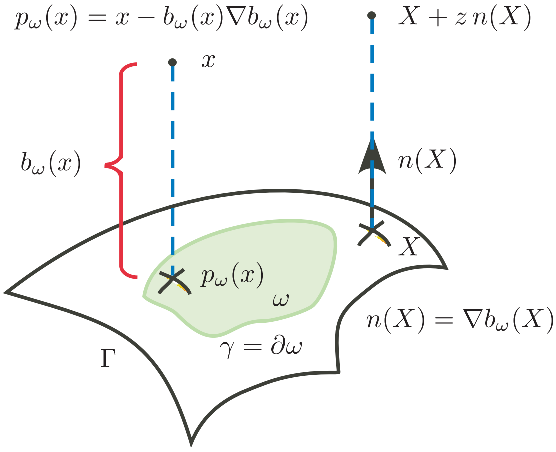


Figure 3. Mapping T with $b_\Omega \in C^{1,1}(\mathbb{S}_h(\omega))$

The global smoothness assumptions on Γ can be relaxed to a local one in a neighborhood of $\overline{\omega}$.

THEOREM 2.2 *Given $\Omega \subset \mathbf{R}^N$ with boundary $\Gamma \neq \varnothing$ and a bounded (relatively) open subset ω of Γ, assume that there exists a neighborhood $N(\omega)$ of $\overline{\omega}$ such that $b_\Omega \in C^{1,1}(N(\omega))$. Then there exists $\bar{h} > 0$ such that $b_\Omega \in C^{1,1}(U_{\bar{h}}(\omega))$ and, for all h, $0 < h < \bar{h}$, the mapping*

$$X, z \mapsto T(X, z) \stackrel{\text{def}}{=} X + z\nabla b_\Omega(X) : \omega \times\,]-h, h[\, \to S_h(\omega) \tag{2.11}$$

is a bi-Lipschitzian bijection and its inverse is given by

$$x \mapsto T^{-1}(x) = (p_\Gamma(x), b_\Omega(x)) : S_h(\omega) \to \omega \times\,]-h, h[\,. \tag{2.12}$$

Proof. Choose $\bar{h} = \inf_{z \in \complement N(\omega)} d_\omega(z)$. From the assumptions, $\bar{h} > 0$, $U_{\bar{h}}(\omega) \subset N(\omega)$, $b_\Omega \in C^{1,1}(U_{\bar{h}}(\omega))$, and the projection p_Γ of each point of $U_{\bar{h}}(\omega)$ onto Γ

is a singleton and is continuous. In particular, $S_{\bar h}(\omega)$ is a well defined open set and $b_\Omega \in C^{1,1}(S_{\bar h}(\omega))$.

Since b_Ω is bounded on bounded subsets and $|\nabla b_\Omega| = 1$, $b_\Omega \in C^{1,1}(\overline{U_{\bar h}(\omega)})$ and the mapping T from $\omega\times\,]-\bar h, \bar h[$ to $\mathbf{R}^N$ is clearly Lipschitzian. Since b_Ω is differentiable in $U_{\bar h}(\omega)$, we know from Delfour and Zolésio (2001, Thm 3.1, Chap. 5, p. 214) that $p_\Gamma(x) = x - b_\Omega(x)\nabla b_\Omega(x)$ in $U_{\bar h}(\omega)$ and for all x and y in $U_{\bar h}(\omega)$

$$\begin{aligned} |p_\Gamma(y) - p_\Gamma(x)| &= |y - x + (b_\Omega(y) - b_\Omega(x))\nabla b_\Omega(y) + b_\Omega(x)(\nabla b_\Omega(y) - \nabla b_\Omega(x))| \\ &\le |y-x| + |y-x| + \|b_\Omega\|_{C(\overline{S_{\bar h}(\omega)})}\, c\,|y-x|, \end{aligned}$$

and $p_\Gamma \in C^{0,1}(\overline{U_{\bar h}(\omega)})$. Therefore, the map $x \mapsto S(x) \stackrel{\text{def}}{=} (p_\Gamma(x), b_\Omega(x)) : S_{\bar h}(\omega) \to \omega\times\,]-\bar h, \bar h[$ is well defined and Lipschitzian. Moreover $T(S(x)) = T(p_\Gamma(x), b_\Omega(x)) = p_\Gamma(x)+b_\Omega(x)n(p_\Gamma(x)) = p_\Gamma(x)+b_\Omega(x)\nabla b_\Omega(x) = x$ by Delfour and Zolésio (2001, Thm 3.1, Chap. 5, p. 214). To show that T is the inverse of S, it remains to show that $\operatorname{Im} T = S_h(\omega)$ for $0 < h < \bar h$.

Let E be the classical Lipschitz extension of ∇b_Ω from $\overline{U_{\bar h}(\omega)}$ to $\mathbf{R}^{\mathrm{N}}$. Since $|\nabla b_\Omega| = 1$ on $U_{\bar h}(\omega)$, there exists a neighborhood W of $\overline{U_{\bar h}(\omega)}$ such that $|E| \ge 3/4$. Let $\psi \in C^\infty(\mathbf{R}^{\mathrm{N}})$ be such that

$$\psi(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \in \overline{U_h(\omega)} \\ \in [0,1], & x \in W\backslash\overline{U_h(\omega)} \cap \{x : |b_\Omega(x)| < \bar h\} \\ 0, & x \in \mathbf{R}^{\mathrm{N}}\backslash W \cup \{x : |b_\Omega(x)| \ge \bar h\}. \end{cases} \tag{2.13}$$

It is readily seen that the vector field $V \stackrel{\text{def}}{=} \psi E/|E|$ is uniformly Lipschitzian in $\mathbf{R}^{\mathrm{N}}$ and that the equation $x'(t) = V(x(t))$, $x(0) = X$ has a unique solution $x(t, X)$. This defines the Lipschitz transformation $T_t(x) \stackrel{\text{def}}{=} x(t, X)$ of $\mathbf{R}^{\mathrm{N}}$ for all t.

Since $p_\Gamma \in C^{0,1}(\overline{U_{\bar h}(\omega)})$, the mapping $p_\Gamma : U_{\bar h}(\omega) \to \overline{\omega}$ is well defined from $\overline{U_{\bar h}(\omega)} \to \overline{\omega}$. Thus $\partial S_{\bar h}(\omega)$, which is made up of two parts: the *lateral boundary* $\Sigma_{\bar h}(\gamma) = \{x \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| < \bar h$ and $p_\Gamma(x) \in \gamma\}$ and the top and bottom parts $S_{\pm\bar h}(\overline{\omega}) = \{x \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| = \bar h$ and $p_\Gamma(x) \in \overline{\omega}\}$. By construction $V = 0$ on $S_{\pm\bar h}(\overline{\omega})$ and $V \in T_x S_{\bar h}(\omega)$ for all $x \in \Sigma_{\bar h}(\gamma)$. By Nagumo's (1942) viability theorem, for all t, $T_t(\overline{S_{\bar h}(\omega)}) \subset \overline{S_{\bar h}(\omega)}$. Since $|V| \le 1$, for all $X \in \omega$ and $|t| < \bar h$,

$$\frac{d}{dt}(b_\Omega \circ T_t) = \nabla b_\Omega \circ T_t \cdot \frac{dT_t}{dt} = (\nabla b_\Omega \cdot V) \circ T_t \quad \Rightarrow |b_\Omega(T_t(X))| \le |t| < \bar h$$

and $T_t(X) \in S_{\bar h}(\omega)$. In particular, for all $X \in \omega$ and $|t| < h$, $T_t(X) \in S_h(\omega)$, $V(T_t(X)) = \nabla b_\Omega(T_t(X))$, and necessarily $b_\Omega(T_t(X)) = t$. Recalling the property that $p_\Gamma \in C^{0,1}(\overline{U_{\bar h}(\omega)})$, we get $Dp_\Gamma = I - \nabla b_\Omega\,(\nabla b_\Omega)^* - b_\Omega D^2 b_\Omega$ and $Dp_\Gamma \nabla b_\Omega = 0$ a.e. in $U_{\bar h}(\omega)$. Finally, for all $X \in \omega$ and $|t| < h$,

$$\frac{d}{dt}(p_\Gamma \circ T_t) = Dp_\Gamma \circ T_t\,\frac{dT_t}{dt} = (Dp_\Gamma \nabla b_\Omega)\circ T_t = 0 \quad \Rightarrow p_\Gamma(T_t(X)) = p_\Gamma(X) = X.$$

From Delfour and Zolésio (2001, Thm 4.3 (i), Chap. 5, p. 219), $\nabla b_\Omega = n \circ p_\Gamma$, where $n = \nabla b_\Omega|_\omega$. So, for all $X \in \omega$ and all $|t| < h$

$$\begin{aligned}&\frac{d}{dt}T_t(X) = \nabla b_\Omega(T_t(X)) = n(p_\Gamma(T_t(X)) = n(p_\Gamma(X)) = n(X)\\ &\Rightarrow T_t(X) = X + t\,n(X), \quad p_\Gamma(T_t(X)) = X, \text{ and } b_\Omega(T_t(X)) = t.\end{aligned}$$

Therefore the map

$$(t, X) \mapsto T_t(X) = X + t\,n(X) : \omega \times\,]-h, h[\,\to S_h(\omega)$$

is well-defined and finally $S(T(X,t)) = (p_\Gamma(T_t(X)), b_\Omega(T_t(X)) = (X, t)$. ∎

Under the hypotheses of Theorem 2.2, it is possible to define the *signed distance function* to the hypersurface $\overline{\omega}$ in the region $S_{\bar{h}}(\omega)$

$$b_\omega(x) \overset{\text{def}}{=} \begin{cases} d_\omega(x), & \text{if } b_\Omega(x) \geq 0 \\ -d_\omega(x), & \text{if } b_\Omega(x) < 0. \end{cases} \tag{2.14}$$

When the projection of a point x onto $\overline{\omega}$ is a singleton, we denote it by $p_\omega(x)$ and necessarily $p_\omega = p_\Gamma$ on $\overline{S_h(\omega)}$. Note the difference between the oriented distance function b_Ω that is always defined everywhere in $\mathbf{R}^{\mathrm{N}}$ and the signed distance function that is defined only in a region where it is possible to distinguish what is *above* from what is *below* ω. Here $b_\omega = b_\Omega \in C^{1,1}(\overline{S_h(\omega)})$ and T and T^{-1} can be rewritten as

$$\begin{aligned}(X, z) \mapsto T(X, z) = X + z\nabla b_\omega(X) : \omega \times\,]-h, h[\,\to S_h(\omega)\\ x \mapsto T^{-1}(x) = (p_\omega(x), b_\omega(x)) : S_h(\omega) \to \omega \times\,]-h, h[\,.\end{aligned} \tag{2.15}$$

2.2. Intrinsic theory of thin and asymptotic shells

In order to complete the references in Anicic, Le Dret and Raoult (2004) on the theory of shells and to provide a broader perspective to the reader, we briefly recall a few results starting with the key paper Delfour (1998) on the use of intrinsic methods in the asymptotic analysis of three models of thin shells for an arbitrary linear 3D constitutive law. They all converge to asymptotic shell models that consist of a coupled system of two variational equations. They only differ in their resulting effective constitutive laws. The first equation yields the generally accepted classical *membrane shell equation* and the Love-Kirchhoff terms. The second is a generalized *bending equation*. It explains that convergence results for the 3D models were only established for plates and in the bending dominated case for shells. From the analysis of the three models, the richer $P(2,1)$-model turns out to be the most pertinent since it converges to the right asymptotic model with the right effective constitutive law. We also show in Delfour (1999a) that models of the Naghdi's type can be obtained directly from the $P(2,1)$-model by a simple elimination of variables without introducing the a priori assumption on the stress tensor $\sigma_{33} = 0$. Bridges are thrown with classical models using local bases or representations. Those results are completed in Delfour (1999a) with the characterization of the space of solution for the $P(2,1)$ thin shell model and the space of solutions of the asymptotic membrane shell equation in Delfour (1999b). This characterization was only known in the case of the plate and uniformly elliptic shells.

In Delfour (2002), a new choice of the projection leads to the disappearance of the coupling term in the second asymptotic equation. After reduction of the number of variables, this new choice changes the form of the second equation to achieve the complete decoupling of the membrane and bending equations without the classical plate or bending dominated assumptions. In the second part of Delfour (2002) we present a dynamical thin shell model for small vibrations and investigate the corresponding dynamical asymptotic model. Those papers complete Delfour (1998) and make the connection with most existing results in the literature, thus confirming the pertinence and the interest of the methods we have developed. Extensions of the $P(2,1)$-model have also been developed for piezoelectric shells (Bernadou and Delfour, 2000; Delfour and Bernadou, 2002) with a complete decoupling of the membrane and bending equations having been also obtained.

3. Parametric representation of hypersurfaces

In § 3.1 we extend the constructions of Anicic, Le Dret and Raoult (2004) from $\mathbf{R}^3$ to $\mathbf{R}^N$ and show that the resulting hypersurface has unique tangent hyperplane and one-dimensional normal field in every point without the assumption that the underlying flat domain in R^{N-1} be Lipschitz and connected. In § 3.2 we give a constructive proof of Theorem 3.9 in Anicic, Le Dret and Raoult (2004) and show that the sandwich $\mathbb{S}_h(\omega)$ as defined in (3.13) is indeed a tubular neighborhood specified by the signed distance function b_ω and the projection p_ω as defined in (2.6) of § 2. We then give the example of a *parametrized* plate that verifies assumptions (H_1), (H_2), and (H_{3L}) that is not a Lipschitzian domain for all thicknesses $h > 0$. In § 3.3 we prove that the hypersurface ω is of class $C^{1,1}$.

3.1. Preliminary results

Let $\{e_i : 1 \le i \le N\}$ be an orthonormal basis in $\mathbf{R}^{\mathrm{N}}$ and denote by $R^{N-1} = \{e_N\}^\perp$ the hyperplane orthogonal to e_N. Generalizing Anicic, Le Dret and Raoult (2004) from $N = 3$ to an arbitrary $N \ge 1$, let U be a bounded open domain in R^{N-1} and $\varphi : U \to \mathbf{R}^{\mathrm{N}}$ be a mapping with the following properties: there exist $c > 0$ and $C > 0$ such that

$$\text{Assumption } (H_1): \quad \forall \xi, \zeta \in U, \quad c\,|\zeta - \xi| \le |\varphi(\zeta) - \varphi(\xi)| \le C\,|\zeta - \xi|, \tag{3.1}$$

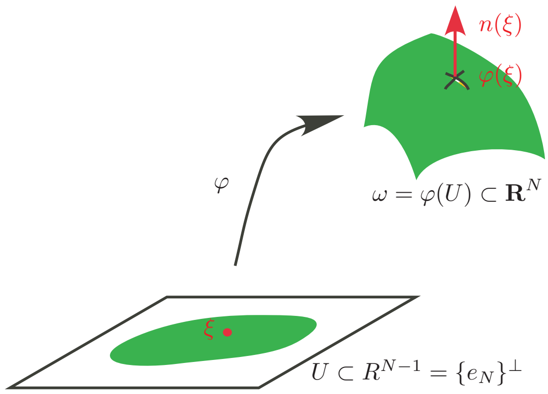


Figure 4. Parametric representation in $\mathbf{R}^N$.

where $c = \inf_{\zeta \neq \xi \in U} |\varphi(\zeta) - \varphi(\xi))/|\zeta - \xi|$. In view of assumption (H_1)[1], $\omega \overset{\text{def}}{=} \varphi(U)$ is a (non self-intersecting) parametric hypersurface in $\mathbf{R}^N$ of dimension $N-1$.

For almost all $\xi \in U$, $D\varphi(\xi)$, $D\varphi(\xi)_{ij} \overset{\text{def}}{=} \partial_j \varphi_i(\xi)$, exists and

$$\forall V \in R^{N-1}, \quad c\,|V| \leq |D\varphi(\xi)V| \leq C\,|V|. \tag{3.2}$$

Therefore, $D\varphi(\xi) : R^{N-1} \to \mathbf{R}^N$ is injective and the $(N-1)$ column vectors $\partial_1\varphi(\xi)$, $\partial_2\varphi(\xi)$, $\ldots$, $\partial_{N-1}\varphi(\xi)$ are linearly independent in $\mathbf{R}^{\mathrm{N}}$. The *surface measure* associated with ω is

$$\int_\omega dH^{N-1} = \int_U J\varphi\, d\xi,$$

where $J\varphi$ is the square root of the sum of the squares of the $(N-1) \times (N-1)$ subdeterminants of $D\varphi$

$$(J\varphi)^2 = \sum_{i=1}^{N} \left[\frac{\partial(\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_N)}{\partial(\xi^1, \ldots, \xi^{N-1})} \right]^2.$$

Choose a unit vector $n(\xi)$ orthogonal to the vectors $\{\partial_i \varphi(\xi)\}$,

$$D\varphi(\xi)^*\, n(\xi) = 0 \text{ and } |n(\xi)| = 1. \tag{3.3}$$

Then the square matrix $[D\varphi(\xi) \vdots n(\xi)]$ is invertible and

$$\det[D\varphi(\xi) \vdots n(\xi)] = b(\xi) \cdot n(\xi) \neq 0 \text{ and } b(\xi)_i = M([D\varphi(\xi) \vdots 0])_{iN}, \quad 1 \leq i \leq N, \tag{3.4}$$

[1] In contrast with the notation in the theory of shells, the greek lower case letters ω and γ are used for the hypersurface and its boundary. The associated flat reference domain in R^{N-1} is denoted U.

where $M(A)$ denotes the *matrix of cofactors* of a square matrix A. In dimension $N = 3$, $b(\xi)$ coincides with the wedge product $\partial_1\varphi(\xi) \wedge \partial_2\varphi(\xi)$. We summarize the main properties.

THEOREM 3.1 *Assume that $\varphi : U \to \mathbf{R}^{\mathrm{N}}$ verifies assumption (H_1) and let $\tilde{U} = \{\xi \in U : \varphi$ is differentiable at $\xi\}$. Then, at each point $\xi \in \tilde{U}$,*

$$n(\xi) = \pm b(\xi)/|b(\xi)|, \ \det\left[D\varphi(\xi) \vdots \frac{b(\xi)}{|b(\xi)|}\right] = |b(\xi| > 0, \ J\varphi(\xi) = |b(\xi)|, \tag{3.5}$$

for all $V \in R^{N-1}$ and $V^N \in \mathbf{R}$

$$c^2|V|^2 + |V^N|^2 \le \left|\left[D\varphi(\xi) \vdots \frac{b(\xi)}{|b(\xi)|}\right]\begin{bmatrix} V \\ V^N \end{bmatrix}\right|^2 \le C^2|V|^2 + |V^N|^2, \tag{3.6}$$

and

$$T_{\varphi(\xi)}\omega = \mathrm{Im}\ D\varphi(\xi) = \{n(\xi)\}^\perp \ \textit{and}\ (T_{\varphi(\xi)}\omega)^* = \mathbf{R}\, n(\xi). \tag{3.7}$$

Proof. (i) We first prove (3.5) and (3.6), The projection of the vector $b = b(\xi)$ onto Im $D\varphi(\xi)$ can be obtained via the following minimization problem:

$$\inf_{V \in R^{N-1}} f(V), \quad f(V) \stackrel{\text{def}}{=} |b - D\varphi(\xi)V|^2.$$

By assumption, we get $c^2\,|V|^2 \le |D\varphi(\xi)V|^2 \le C^2\,|V|^2$ and the symmetrical matrix $D\varphi(\xi)^*\,D\varphi(\xi)$ is positive definite. So, there exists a unique $\hat{V}$ such that

$$\forall V \in R^{N-1}, \quad (D\varphi(\xi)\hat{V} - b)\cdot D\varphi(\xi)V = 0.$$

Hence, $D\varphi(\xi)\hat{V} - b \in (\mathrm{Im}\ D\varphi(\xi))^\perp = \mathbf{R}\,n$, $n = n(\xi)$: there exists $\beta \neq 0$ such that

$$\begin{aligned}
&b - D\varphi(\xi)\,\hat{V} = \beta\,n \\
&\Rightarrow \quad [D\varphi(\xi)\,(\beta\,n - b)]\begin{bmatrix} \hat{V} \\ 1 \end{bmatrix} = D\varphi(\xi)\,\hat{V} + \beta\,n - b = 0, \\
&\Rightarrow\ 0 = \det[D\varphi(\xi)\,(\beta\,n - b)] = b\cdot(\beta\,n - b) = \beta\,b\cdot n - |b|^2.
\end{aligned}$$

Finally

$$\begin{aligned}
&0 = n\cdot(D\varphi(\xi)\,\hat{V} + \beta\,n - b) = n\cdot(\beta\,n - b) = \beta - n\cdot b \\
&\Rightarrow \quad |n| = 1 \text{ and } |b\cdot n| = |b| \quad \Rightarrow n(\xi) = \pm b(\xi)/|b(\xi)|.
\end{aligned}$$

By definition, $J\varphi(\xi) = |b(\xi)|$.

(ii) We now prove (3.7). Let $T = T_{\varphi(\xi)}\omega$. Since φ is differentiable at ξ, for any sequence $\varepsilon_n \searrow 0$

$$(\varphi(\xi + \varepsilon_n V) - \varphi(\xi))/\varepsilon_n \to D\varphi(\xi)V \quad \Rightarrow \mathrm{Im}\ D\varphi(\xi) \subset T.$$

Conversely, given $\tau \in T$, there exist$\{\xi_n\} \subset U$ and $\varepsilon_n \searrow 0$ such that $(\varphi(\xi_n) - \varphi(\xi))/\varepsilon_n \to \tau$. By assumption (H_1)

$$c\left|\frac{\xi_n - \xi}{\varepsilon_n}\right| \le \left|\frac{\varphi(\xi_n) - \varphi(\xi)}{\varepsilon_n}\right|$$

and the quotient $|\xi_n - \xi|/\varepsilon_n$ is bounded. There exists $V \in R^{N-1}$ and a subsequence of $\{(\xi_n - \xi)/\varepsilon_n\}$, still denoted $\{(\xi_n - \xi)/\varepsilon_n\}$, that converges to V and

$$\frac{\varphi(\xi_n) - \varphi(\xi) - D\varphi(\xi)(\xi_n - \xi)}{\varepsilon_n} = \frac{\varphi(\xi_n) - \varphi(\xi) - D\varphi(\xi)(\xi_n - \xi)}{|\xi_n - \xi|}\,\frac{|\xi_n - \xi|}{\varepsilon_n} \to 0$$

$$\Rightarrow \frac{\varphi(\xi_n) - \varphi(\xi)}{\varepsilon_n} - D\varphi(\xi)\frac{\xi_n - \xi}{\varepsilon_n} \to \tau - D\varphi(\xi)V = 0.$$

So $T \subset \text{Im } D\varphi(\xi)$. Finally, by construction of $n(\xi)$, $T = \text{Im } D\varphi(\xi) = \{n(\xi)\}^{\perp}$ and its dual cone is $T^* = \mathbf{R}\, n(\xi)$. ∎

The normal field to ω in $\varphi(\xi)$ is specified by $n(\xi) = \pm b(\xi)/|b(\xi)|$ in each point of the subset $\tilde{U}$ of U where φ is differentiable. We now choose

$$a(\xi) \stackrel{\text{def}}{=} b(\xi)/|b(\xi)|$$

in order to have the determinant of $[D\varphi\vdots a]$ positive and equal to $J\varphi$. Following Anicic, Le Dret and Raoult (2004), it is now assumed that the resulting normal mapping $a(\xi)$ is uniformly Lipschitz on $\tilde{U}$:

Assumption (H_2): $\exists \alpha > 0$ such that $\forall \xi, \zeta \in \tilde{U},\ |a(\zeta) - a(\xi)| \le \alpha\,|\zeta - \xi|$. (3.8)

Since $\overline{\tilde{U}} = \overline{U}$, a extends to a unique uniformly Lipschitz function, still denoted a, on $\overline{U}$: a verifies assumption (H_2) on $\overline{U}$. This very strong assumption “orients" the hypersurface ω that no longer “see" the singularities of its bi-Lipschitzian representation.

Theorem 3.2 *Assume that φ and a verify assumptions (H_1) and (H_2). Then*

$$\forall \xi \in U, \quad T_{\varphi(\xi)}\omega = \{a(\xi)\}^{\perp} \text{ and } (T_{\varphi(\xi)}\omega)^* = \mathbf{R}\, a(\xi). \tag{3.9}$$

and the parametric hypersurface ω has a unique tangent hyperplane in each point with Lipschitzian normal $a \circ \varphi^{-1}$.

Proof. Given $\xi \in U$, there exists ρ, $0 < \rho < \min\{c/(3C\alpha)$, such that $B_{3\rho}(\xi) \subset U$. The proof proceeds in two steps. Part (i) is a global version of Anicic, Le Dret and Raoult (2004, Lemma 3.5) in a ball.

(i) Consider the Lipschitz function $f_\xi(\zeta) = [\varphi(\zeta) - \varphi(\xi)] \cdot a(\xi)$. For almost all $\zeta \in U$, $\nabla f_\xi(\zeta) = D\varphi(\zeta)^* a(\xi)$. By construction, $D\varphi(\zeta)^* a(\zeta) = 0$ almost everywhere in U. As a result $\nabla f_\xi(\zeta) = D\varphi(\zeta)^*[a(\xi) - a(\zeta)]$ and for all $\zeta \in B_{3\rho}(\xi)$

$$|\nabla f_\xi(\zeta)| \le |D\varphi(\zeta)|\,|a(\xi) - a(\zeta)| \le |D\varphi(\zeta)|\,\alpha\,|\xi - \zeta| \le C\,\alpha\,3\rho.$$

Since the ball $B_{3\rho}(\xi)$ is contained in U, the geodesic distance in $B_{3\rho}(\xi)$ is the usual distance and for all $\zeta_2, \zeta_1 \in B_\rho(\xi)$,

$$\begin{aligned}|f_\xi(\zeta_2) - f_\xi(\zeta_1)| &\leq \|\nabla f_\xi\|_{L^\infty(B_{|\zeta_2-\zeta_1|}(\zeta_1))}|\zeta_2 - \zeta_1| \\ &\leq \|\nabla f_\xi\|_{L^\infty(B_{3\rho}(\xi))}|\zeta_2 - \zeta_1| \leq C\alpha\, 3\rho\, |\zeta_2 - \zeta_1|,\end{aligned}$$

since $B_{|\zeta_2-\zeta_1|}(\zeta_1) \subset B_{3\rho}(\xi)$.

(ii) Define the map

$$\zeta \mapsto \varphi_\xi(\zeta) \stackrel{\text{def}}{=} \varphi(\zeta) - a(\xi) \cdot (\varphi(\zeta) - \varphi(\xi))\, a(\xi) : B_\rho(\xi) \to \varphi(\xi) + \{a(\xi)\}^\perp. \tag{3.10}$$

From above, $|\varphi_\xi(\zeta_2) - \varphi_\xi(\zeta_1)| \geq (c - 3C\alpha\rho)|\zeta_2 - \zeta_1|$ and by choice of $\rho < c/(3C\alpha)$, $\varphi_\xi : B_\rho(\xi) \to \varphi_\xi(B_\rho(\xi)) \subset \varphi(\xi) + \{a(\xi)\}^\perp$ is a bi-Lipschitzian bijection. Thus $\varphi_\xi(B_\rho(\xi))$ is a neighborhood of $\varphi(\xi)$ in $\varphi(\xi) + \{a(\xi)\}^\perp$. So, for any $\tau \in \{a(\xi)\}^\perp$, there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, $\varphi(\xi) + \varepsilon\,\tau \in \varphi_\xi(B_\rho(\xi))$. Moreover, there exists $\xi_\varepsilon \in B_\rho(\xi) \subset U$ such that $\varphi_\xi(\xi_\varepsilon) = \varphi(\xi) + \varepsilon\,\tau$ since $\varphi_\xi(\xi) = \varphi(\xi)$. In particular

$$(c - 3C\alpha\rho)\,|\xi_\varepsilon - \xi| \leq |\varphi_\xi(\xi_\varepsilon) - \varphi_\xi(\xi)| = \varepsilon\,|\tau| \quad \Rightarrow |\xi_\varepsilon - \xi|/\varepsilon \leq |\tau|/(c - 3C\alpha\rho).$$

As a result, as $\varepsilon \searrow 0$, $\varphi(\xi_\varepsilon)$ goes to $\varphi(\xi)$ and falls in the ball $B_\rho(\xi)$. Moreover

$$\frac{\varphi(\xi_\varepsilon) - \varphi(\xi)}{\varepsilon} = \frac{\varphi_\xi(\xi_\varepsilon) - \varphi_\xi(\xi)}{\varepsilon} + \frac{|\xi_\varepsilon - \xi|}{\varepsilon}\, a(\xi) \cdot \left\{\frac{\varphi(\xi_\varepsilon) - \varphi(\xi)}{|\xi_\varepsilon - \xi|}\right\} a(\xi).$$

If we could show that $a(\xi) \cdot \{(\varphi(\xi_\varepsilon) - \varphi(\xi))/|\xi_\varepsilon - \xi|\} \to 0$ as $\varepsilon \searrow 0$, then $(\varphi(\xi_\varepsilon) - \varphi(\xi))/\varepsilon \to \tau$ and $\{a(\xi)\}^\perp \subset T_{\varphi(\xi)}\omega$. Going back to part (i), we have already shown that for $\zeta \in B_{3\rho}(\xi)$

$$|\nabla f_\xi(\zeta)| \leq |D\varphi(\zeta)|\, |a(\xi) - a(\zeta)| \leq 3C\,\alpha\, |\xi - \zeta|.$$

Since the ball $B_{3\rho}(\xi)$ is contained in U, for all $\xi_\varepsilon \in B_\rho(\xi)$

$$|f_\xi(\xi_\varepsilon)| = |f_\xi(\xi_\varepsilon) - f_\xi(\xi)| \leq |\nabla f_\xi|_{L^\infty(B_{|\xi_\varepsilon - \xi|}(\xi))}|\xi_\varepsilon - \xi| \leq 3C\,\alpha\, |\xi_\varepsilon - \xi|^2. \tag{3.11}$$

Hence $a(\xi) \cdot (\varphi(\xi_\varepsilon) - \varphi(\xi))/|\xi_\varepsilon - \xi| \to 0$ and we get the result.

Conversely, for $\tau \in T_{\varphi(\xi)}\omega$, there exist $\{\xi_n\} \subset U$ and $\varepsilon_n \searrow 0$ such that $(\varphi(\xi_n) - \varphi(\xi))/\varepsilon_n \to \tau$. Hence, $\{(\xi_n - \xi)/\varepsilon_n\}$ is bounded and

$$a(\xi) \cdot \frac{\varphi(\xi_n) - \varphi(\xi)}{\varepsilon_n} = a(\xi) \cdot \frac{\varphi(\xi_n) - \varphi(\xi)}{|(\xi_n - \xi|}\, \frac{|(\xi_n - \xi|}{\varepsilon_n} \quad \Rightarrow a(\xi) \cdot \tau = 0$$

by using (3.11). Finally $\{a(\xi)\}^\perp = T_{\varphi(\xi)}\omega$ and $(T_{\varphi(\xi)}\omega)^* = \mathbf{R}\, a(\xi)$. ∎

3.2. The “sandwich" $\mathbb{S}_h(\omega)$

As in Anicic, Le Dret and Raoult (2004), consider for an arbitrary $k > 0$ the Lipschitz continuous mapping

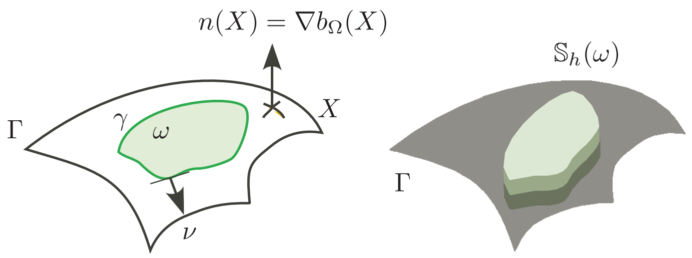
$$\tilde{\xi} \stackrel{\text{def}}{=} (\xi, \xi^N) \mapsto \Phi(\tilde{\xi}) \stackrel{\text{def}}{=} \varphi(\xi) + \xi^N\, a(\xi) : \overline{U} \times [-k, k] \to \mathbf{R}^N \tag{3.12}$$

and the associated “sandwich" of thickness $2h$ around ω

$$\mathbb{S}_k(\omega) \stackrel{\text{def}}{=} \Phi(U \times]-k, k[\,). \tag{3.13}$$

Under assumptions (H_1) and (H_2), Φ is Lipschitzian on $\overline{U} \times [-k, k]$,

$$|\Phi(\xi_2, \xi_2^N) - \Phi(\xi_1, \xi_1^N)| \leq (C + k\,\alpha)|\xi_2 - \xi_1| + |\xi_2^N - \xi_1^N|$$
$$\text{and } \overline{\mathbb{S}_k(\omega)} = \Phi(\overline{U} \times [-k, k]).$$



(a) ω with boundary γ (b) Sandwich $\mathbb{S}_h(\omega)$

Figure 5. Sandwich or tubular neighborhood $\mathbb{S}_h(\omega)$

Associate with Φ the intrinsic Lipschitzian mapping

$$(\mathrm{X}, z) \mapsto \tilde{\Phi}(\mathrm{X}, z) \stackrel{\text{def}}{=} \Phi(\varphi^{-1}(X), z) = X + z\,a(\varphi^{-1}(X)) : \overline{\varpi} \times [-k, k] \to \mathbf{R}^N\,. \tag{3.14}$$

Define the following *signed distance function* to the hypersurface $\overline{\varpi}$ in the region $\overline{\mathbb{S}_h(\omega)}$

$$b_\omega(\Phi(\zeta, \zeta^N)) \stackrel{\text{def}}{=} \begin{cases} d_\omega(\Phi(\zeta, \zeta^N)), & \text{if } \zeta^N \geq 0, \\ -d_\omega(\Phi(\zeta, \zeta^N)), & \text{if } \zeta^N < 0. \end{cases} \tag{3.15}$$

When the set of projections $\Pi_\omega(y)$ of y onto $\overline{\varpi}$ is a singleton, we denote it by $p_\omega(y)$.

We now give a constructive proof of Theorem 3.9 in Anicic, Le Dret and Raoult (2004) and, in addition, the expression of the inverse of Φ in terms of

p_ω and b_ω. To do that, we need the following additional assumption on the bounded open subset U of R^{N-1}:

$$(H_3) \quad \left| \begin{array}{l} U \text{ is connected and} \\ \exists C_U \text{ such that } \forall \xi, \zeta \in \overline{U}, \quad d_U(\xi,\zeta) \leq C_U\, |\xi - \zeta|, \end{array} \right. \tag{3.16}$$

where d_U denotes the geodesic distance in U. Recalling the following lemma, we could use the stronger condition of Anicic, Le Dret and Raoult (2004) but we shall see later on that it is not necessary:

$$(H_{3L}) \quad U \text{ is connected and Lipschitzian.} \tag{3.17}$$

Lemma 3.1 *(Anicic, Le Dret and Raoult, 2004, Proposition A.1) Assume that U is a bounded, open, connected, and Lipschitzian domain in R^{N-1}. Then U verifies assumption (H_3).*

We now need a few key properties.

Lemma 3.2 *Assume that (H_1), (H_2), and (H_3) are verified.*

(i) *For all $\xi, \zeta \in \overline{U}$,*

$$|(\varphi(\zeta) - \varphi(\xi)) \cdot a(\xi)| \leq C_U^2 C\alpha\, |\zeta - \xi|^2 . \tag{3.18}$$

(ii) *For all $\xi \in \overline{U}$ and $|\xi^N| < \bar{h} \stackrel{\text{def}}{=} c^2/(2C_U^2 C\alpha)$*

$$\Pi_\omega(\Phi(\xi,\xi^N)) = \{\varphi(\xi)\},\ p_\omega(\Phi(\xi,\xi^N)) = \varphi(\xi),\ \textit{and}\ b_\omega(\Phi(\xi,\xi^N)) = \xi^N. \tag{3.19}$$

(iii) *For all h, $0 < h < \bar{h}$, $p_\omega \in C^{0,1}(\overline{\mathbb{S}_h(\omega)})^N$ and $b_\omega \in C^{1,1}(\overline{\mathbb{S}_h(\omega)})$. Moreover,*

$$\nabla b_\omega = \nabla b_\omega \circ p_\omega = a \circ \varphi^{-1} \circ p_\omega \ \textit{in}\ \mathbb{S}_h(\omega). \tag{3.20}$$

Recalling that $\overline{\mathbb{S}_h(\omega)} = \Phi(\overline{U} \times [-h,h])$, we get the following results.

Theorem 3.3 *Assume that (H_1), (H_2), and (H_3) are verified and let h, $0 < h < \bar{h} \stackrel{\text{def}}{=} c^2/(2C_U^2 C\alpha)$. The mapping $\Phi : \overline{U} \times [-h,h] \to \overline{\mathbb{S}_h(\omega)}$ is bijective and bi-Lipschitzian, and*

$$y \mapsto \Phi^{-1}(y) = (\varphi^{-1}(p_\omega(y)), b_\omega(y)) : \overline{\mathbb{S}_h(\omega)} \to \overline{U} \times [-h,h]. \tag{3.21}$$

Remark 3.1 As we have seen in the previous section, Assumption (H_2) on the normal field a effectively controls the smoothness of the hypersurface ω in the normal direction. There is a unique tangent plane and a unique one-dimensional normal field at *every* point *without* the additional assumption (H_3). In other words, ω ignores the singularities of the mapping φ in the normal direction. Yet, in the tangential direction, the choice of a bi-Lipschitzian parametrization φ of ω and the assumptions (H_{3L}) that U be Lipschitzian and

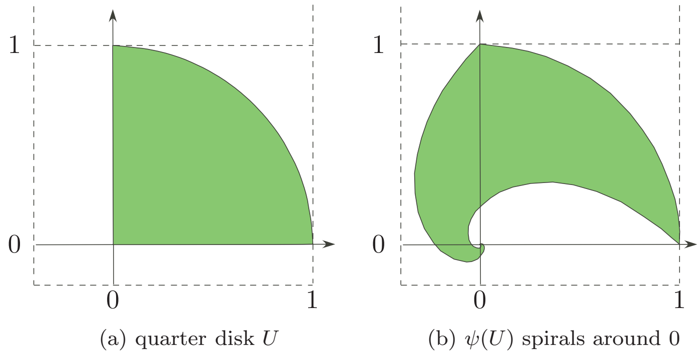


Figure 6. Bi-Lipschitzian transformation of Adams, Aronszajn and Smith (1967)

connected are not sufficient to make the *lateral boundary* of $\mathbb{S}_h(\omega)$ a Lipschitzian hypersurface even in the case of a plate in $\mathbf{R}^3$. Recall the following example from Adams, Aronszajn and Smith (1967), that shows that the transformation of a Lipschitzian domain by a bi-Lipschitzian mapping is not necessary a Lipschitzian domain.

Example 3.1 (Fig. 6) Consider the open convex connected domain $U = \{\rho\, e^{i\theta} : 0 < \rho < 1, 0 < \theta < \pi/2\}$ in $\mathbf{R}^2$ and its image $\omega = \psi(U)$ in $\mathbf{R}^2$ by the following bi-Lipschitzian mapping ψ from $\mathbf{R}^2$ to $\mathbf{R}^2$:

$$\psi(\rho\, e^{i\theta}) \stackrel{\text{def}}{=} \rho\, e^{i(\theta - \log \rho)}, \quad \psi^{-1}(\rho\, e^{i\theta}) = \rho\, e^{i(\theta + \log \rho)} \quad \text{(in polar coordinates).}$$

It is readily seen that as ρ goes to zero the image of the two parts of the boundary of U corresponding to $\theta = 0$ and $\theta = \pi/2$ begin to spiral around the origin. As a result ω is not locally the epigraph of a function (and a fortiori of a Lipschitzian function) at the origin as shown in Fig. 6. Choose $\varphi : R^2 \to \mathbf{R}^3$, $\varphi(\zeta) = (\psi(\zeta), 0)$. This is a bi-Lipschitzian representation of the plane and the normal field is constant: $a(\zeta) = (0, 0, 1)$. So the image ω of the Lipschitzian domain U lies in the plane and all three assumptions (H_1), (H_2), and (H_{3L}) are verified.

Remark 3.2 Even if Φ is a bi-Lipschitzian mapping under assumptions (H_1), (H_2), and (H_{3L}) for all h, the *lateral boundary* of the open set $\mathbb{S}_h(\omega)$ is generally not Lipschitzian as illustrated for the plate of thickness $2h$ of Example 3.1. Results from the *theory of linear elasticity* assuming a Lipschitzian elastic body cannot be directly used. This contradicts the following statement from Anicic, Le Dret and Raoult (2004, Remark in page 1290): *"Note that this result shows that the boundary of the three-dimensional shell is Lipschitz, hence the three-dimensional linearized elasticity problem is well-posed".*

Proof of Lemma 3.2 (i) The proof is based on the argument of Anicic, Le Dret and Raoult (2004, Lemma 3.5). Fix $\xi \in \overline{U}$ and consider the Lipschitz function $\zeta \mapsto f_\xi(\zeta) = [\varphi(\zeta) - \varphi(\xi)] \cdot a(\xi) : \overline{U} \to \mathbf{R}$. For almost all $\zeta \in U$, $\nabla f_\xi(\zeta) = D\varphi(\zeta)^* a(\xi)$. Since, by construction, $D\varphi(\zeta)^* a(\zeta) = 0$ a. e. in U, $\nabla f_\xi(\zeta) = D\varphi(\zeta)^*[a(\xi) - a(\zeta)]$ and for almost all $\zeta \in U$

$$|\nabla f_\xi(\zeta)| \leq |D\varphi(\zeta)|\,|a(\xi) - a(\zeta)| \leq \alpha\,|D\varphi(\zeta)|\,|\xi - \zeta| \leq \alpha\,C|\,|\xi - \zeta|.$$

Since Assumption (H_3) on U is verified, for all $\xi' \in \overline{U}$, the geodesic path between ξ' and ξ is contained in the closure of the ball $B_{d_U(\xi',\xi)}(\xi)$ and

$$|f_\xi(\xi') - f_\xi(\xi)| \leq \|\nabla f_\xi\|_{L^\infty(B_{d_U(\xi',\xi)}(\xi)\cap U)} d_U(\xi',\xi) \leq C_U^2 C\alpha|\xi' - \xi|^2.$$

(ii) Let $y = \Phi(\xi, \xi^N)$, for some $\xi \in \overline{U}$. We show that for $|\xi^N| < c^2/(2C_U^2 C\alpha)$, $\varphi(\xi)$ is the unique projection of y onto $\overline{\omega}$. Given any $\zeta \in \overline{U}$, $\zeta \neq \xi$, consider the difference

$$\begin{aligned}
|y - \varphi(\zeta)|^2 - |y - \varphi(\xi)|^2 &= |\varphi(\zeta) - \varphi(\xi)|^2 + 2\xi^N\, a(\xi) \cdot (\varphi(\xi) - \varphi(\zeta)) \\
&\geq c^2|\xi - \zeta|^2 - 2|\xi^N|\;|a(\xi) \cdot (\varphi(\xi) - \varphi(\zeta))| \\
&\geq c^2|\xi - \zeta|^2 - 2|\xi^N|\, C_U^2 C\alpha\,|\xi - \zeta|^2 \\
&= \left(c^2 - 2\,C_U^2 C\alpha\,|\xi^N|\right)|\xi - \zeta|^2 > 0
\end{aligned}$$

from part (i) and the fact that $|\xi^N| < c^2/(2C_U^2 C\alpha)$. Hence $\varphi(\xi)$ is the unique projection of y onto $\overline{\omega}$ and $p_\omega(\Phi(\xi, \xi^N)) = \varphi(\xi)$. Moreover, $d_\omega(\Phi(\xi, \xi^N)) = |\xi^N|$ and by definition of b_ω, $b_\omega(\Phi(\xi, \xi^N)) = \xi^N$.

(iii) For completeness we provide the proof that p_ω is Lipschitzian on $\overline{\mathbb{S}_h(\omega))}$ (see Delfour and Zolésio, 2001, Thm 7.1, Ch. 4, p. 192–193, and Federer, 1959). We proceed in two steps.

(Step 1) We first prove a local Lipschitz property. For all $x \in \overline{\omega}$, $y \in \Phi(\overline{U} \times [-h, h])$, and $t > 0$ such that $td_\omega(y) < \bar{h}$

$$\begin{aligned}
&|x - [p_\omega(y) + t(y - p_\omega(y))]|^2 \geq d_\Omega(p_\omega(y) + t(y - p_\omega(y))^2 = t^2 d_\omega(y)^2 \\
&|x - p_\omega(y)|^2 + t^2|y - p_\omega(y)|^2 - 2t(y - p_\omega(y)) \cdot (x - p_\omega(y)) \geq t^2 d_\omega(y)^2 \\
\Rightarrow \quad &-\frac{1}{2t}|x - p_\omega(y)|^2 \leq -(y - p_\omega(y)) \cdot (x - p_\omega(y)).
\end{aligned} \tag{3.22}$$

Fix $y \in \Phi(\overline{U} \times [-h, h])$. Since $h < \bar{h}$, there exists $\rho > 0$ such that $d_\omega(y) + \rho < \bar{h}$. Pick $\rho = (\bar{h} - h)/2$. Let $t = \bar{h}/(d_\omega(y) + \rho) =$ (which is strictly greater than 1). For all $z \in B_\rho(y)$, $d_\omega(z) < d_\omega(y) + \rho$ and $td_\omega(z) < \bar{h}$. Hence, from inequality (3.22), for all y_1 and y_2 in $B_\rho(y)$, with the pairs $(y_1, p_\omega(y_2))$ and $(y_2, p_\omega(y_1))$

$$|p_\omega(y_2) - p_\omega(y_1)| \leq \frac{\bar{h}}{\bar{h} - (d_\omega(y) + \rho)}\,|y_2 - y_1| = \frac{2\bar{h}}{\bar{h} - h}\,|y_2 - y_1|\,.$$

(Step 2) By contradiction. Assume that there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\Phi(\overline{U} \times [-h,h])$ such that

$$\forall n \geq 1, \quad (|p_\omega(y_n) - p_\omega(x_n))/|y_n - x_n| \geq n. \tag{3.23}$$

There exist subsequences, still indexed by n, and points x and y in $\Phi(\overline{U}\times[-h,h])$ such that $y_n \to y$ and $x_n \to x$ in $\Phi(\overline{U} \times [-h,h])$ since $\Phi(\overline{U} \times [-h,h])$ is closed. But in view of our hypothesis (3.23) $y = x$. Associate with y the ρ of Step 1. There exists N such that for all $n > N$, x_n and y_n belong to $B_\rho(y)$ and from Step 1 we get the following contradiction

$$\forall n > N, \quad n \leq (|p_\omega(y_n) - p_\omega(x_n)|)/|y_n - x_n| \leq \bar{h}/(\bar{h} - (d_\omega(y) + \rho)).$$

This proves that $p_\omega \in C^{0,1}(\overline{\mathbb{S}_h(\omega)})$.

By construction for each $y \in \overline{\mathbb{S}_h(\omega)})$, there exists a unique $(\xi, \xi^N) \in \overline{U} \times [-h,h]$ such that $y = \Phi(\xi, \xi^N) = \varphi(\xi) + \xi^N a(\xi)$. From part (ii), $p_\omega(y) = \varphi(\xi)$ and $b_\omega(y) = \xi^N$. Therefore

$$b_\omega(y) = (y - p_\omega(y)) \cdot a(\xi) = (y - p_\omega(y)) \cdot a(\varphi^{-1}(p_\omega(y)))$$

and $b_\omega \in C^{0,1}(\overline{\mathbb{S}_h(\omega)})$ as the inner product of two Lipschitzian vector functions. In addition, since the projection exists,

$$\begin{aligned}
&p_\omega(y) = y - \frac{1}{2}\nabla b^2_\omega(y) \quad \Rightarrow b_\omega(y) a(\varphi^{-1}(p_\omega(y))) = \xi^N a(\xi) = \frac{1}{2}\nabla b^2_\Omega(y)\\
&\Rightarrow \forall y \notin \overline{\omega}, \quad \nabla b_\omega(y) = a(\varphi^{-1}(p_\omega(y))).
\end{aligned}$$

Since Φ is bi-Lipschitzian, the image $\overline{\omega}$ of $\overline{U} \times \{0\}$ has zero measure in $\mathbf{R}^N$ and the L^2-vector function ∇b_ω is almost everywhere equal to the Lipschitzian vector function $a \circ \varphi^{-1} \circ p_\omega$. Therefore, $b_\omega \in C^{1,1}(\overline{\mathbb{S}_h(\omega)})$. In particular, $\nabla b_\omega = a \circ \varphi^{-1}$ on $\overline{\omega}$ and hence $\nabla b_\omega = \nabla b_\omega \circ p_\omega$ in $\mathbb{S}_h(\omega)$. ∎

Proof of Theorem 3.3 The mapping $\Phi : \overline{U}\times\,]-\bar{h},\bar{h}[\to \mathbf{R}^N$ is well-defined and Lipschitzian. Given two points (ξ,ξ^N) and (ζ,ζ^N) in $\overline{U} \times [-h,h]$, such that $\Phi(\xi,\xi^N) = y = \Phi(\zeta,\zeta^N)$, by uniqueness of the projection (Lemma 3.2 (ii)) $\varphi(\xi) = p_\omega(y) = \varphi(\zeta)$. But φ is one-to-one and hence $\xi = \zeta \in \overline{U}$. Moreover, always from Lemma 3.2 (ii), $\xi^N = b_\omega(y) = \zeta^N \in [-h,h]$. This proves the injectivity of the mapping. Moreover, from Lemma 3.2 (iii) the map

$$y \mapsto \Psi(y) \stackrel{\text{def}}{=} (\varphi^{-1}(p_\omega(y)), b_\omega(y)) : \Phi(\overline{U} \times [-h,h]) \to \overline{U} \times [-h,h] \tag{3.24}$$

is well defined and Lipschitzian. It is easy to verify that

$$\Psi(\Phi(\xi,\xi^N)) = (\varphi^{-1}(p_\omega(\Phi(\xi,\xi^N))), b_\omega(\Phi(\xi,\xi^N)) = (\varphi^{-1}(\varphi(\xi)),\xi^N) = (\xi,\xi^N).$$

Similarly, given y, there exists (ξ,ξ^N) such that $y = \Phi(\xi,\xi^N)$, and

$$\Phi(\Psi(y)) = \Phi(\Psi(\Phi(\xi,\xi^N))) = \Phi(\xi,\xi^N) = y.$$

Hence $\Phi^{-1} = \Psi$. This proves that Φ is bi-Lipschitzian. ∎

3.3. The hypersurface ω is of class $C^{1,1}$

We now complete the characterization of the hypersurface ω and make the connection between Φ and $\tilde{\Phi}$ and the intrinsic mapping T defined by (2.11) in Theorem 2.2 and between the sets $\mathbb{S}_h(\omega)$ and Im T.

THEOREM 3.4 *Assume that assumptions* (H_1), (H_2), *and* (H_3) *are verified and let* h, $0 < h < \bar{h}$. *Denote by* Ω *the open domain* $\Phi(U\times\,]-h,0[\,)$ *and by* Γ *its boundary. The hypersurface* ω *is locally* $C^{1,1}$, *that is, for each* $x \in \omega$, *there exists* $r(x) > 0$ *such that* $b_\Omega \in C^{1,1}(B_{r(x)}(x))$ *and, hence,* $\omega \cap B_{r(x)}(x)$ *is of class* $C^{1,1}$ *in* $B_{r(x)}(x)$. *Moreover, its normal and second fundamental form are given by*

$$\nabla b_\Omega|_\omega = a \circ \varphi^{-1} = \nabla b_\omega|_\omega \in C^{0,1}(\overline{\omega})^N \tag{3.25}$$

$$D^2 b_\Omega|_\omega = \left\{Da\,[(D\varphi)^* D\varphi]^{-1}(D\varphi)^*\right\} \circ \varphi^{-1} = D^2 b_\omega|_\omega \in L^\infty(\omega)^{N\times N}. \tag{3.26}$$

In addition, $\tilde{\Phi} = T$ *on* $\omega\times\,]-h,h[$ *and* $\mathrm{Im}\,T = \mathrm{Im}\,\tilde{\Phi} = \mathrm{Im}\,\Phi = \mathbb{S}_h(\omega)$.

Note that no gain is achieved through this parametrization (see Theorem 2.2 with the mapping (2.11) replaced by the mapping (2.15)).

Proof. The boundary Γ of Ω is equal to $\overline{\omega} \cup \Phi(U \times \{-h\}\,) \cup \Phi(\partial U \times [-h,0[\,)$. Given $x \in \omega$, there exists $\xi \in U$ such that $x = \varphi(\xi)$ and $r = r(x) > 0$ such that $B_{3r}(\varphi(\xi)) \subset \mathbb{S}_h(\omega)$. Consider a point $y \in B_r(\varphi(\xi))$. The set of projections $\Pi_\Gamma(y)$ of y onto Γ lies in the ball $B_{2r}(\varphi(\xi))$ and the points of $\Pi_\Gamma(B_r(\varphi(\xi)))$ are at least at a distance r from the boundary of $\mathbb{S}_h(\omega)$, and a fortiori from the points of $\Phi(U \times \{-h\}\,) \cup \Phi(\partial U \times [-h,0])$. Therefore $\Pi_\Gamma(B_r(\varphi(\xi))) \subset \omega$. So, from Lemma 3.2 (ii), for any $y \in B_r(\varphi(\xi))$, $\Pi_\Gamma(y) = \Pi_\omega(y)$ is a singleton, $b_\Omega(y) = b_\omega(y)$, and $p_\Gamma = p_\omega$. From Lemma 3.2 (iii), $b_\Omega = b_\omega \in C^{1,1}(\overline{B_r(x)})$ From Delfour and Zolésio (1994) or Delfour and Zolésio (2001, Chapter 5, §4.2), this implies that, for each $x \in \omega$, ω is a $C^{1,1}$ hypersurface in the neighborhood $B_r(x)$. The restrictions of $\nabla b_\Omega = a \circ \varphi^{-1} \in C^{0,1}(\overline{\omega})^N$ and $D^2 b_\Omega \in L^\infty(\omega)^{N\times N}$ to the $C^{1,1}$-subset $\omega \subset \Gamma$ coincide with the normal and the second fundamental form of ω.

If T is the mapping defined by (2.11) in Theorem 2.2 of § 2, we get

$$T(X,z) = X + z\nabla b_\Omega(X) = X + z(a \circ \varphi^{-1}(X) = \tilde{\Phi}(X,z)$$

by the definition (3.14) of $\tilde{\Phi}$. Hence $T(\omega\times\,]-h,h[\,) = \tilde{\Phi}(\omega\times\,]-h,h[\,) = \Phi(\omega\times\,]-h,h[\,) = \mathbb{S}_h(\omega)$. ∎

4. Hypersurfaces defined from a domain with facets

Since it is not possible to represent a sphere or a torus from a bilipschitzian mapping φ from some domain U in the hyperplane R^{N-1}, we replace U by a domain with facets. Each facet will lie in an $(N-1)$-dimensional affine

Figure 7. Example of a domain U with facets

subspace A of $\mathbf{R}^N$ allowing different angles between facets. The good news is that everything we have proved in § 3 will remain true.

Definition 4.1 (i) *A* facet U *in* $\mathbf{R}^N$ *is a bounded, open, connected subset of an* $(N-1)$*-dimensional affine subspace of* $\mathbf{R}^N$ *such that its geodesic distance satisfies the condition*

$$\exists C_U,\, \forall \zeta, \xi \in U, \quad d_U(\zeta, \xi) \leq C_U\, |\zeta - \xi|,$$

where d_U *denotes the geodesic distance in* $\overline{U}$*.*

(ii) *Given* $n \geq 1$ *facets* U_i*,* $1 \leq i \leq n$*, in* $\mathbf{R}^N$ *such that*

a) $\forall i \neq j$, $U_i \cap U_j = \varnothing$,

b) *for all pairs* $i \neq j$ *such that* $\overline{U_i} \cap \overline{U_j} \neq \varnothing$, $H_{N-1}(\overline{U_i} \cap \overline{U_j}) = 0$, *where* H_{N-1} *is the* $(N-1)$*-dimensional Hausdorff measure in* $\mathbf{R}^N$,

c) $\overline{\cup_{i=1}^n U_i} = \cup_{i=1}^n \overline{U_i}$,

we say that the set

$$U \stackrel{\text{def}}{=} rel\ int \cup_{i=1}^n \overline{U_i} \tag{4.1}$$

is a domain with n facets*.*

From the above definition

$$\overline{\cup_{i=1}^n U_i} = \overline{U} = \cup_{i=1}^n \overline{U_i} \text{ and } H_{N-1}\left(\cup_{\substack{i,j=1,\dots,n \\ i\neq j}} \overline{U_i} \cap \overline{U_j}\right) = 0. \tag{4.2}$$

Remark 4.1 In the previous sections all the results have been established for a single facet U in the hyperplane $\{e_N\}^\perp$. They still hold for a domain U in an $(N-1)$-dimensional affine subspace A of $\mathbf{R}^N$ by choosing an orthonormal basis $\{e_i : 1 \leq i \leq N\}$ in $\mathbf{R}^N$ such that e_N is normal to A. So, there exists a scalar

c such that $A = \{\sum_{i=1}^{N-1} \xi_i e_i + c e_N : \forall \xi_i \in \mathbf{R},\, 1 \le i \le N-1\}$ and A can be identified with $R^{N-1} = \{e_N\}^\perp$ through the diffeomorphism $x \mapsto x - c e_N : A \to R^{N-1}$. This translates the domain U, but does not change the properties of the vector functions φ and a.

Let U be a bounded domain in $\mathbf{R}^N$ with facets and $\varphi : U \to \mathbf{R}^{\mathrm{N}}$ be a mapping with the following properties: there exist $c > 0$ and $C > 0$ such that

$$\text{Assumption } (H_1)\text{:}\ \ \forall \xi, \zeta \in U,\ \ c\,|\zeta - \xi| \le |\varphi(\zeta) - \varphi(\xi)| \le C\,|\zeta - \xi|, \tag{4.3}$$

where $c = \inf_{\zeta \ne \xi \in U} |\varphi(\zeta) - \varphi(\xi))/|\zeta - \xi|$. Let φ_i be the restriction of φ to $\overline{U_i}$. For almost all $\xi \in U_i$, we can construct a normal a and a surface density in U and we get a generalization of Theorem 3.1.

THEOREM 4.1 *Let U be a bounded domain in $\mathbf{R}^N$ with facets. Assume that $\varphi : U \to \mathbf{R}^{\mathrm{N}}$ verifies assumption (H_1) and let $\tilde{U} = \{\xi \in \cup_{i=1}^n U_i : \varphi$ is differentiable at $\xi\}$. Then, at each point $\xi \in \tilde{U}$,*

$$n(\xi) = \pm b(\xi)/|b(\xi)|,\ \ \det\left[D\varphi(\xi) \vdots \frac{b(\xi)}{|b(\xi)|}\right] = |b(\xi| > 0,\ \ J\varphi(\xi) = |b(\xi)|, \tag{4.4}$$

for all i, $V \in \{d_i\}^\perp$, and $V^N \in \mathbf{R}$

$$c^2|V|^2 + |V^N|^2 \le \left|\left[D\varphi(\xi) \vdots \frac{b(\xi)}{|b(\xi)|}\right]\begin{bmatrix} V \\ V^N \end{bmatrix}\right|^2 \le C^2|V|^2 + |V^N|^2, \tag{4.5}$$

and

$$T_{\varphi(\xi)}\omega = \mathrm{Im}\ D\varphi(\xi) = \{n(\xi)\}^\perp \text{ and } (T_{\varphi(\xi)}\omega)^* = \mathbf{R}\, n(\xi). \tag{4.6}$$

Proof. Same proof as Theorem 3.1 on each U_i. ∎

Now assume that the resulting normal mapping $a(\xi)$ is uniformly Lipschitz on $\tilde{U}$:

$$\text{Assumption } (H_2)\text{: } \exists \alpha > 0 \text{ such that } \forall \xi, \zeta \in \tilde{U},\ |a(\zeta) - a(\xi)| \le \alpha\,|\zeta - \xi|. \tag{4.7}$$

From assumptions b) and c), $\overline{\tilde{U}} = \overline{U}$ and a extends to a (unique) uniformly Lipschitz function, still denoted a, on $\overline{U}$ that verifies assumption (H_2) on $\overline{U}$.

THEOREM 4.2 *Let U be a bounded domain in $\mathbf{R}^N$ with facets. Assume that φ and a verify assumptions (H_1) and (H_2). Then*

$$\forall \xi \in U,\quad T_{\varphi(\xi)}\omega = \{a(\xi)\}^\perp \text{ and } (T_{\varphi(\xi)}\omega)^* = \mathbf{R}\, a(\xi). \tag{4.8}$$

and the parametric surface ω has a unique tangent hyperplane in each point with Lipschitzian normal $a \circ \varphi^{-1}$.

Proof. Same proof as in Theorem 3.2. ∎

Now define the Lipschizian mapping Φ, the intrinsic Lipschizian mapping (3.14), and the *signed distance function* to the hypersurface $\overline{\omega}$ in the region $\mathbb{S}_h(\omega)$ as in § 3.2. Finally, introduce the following assumption on the underlying domain U with facets:

$$(H_3)\quad \left|\begin{array}{l} U \text{ is connected and} \\ \exists C_U \text{ such that } \forall \xi,\zeta \in \overline{U}, \quad d_U(\xi,\zeta) \le C_U\,|\xi-\zeta|, \end{array}\right. \tag{4.9}$$

where d_U denotes the geodesic distance in U. We get the generalization of Lemma 3.2 and Theorems 3.3 and 3.4 using essentially the same proofs.

Theorem 4.3 *Let U be a bounded domain in $\mathbf{R}^N$ with facets. Assume that (H_1), (H_2), and (H_3) are verified and let h, $0 < h < \bar{h} \stackrel{\text{def}}{=} c^2/(2C_U^2 C\alpha)$. The mapping $\Phi : \overline{U} \times [-h,h] \to \overline{\mathbb{S}_h(\omega)}$ is bijective and bi-Lipschitzian, and*

$$y \mapsto \Phi^{-1}(y) = (\varphi^{-1}(p_\omega(y)), b_\omega(y)) : \overline{\mathbb{S}_h(\omega)} \to \overline{U} \times [-h,h]. \tag{4.10}$$

Theorem 4.4 *Let U be a bounded domain in $\mathbf{R}^N$ with facets. Assume that assumptions (H_1), (H_2), and (H_3) are verified and let h, $0 < h < \bar{h}$. Denote by Ω the open domain $\Phi(U\times\,]-h,0[)$ and by Γ its boundary. The hypersurface ω is locally $C^{1,1}$, that is, for each $x \in \omega$, there exists $r(x) > 0$ such that $b_\Omega \in C^{1,1}(B_{r(x)}(x))$ and, hence, $\omega \cap B_{r(x)}(x)$ is of class $C^{1,1}$ in $B_{r(x)}(x)$. Moreover, its normal and second fundamental form are given by*

$$\nabla b_\Omega|_\omega = a \circ \varphi^{-1} = \nabla b_\omega|_\omega \in C^{0,1}(\overline{\omega})^N \tag{4.11}$$

$$D^2 b_\Omega|_\omega = \left\{Da\,[(D\varphi)^* D\varphi]^{-1}(D\varphi)^*\right\} \circ \varphi^{-1} = D^2 b_\omega|_\omega \in L^\infty(\omega)^{N\times N}. \tag{4.12}$$

Moreover, $\tilde{\Phi} = T$ on $\omega\times\,]-h,h[$ and $\operatorname{Im} T = \operatorname{Im}\tilde{\Phi} = \operatorname{Im}\Phi = \mathbb{S}_h(\omega)$.

5. G_1-joins of K-regular and $C^{1,1}$-patches

For completeness we first recall and add some definitions.

Definition 5.1 *Given an open subset U of R^{N-1} and a mapping $\varphi : \overline{U} \to \mathbf{R}^N$, we say that the set $\varphi(\overline{U})$ is* not self-intersecting *if φ is injective.*

Following the terminology of Le Dret (2004, Definition 2.4) in dimension three, a *K-regular patch* $\overline{\omega}$ is an hypersurface specified by the two mappings φ and a from $U \subset R^{N-1} \to \mathbf{R}^N$ that verify assumptions (H_1), (H_2), and (H_{3L}) with constants c, C, and α. From assumption (H_1), such surfaces are not self-intersecting and, from assumptions (H_1), (H_2), and (H_{3L}), they are $C^{1,1}$-hypersurfaces by Theorem 3.4. So, we suggest to use the more descriptive and general terminology *$C^{1,1}$-patch* that emphasizes the purely geometric property which is not only specific of the theory of shells.

Definition 5.2 *A* $C^{1,1}$-patch *is a parametric hypersurface specified by the two mappings* φ *and* a *from* $U \subset R^{N-1} \to \mathbf{R}^N$ *that verify assumptions* (H_1) *to* (H_3) *with constants* c, C, *and* α.

By definition, a K-regular patch is a $C^{1,1}$-patch since assumption (H_{3L}) implies assumption (H_3).

One important contribution in the paper of Le Dret is an accurate definition of a G_1-join (Le Dret, 2004, Definition 2.6) and the proof that for two contiguous K-regular patches $\overline{\omega}_1 = \varphi_1(\overline{U}_1)$ and $\overline{\omega}_2 = \varphi_2(\overline{U}_2)$ such that $\overline{\omega}_1 \cup \overline{\omega}_2$ is not self-intersecting with a G_1-join along $\overline{U}_1 \cap \overline{U}_2$, rel int $\overline{\omega}_1 \cup \overline{\omega}_2$ satisfies assumptions (H_1), (H_2), and (H_{3L}) (Le Dret, 2004, Lemma 3.2), that is, *it is a* $C^{1,1}$*-hypersurface and the* G_1*-join along* $\overline{U}_1 \cap \overline{U}_2$ *is in fact a* $C^{1,1}$*-join.*

To complete this section we extend this last result to a finite number of $C^{1,1}$-patches in $\mathbf{R}^N$ defined on a domain with facets.

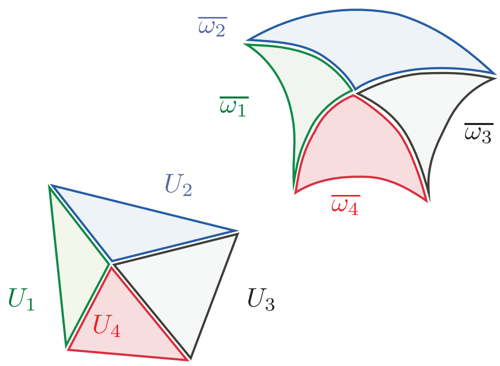


Figure 8. From $C^{1,1}$-patches $\{\omega_i\}$ to a globally $C^{1,1}$-patch ω

Theorem 5.1 *Let* U *be a bounded connected domain in* $\mathbf{R}^N$ *with facets. Assume that the facets* $\omega_i = \varphi_i(U_i)$ *specified by* (U_i, φ_i, a_i) *are* $C^{1,1}$*-patches such that for all* $i \neq j$ *such that* $\overline{\omega}_i \cap \overline{\omega}_j \neq \varnothing$

a) *there exists* C_{ij} *such that for all* ξ *and* ζ *in* $\overline{U}_i \cup \overline{U}_j$, $d_{\overline{U}_i \cup \overline{U}_j}(\xi, \zeta) \leq C_{ij}\,|\xi - \zeta|$,

b) $\varphi_i(\xi) = \varphi_j(\xi)$, *for all* $\xi \in \overline{\omega}_i \cap \overline{\omega}_j$,

c) $a_i(\xi) = a_j(\xi)$, *for all* $\xi \in \overline{\omega}_i \cap \overline{\omega}_j$,

d) $\overline{\omega}_i \cap \overline{\omega}_j \subset \varphi(\overline{U_i} \cap \overline{U_j})$,

e) *given any sequences* $\{\zeta_{in}\} \subset U_i$ *and* $\{\zeta_{jn}\} \subset U_j$ *that converge to some point* $\xi \in \overline{U}_i \cap \overline{U}_j$ *and a corresponding sequence* $\{\xi_n\} \subset \overline{U}_i \cap \overline{U}_j$ *such that* ξ_n *lie on the geodesic between* ζ_{ni} *and* ζ_{nj}, *the angle between any limit vectors* τ_i *and* τ_j *of the sequences* $(\varphi_i(\zeta_{in}) - \varphi_i(\xi_n))/|\zeta_{in} - \xi_n|$ *and* $(\varphi_j(\zeta_{jn}) - \varphi_j(\xi_n))/|\zeta_{jn} - \xi_n|$ *is nonzero.*

Then
(i) *U satisfies assumption (H_3),*
(ii) *the maps φ and $a : \overline{U} \to \mathbf{R}^N$,*

$$\varphi(\zeta) = \varphi_i(\zeta), \quad \text{if } \zeta \in \overline{U_i}, \quad a(\zeta) = a_i(\zeta), \quad \text{if } \zeta \in \overline{U_i},$$

are well-defined and Lipschitz continuous on $\overline{U}$,
(iii) *$\overline{\omega}$ is not self-intersecting (φ is injective),*
(iv) *φ satisfies assumption (H_1) and a satisfies assumption (H_2).*
In particular, $\omega = \varphi(U)$ satisfies assumptions (H_1), (H_2), and (H_3), that is ω is a $C^{1,1}$-patch.

Proof. (i) (U verifies assumption (H_3)). By assumption, U is connected. If ξ and ζ both belong to the same $\overline{U}_i$, then $d_U(\xi,\zeta)/|\xi-\zeta|$ is bounded by C_{U_i} and, a fortiori, by $\max_{1\le i\le n} C_{U_i}$. Otherwise, there is a continuous path in $\overline{U}$ successively going through $\overline{U}_{i_0}$, ..., $\overline{U}_{i_\ell}$, ..., $\overline{U}_{i_k}$, and points $\xi_{i_0} \overset{\text{def}}{=} \xi \in \overline{U}_{i_0}$, $\xi_{i_\ell} \in \overline{U}_{i_{\ell-1}} \cap \overline{U}_{i_\ell}$, $1 \le \ell \le k$, $\xi_{i_{k+1}} \overset{\text{def}}{=} \zeta \in \overline{U}_{i_k}$. Therefore

$$d_U(\xi,\zeta) \le \sum_{\ell=0}^{k} d_{U_{i_\ell}}(\xi_{i_\ell},\xi_{i_{\ell+1}}) \le \sum_{\ell=0}^{k} C_{U_{i_\ell}} |\xi_{i_\ell} - \xi_{i_{\ell+1}}| \le \sum_{i=1}^{n} C_{U_i} \operatorname{diam}(U_i)$$

and the geodesic distance is uniformly bounded. We now proceed by contradiction. Assume that there exist sequences $\{\xi_n\}$ and $\{\zeta_n\}$ in $\overline{U}$, $\xi_n \ne \zeta_n$, such that $d_U(\xi_n,\zeta_n)/|\xi_n-\zeta_n|$ goes to $+\infty$. If there exists $N \ge 1$ such that for all $n \ge N$, both ξ_n and ζ_n belong to $\overline{U}_i$, then $d_U(\xi_n,\zeta_n)/|\xi_n-\zeta_n| \le d_{U_i}(\xi_n,\zeta_n)/|\xi_n-\zeta_n| \le C_{U_i}$ and there is a contradiction. Since the number of patches is finite, there exist $i \ne j$ such that for all $k \ge 1$, there exist $n_k \ge k$ such that $\xi_{n_k} \in \overline{U}_i$ and $\zeta_{n_k} \in \overline{U}_j$. By going to sub-subsequences that we relabel with the index n, $\{\xi_n\} \subset \overline{U}_i$ and $\{\zeta_n\} \subset \overline{U}_j$ and there exist $\xi \in \overline{U}_i$ and $\zeta \in \overline{U}_j$ such that $\xi_n \to \xi$ and $\zeta_n \to \zeta$; if $\xi \ne \zeta$, then

$$\frac{c}{|\xi-\zeta|} \leftarrow \frac{c}{|\xi_n-\zeta_n|} \ge \frac{d_U(\xi_n,\zeta_n)}{|\xi_n-\zeta_n|} \to \infty,$$

a contradiction. Therefore $\xi = \zeta \in \overline{U}_i \cap \overline{U}_j$ and by assumption a)

$$C_{ij} \ge \frac{d_{\overline{U}_i\cup\overline{U}_j}(\xi_n,\zeta_n)}{|\xi_n-\zeta_n|} \ge \frac{d_U(\xi_n,\zeta_n)}{|\xi_n-\zeta_n|} \to \infty$$

another contradiction.

(ii) By assumptions b) and c), φ and a are well-defined on $\overline{U}$. If ξ and ζ both belong to some $\overline{U}_i$ then $|\varphi(\xi)-\varphi(\zeta)| = |\varphi_i(\xi)-\varphi_i(\zeta)| \le C_i|\xi-\zeta|$ and, a fortiori, $|\varphi(\xi)-\varphi(\zeta)| \le \max_{1\le i\le n} C_i|\xi-\zeta|$. Otherwise, since U is connected, there is a continuous path in $\overline{U}$ successively going through the facets $\overline{U}_{i_0}$, ...

$\overline{U}_{i_\ell}$, ..., $\overline{U}_{i_k}$ and some points $\xi_{i_0} \stackrel{\text{def}}{=} \xi \in \overline{U}_{i_0}$, $\xi_{i_\ell} \in \overline{U}_{i_{\ell-1}} \cap \overline{U}_{i_\ell}$, $1 \leq \ell \leq k$, $\xi_{i_{k+1}} \stackrel{\text{def}}{=} \zeta \in \overline{U}_{i_k}$. Therefore

$$\begin{aligned}
|\varphi(\xi) - \varphi(\zeta)| &\leq \sum_{\ell=0}^{k} |\varphi(\xi_{i_\ell}) - \varphi(\xi_{i_{\ell+1}})| = \sum_{\ell=0}^{k} |\varphi_{i_\ell}(\xi_{i_\ell}) - \varphi_{i_\ell}(\xi_{i_{\ell+1}})| \\
&\leq \sum_{\ell=0}^{k} C_{i_\ell} |\xi_{i_\ell} - \xi_{i_{\ell+1}}| \leq \sup_{i=1,\dots,n} C_i \sum_{\ell=0}^{k} |\xi_{i_\ell} - \xi_{i_{\ell+1}}| \\
&\leq \sup_{i=1,\dots,n} C_i \sum_{\ell=0}^{k} d_{U_{i_\ell}}(\xi_{i_\ell}, \xi_{i_{\ell+1}}).
\end{aligned}$$

Now taking the infimum over all such paths

$$|\varphi(\xi) - \varphi(\zeta)| \leq \sup_{i=1,\dots,n} C_i \, d_U(\xi, \zeta) \leq \sup_{i=1,\dots,n} C_i \, C_U \, |\xi - \zeta|,$$

from part (i). Same proof for the mapping a.

(iii) (φ is injective). Consider two points ζ and ξ in $\overline{U}$ such that $\varphi(\zeta) = \varphi(\xi)$. If both points belong to $\overline{U}_i$ for some i, $\varphi_i(\zeta) = \varphi_i(\xi)$ and, by injectivity of φ_i, $\zeta = \xi$. If $\zeta \in \overline{U}_i$ and $\xi \in \overline{U}_j$ for some $i \neq j$, then $\varphi(\zeta) = \varphi_i(\zeta) \in \overline{\omega}_i$, $\varphi(\xi) = \varphi_j(\xi) \in \overline{\omega}_j$, and $\varphi(\zeta) \in \overline{\omega}_i \cap \overline{\omega}_j \subset \varphi_i(\overline{U}_i \cap \overline{U}_j)$ by assumption d). So, there exists $\delta \in \overline{U}_i \cap \overline{U}_j$ such that $\varphi_i(\zeta) = \varphi_i(\delta) = \varphi_j(\delta) = \varphi_j(\xi)$. But, by injectivity of both φ_i and φ_j, $\zeta = \delta = \xi$ and φ is injective.

(iv) (φ is bi-Lipschitzian). By hypothesis and from part (i), U verifies assumption (H_3). We have proved that φ is Lipschitzian and injective on $\overline{U}$. So, in order to verify assumption (H_1), it remains to prove that

$$\inf_{\zeta_1 \neq \zeta_2 \in \overline{U}} |\varphi(\zeta_2) - \varphi(\zeta_1)|/|\zeta_2 - \zeta_1| > 0. \tag{5.1}$$

Following Le Dret (2004, Lemma 3.2), we proceed by contradiction. Assume that there exist sequences $\{\zeta_n\}$ and $\{\xi_n\}$, $\xi_n \neq \zeta_n$, such that

$$\forall n, \quad |\varphi(\zeta_n) - \varphi(\xi_n)| \leq |\zeta_n - \xi_n|/n. \tag{5.2}$$

Let $c \stackrel{\text{def}}{=} \min_{i=1}^{n} \{c_i\}$. For $n > 1/c$, ζ_n and ξ_n cannot belong to the same $\overline{U_i}$. So, without loss of generality, there exist $i \neq j$, an infinite subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ in $\overline{U}_i$ that converges to some $\zeta \in \overline{U_i}$, and an infinite subsequence of $\{\xi_{n_k}\}$ in $\overline{U_j}$ that converges to some $\xi \in \overline{U_i}$. We relabel the two sub-sub-subsequences $\{\zeta_n\} \subset \overline{U}_i$ and $\{\xi_n\} \subset \overline{U_j}$ such that $\zeta_n \to \zeta \in \overline{U_i}$, and $\xi_n \to \xi \in \overline{U_j}$. Clearly $\varphi(\zeta) = \varphi(\xi)$ and, by injectivity of φ in part (iii), $\zeta = \xi \in \overline{U_i} \cap \overline{U_j}$.

To each pair (ζ_n, ξ_n), associate $\delta_n \in \overline{U_i} \cap \overline{U_j}$ such that $d_{\overline{U}_i \cup \overline{U}_j}(\zeta_n, \xi_n) = d_{U_i}(\zeta_n, \delta_n) + d_{U_j}(\xi_n, \delta_n)$. In view of assumption a),

$$C_{ij} \geq \frac{d_{\overline{U}_i \cup \overline{U}_j}(\zeta_n, \xi_n)}{|\zeta_n - \xi_n|} = \frac{d_{U_i}(\zeta_n, \delta_n)}{|\zeta_n - \xi_n|} + \frac{d_{U_j}(\xi_n, \delta_n)}{|\zeta_n - \xi_n|} \geq \frac{|\zeta_n - \delta_n|}{|\zeta_n - \xi_n|} + \frac{|\xi_n - \delta_n|}{|\zeta_n - \xi_n|}.$$

So there exist $\beta_i \geq 0$, $i = 1, 2$, and subsequences such that

$$C_{ij} \geq \frac{|\zeta_n - \delta_n|}{|\zeta_n - \xi_n|} + \frac{|\xi_n - \delta_n|}{|\zeta_n - \xi_n|} \to \beta_i + \beta_j.$$

In the other direction,

$$\begin{aligned}&|\zeta_n - \xi_n| \leq d_U(\zeta_n, \xi_n) \leq d_{U_i}(\zeta_n, \delta_n) + d_{U_j}(\xi_n, \delta_n) \leq C_{U_i}|\zeta_n - \delta_n| + C_{U_j}|\delta_n - \xi_n| \\ &\Rightarrow \quad 1 \leq C_{U_i}\beta_i + C_{U_j}\beta_j.\end{aligned}$$

Finally, there exist $\tau_i \neq 0$ and $\tau_j \neq 0$ such that

$$\frac{\varphi_i(\zeta_n) - \varphi_i(\delta_n)}{|\zeta_n - \xi_n|} \to \tau_i \text{ and } \frac{\varphi_j(\xi_n) - \varphi_j(\delta_n)}{|\zeta_n - \xi_n|} \to \tau_j.$$

But

$$\frac{\varphi(\zeta_n) - \varphi(\xi_n)}{|\zeta_n - \xi_n|} = \frac{\varphi(\zeta_n) - \varphi(\delta_n)}{|\zeta_n - \delta_n|}\frac{|\zeta_n - \delta_n|}{|\zeta_n - \xi_n|} - \frac{\varphi(\xi_n) - \varphi(\delta_n)}{|\xi_n - \delta_n|}\frac{|\xi_n - \delta_n|}{|\zeta_n - \xi_n|}$$

and finally, by going to the limit,

$$0 = \beta_i\tau_i - \beta_j\tau_j.$$

If $\beta_i = 0$, then $1 \leq C_{U_i}\beta_i + C_{U_j}\beta_j$ implies that $\beta_j > 0$, $\beta_j\tau_j \neq 0$, and $\tau_j = 0$, a contradiction. So necessarily $\beta_i > 0$ and $\beta_j > 0$. The vectors τ_i and τ_j have a zero angle and this contradicts condition e). ∎

Remark 5.1 It is readily checked that Le Dret (2004, Lemma 3.2) is a special case of this Theorem for $k = 1$.

6. Decomposition of a $C^{1,1}$-hypersurface into $C^{1,1}$-patches over a domain with facets

Conversely, it is also possible to decompose a $C^{1,1}$-hypersurface ω into $C^{1,1}$-patches defined over a domain with facets as long as the size of each facet is sufficiently small.

The construction follows the following scheme.

(A-1) Assumptions of Theorem 2.2 or assumptions (H_1) to (H_3).

Under assumption (A-1), there exists $h > 0$ such that, the mapping

$$X, z \mapsto T(X, z) = X + z\,\nabla b_\omega(X) : \overline{\omega} \times [-h, h] \to \overline{\mathbb{S}_h(\omega)}$$

is bi-Lipschitzian. If ω has no boundary, then we proceed as in Frey and George (2008) but the *convex polytopes* (triangles in dimension $N = 3$) have to be chosen sufficiently small in view of the curvature of the surface.

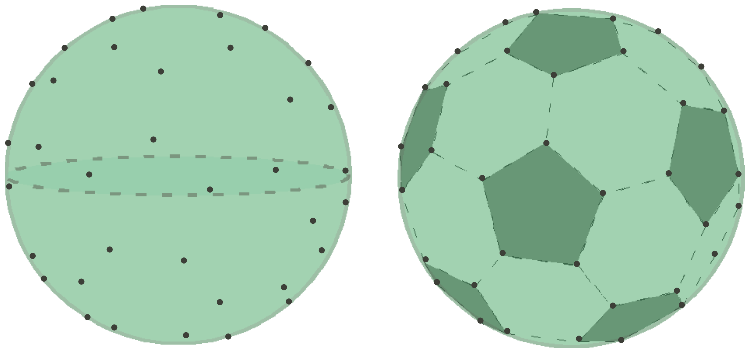


Figure 9. Points on the sphere and associated domain with facets for $N = 3$

(A-2) Choose N neighboring points $\xi_1, \xi_2, \ldots, \xi_N$ on the surface ω such that the vectors $\{\xi_i - \xi_N; i = 1, \ldots N-1\}$ be linearly independent and such that the convex polytope $\Delta = \text{co}\{\xi_1, \xi_2, \ldots \xi_N\}$ with vertices $\xi_1, \xi_2, \ldots, \xi_N$, lies in $\mathbb{S}_h(\omega)$. Denote by ν the normal to the affine subspace $A = \{\xi : (\xi - \xi_N) \cdot \nu = 0\}$ generated by Δ.

This defines the patch $\overline{\omega}_\Delta \overset{\text{def}}{=} p_\omega(\Delta)$. Since $\Delta \subset \mathbb{S}_h(\omega)$, the mapping

$$\xi \mapsto p_\omega(\xi) : \Delta \to \overline{\omega}_\Delta \subset \mathbf{R}^3$$

is well-defined and Lipschitzian. To make p_ω bi-Lipschitzian on Δ, we need to further reduce the size of Δ.

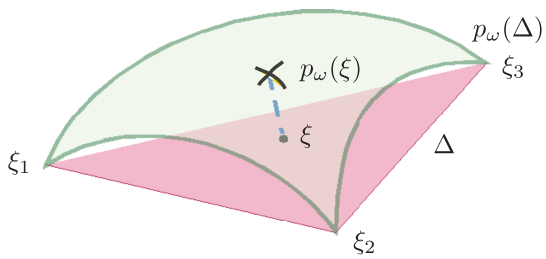


Figure 10. Triangle Δ and $C^{1,1}$-patch $p_\omega(\Delta)$ for $N = 3$

(A-3) Assume that the convex polytope Δ is sufficiently small so that

$$m \overset{\text{def}}{=} \min_{\xi \in \Delta} |n \cdot \nabla b_\omega(\xi)| > 0. \tag{6.1}$$

Remark 6.1 If the convex polytope Δ is chosen in such a way that there exists $\hat{\xi} \in \operatorname{int} \Delta$ such that $b_\omega(\hat{\xi})$ is an extremum of b_ω over Δ, then

$$\begin{aligned}
&\nabla b_\omega(\hat{\xi}) \cdot (\xi - \hat{\xi}) = 0, \quad \forall \xi \in \Delta \\
&\Rightarrow \nabla b_\omega(\hat{\xi}) \cdot (\xi_i - \xi_N) = 0,\, i = 1, \dots, N-1 \quad \Rightarrow \nabla b_\omega(\hat{\xi}) = \pm \nu
\end{aligned}$$

and we can choose $\nu = \nabla b_\omega(\hat{\xi})$. As a result

$$\begin{aligned}
&|\nabla b_\omega(\xi) \cdot \nu| \geq |\nabla b_\omega(\hat{\xi}) \cdot \nu| - |\nabla b_\omega(\hat{\xi}) - \nabla b_\omega(\xi)| \\
&|\nabla b_\omega(\xi) \cdot \nu| \geq 1 - c|\hat{\xi} - \xi| \geq 1 - c \max_{\xi \in \Delta} |\hat{\xi} - \xi|
\end{aligned}$$

and for a sufficiently small triangle Δ, $|\nabla b_\omega(\xi) \cdot \nu|$ is bounded below by a strictly positive constant.

In general, under assumption (A-3), for each $\xi \in \Delta$, $\xi = p_\omega(\xi) + b_\omega(\xi) \nabla b_\omega(p_\omega(\xi))$. Therefore, for all ξ and ζ in Δ

$$\begin{aligned}
&|\xi - \zeta| \\
&\leq |p_\omega(\xi) - p_\omega(\zeta)| + |b_\omega(\xi) \nabla b_\omega(p_\omega(\xi)) - b_\omega(\zeta) \nabla b_\omega(p_\omega(\zeta))| \\
&\leq |p_\omega(\xi) - p_\omega(\zeta)| + |b_\omega(\xi)|\, |\nabla b_\omega(p_\omega(\xi)) - \nabla b_\omega(p_\omega(\zeta))| + |b_\omega(\xi) - b_\omega(\zeta)|\, |\nabla b_\omega(p_\omega(\zeta))| \\
&\leq |p_\omega(\xi) - p_\omega(\zeta)| + hc\, |p_\omega(\xi) - p_\omega(\zeta)| + |b_\omega(\xi) - b_\omega(\zeta)|\,.
\end{aligned}$$

But under assumption (A-3), for each $\xi \in \Delta$, $\nu \cdot \nabla b_\omega(p_\omega(\xi)) \neq 0$ and

$$\nu \cdot (p_\omega(\xi) + b_\omega(\xi) \nabla b_\omega(p_\omega(\xi)) - \xi_N) = 0 \quad \Rightarrow b_\omega(\xi) = -\frac{\nu \cdot (p_\omega(\xi) - \xi_N)}{\nu \cdot \nabla b_\omega(p_\omega(\xi))}.$$

Always by assumption (A-3), the absolute value of the denominator is bounded below and, using the expression for b_ω,

$$\begin{aligned}
&|b_\omega(\xi) - b_\omega(\zeta)| \\
&\leq \left| \frac{\nu \cdot (p_\omega(\xi) - \xi_N)}{\nu \cdot \nabla b_\omega(p_\omega(\xi))} - \frac{\nu \cdot (p_\omega(\zeta) - \xi_N)}{\nu \cdot \nabla b_\omega(p_\omega(\zeta))} \right| \\
&\leq \left| \frac{\nu \cdot (p_\omega(\xi) - \xi_N)\, \nu \cdot \nabla b_\omega(p_\omega(\zeta)) - \nu \cdot (p_\omega(\zeta) - \xi_N)\, \nu \cdot \nabla b_\omega(p_\omega(\xi))}{\nu \cdot \nabla b_\omega(p_\omega(\xi))\, \nu \cdot \nabla b_\omega(p_\omega(\zeta))} \right| \\
&\leq \left| \frac{\nu \cdot (p_\omega(\xi) - p_\omega(\zeta))\, \nu \cdot \nabla b_\omega(p_\omega(\zeta))}{\nu \cdot \nabla b_\omega(p_\omega(\xi))\, \nu \cdot \nabla b_\omega(p_\omega(\zeta))} \right| \\
&\quad + \left| \frac{\nu \cdot (p_\omega(\zeta) - \xi_N)\, \nu \cdot (\nabla b_\omega(p_\omega(\xi)) - \nabla b_\omega(p_\omega(\zeta)))}{\nu \cdot \nabla b_\omega(p_\omega(\xi))\, \nu \cdot \nabla b_\omega(p_\omega(\zeta))} \right| \\
&\leq \frac{1}{m} |p_\omega(\xi) - p_\omega(\zeta)| + \frac{1}{m^2} c\, |p_\omega(\xi) - p_\omega(\zeta)|\, |p_\omega(\zeta) - p_\omega(\xi_N)| \\
&\leq \left(\frac{1}{m} + \frac{1}{m^2} c \sup_{\zeta \in \Delta} |\zeta - \xi_N| \right) |p_\omega(\xi) - p_\omega(\zeta)|\,.
\end{aligned}$$

Finally, there exists a constant c' such that

$$\forall \zeta, \xi \in \Delta, \quad |\xi - \zeta| \le c' \, |p_\omega(\xi) - p_\omega(\zeta)|$$

and $p_\omega : \Delta \to p_\omega(\Delta)$ is bi-Lipschitzian.

As a result, for a $C^{1,1}$-surface without boundary, $p_\omega(\Delta)$ is a $C^{1,1}$-patch since assumption (H_1) is verified with $U = \Delta$ and $\varphi = p_\omega|_\Delta$, assumption (H_2) is verified with $a = \nabla b_\omega \circ p_\omega$, and assumption (H_3) is verified for the triangle $U = \Delta$.

We summarize the above discussion in the following theorem.

THEOREM 6.1 *Let Ω be a bounded set of class $C^{1,1}$ in $\mathbf{R}^N$ and set $\omega = \Gamma$. Around each point $\hat{\xi} \in \omega$ there exist $(N-1)$ points $\{\xi_i \in \omega : 1 \le i \le N-1\}$ such that $\{\xi_i - \hat{\xi} : 1 \le i \le N-1\}$ be linearly independent and assumptions* (A-1) *to* (A-3) *be verified for the convex polytope*

$$\Delta \stackrel{\text{def}}{=} \text{co}\{\xi_1, \ldots, \xi_{N-1}, \hat{\xi}\}$$

with unit normal ν. The mapping

$$\xi \mapsto p_\omega(\xi) : \Delta \to \overline{\omega}_\Delta \subset \mathbf{R}^3$$

is bi-Lipschitzian and its inverse is given by

$$p_\omega^{-1}(X) = X - \frac{\nu \cdot (X - \hat{\xi})}{\nu \cdot \nabla b_\omega(X)} \nabla b_\omega(X).$$

REMARK 6.2 It is quite interesting to observe that the b-Lipschitzian mapping φ of § 4 (Theorems 4.1, 4.2, 4.3, and 4.4) is precisely the projection p_ω which is the same for all facets. This confirms the meaningfulness of the set of assumptions used in § 4.

Proof. (i) Since Ω is bounded and of class $C^{1,1}$ there exists $h > 0$ such that $b_\Omega \in C^{1,1}(S_h(\omega))$ and assumption (A-1) is verified.

(ii) Fix $\hat{\xi} \in \omega$. There exist ε, $0 < \varepsilon < h$, and a $C^{1,1}$-diffeomorphism

$$\mathcal{T} : B_\varepsilon(\hat{\xi}) \to \mathcal{T}(B_\varepsilon(\hat{\xi})) \subset \mathbf{R}^N$$

such that $\mathcal{T}(\hat{\xi}) = 0$ and $\mathcal{T}(B_\varepsilon(\hat{\xi}) \cap \omega) \subset \mathbf{R}^{N-1} \times \{0\}$.

Choose an orthonormal basis $\{e_1, \ldots, e_{N-1}\}$ in $\mathbf{R}^{N-1} \times \{0\}$ and scalars $c_i \neq 0$ such that $\zeta_i \stackrel{\text{def}}{=} c_i e_i \in \mathcal{T}(B_\varepsilon(\hat{\xi}))$. The points $\xi_i \stackrel{\text{def}}{=} \mathcal{T}^{-1}(\zeta_i) \in B_\varepsilon(\hat{\xi})$ and the vectors $\{\xi_i - \hat{\xi} : 1 \le i \le N-1\}$ are linearly independent. The tangent space to ω in $\hat{\xi}$ is generated by

$$\{D\mathcal{T}^{-1}(\hat{\xi}) e_i : 1 \le i \le N-1\}.$$

Finally, assumption (A-2) is verified with the set $\Delta \stackrel{\text{def}}{=} \text{co}\{\xi_1, \ldots, \xi_{N-1}, \hat{\xi}\} \subset B_\varepsilon(\hat{\xi}) \subset S_h(\omega)$. Denote by ν the unit normal to Δ.

(iii) As for the last hypothesis, we proceed by contradiction. Assume that there exists a sequence of sets of points $\{\xi_{in} : 1 \leq i \leq N-1\} \subset B_{\varepsilon/n}(\hat{\xi}) \cap \Gamma$ (as constructed in part (ii)) such that for each $n \geq 1$, there exists $\xi_n \in \Delta_n$ such that $\nabla b_\Omega(\xi_n) \cdot \nu_n = 0$. The vectors

$$\tau_{in} \stackrel{\text{def}}{=} \frac{\xi_{in} - \hat{\xi}}{|\xi_{in} - \hat{\xi}|}, \quad 1 \leq i \leq N-1,$$

are tangent to Δ, that is $\tau_{in} \cdot \nu_n = 0$, $1 \leq i \leq N-1$. Since the set of vectors $\{\tau_{in} : 1 \leq i \leq N-1\}$ is linearly independent and orthogonal to ν_n, there exist unique constants $\{\beta_{in} : 1 \leq i \leq N-1\}$ such that

$$\nabla b_\Omega(\xi_n) = \sum_{i=1}^{N-1} \beta_{in} \tau_{in}.$$

Moreover, there exist constants $0 < \alpha_{in} < 1$ such that

$$0 = \frac{b_\Omega(\xi_{in}) - b_\Omega(\hat{\xi})}{|\xi_{in} - \hat{\xi}|} = \nabla b_\Omega(\hat{\xi} + \alpha_{in}(\xi_{in} - \hat{\xi})) \cdot \frac{\xi_{in} - \hat{\xi}}{|\xi_{in} - \hat{\xi}|} = \nabla_\Omega(\hat{\xi} + \alpha_{in}(\xi_{in} - \hat{\xi})) \cdot \tau_{in}.$$

By construction $\xi_{in} \to \hat{\xi}$ and $\xi_n \to \hat{\xi}$ and there exists a subsequence, still denoted with the index n, such that $\tau_{in} \to \tau$ and $\nu_n \to \nu$ for some τ and ν of norm 1. By choice of the basis in part (i)

$$\tau_{in} \to \frac{D\mathcal{T}^{-1}(\hat{\xi})e_i}{|D\mathcal{T}^{-1}(\hat{\xi})e_i|},$$

the coefficients β_{in} will also converge to some β_i. Finally by continuity of ∇b_Ω

$$\begin{aligned}
&0 = \nabla b_\Omega(\hat{\xi} + \alpha_{in}(\xi_{in} - \hat{\xi})) \cdot \tau_{in} \quad \Rightarrow 0 = \nabla b_\Omega(\hat{\xi}) \cdot \tau_i, \quad 1 \leq i \leq N-1, \\
&\nabla b_\Omega(\xi_n) = \sum_{i=1}^{N-1} \beta_{in} \tau_{in} \quad \Rightarrow \nabla b_\Omega(\hat{\xi}) = \sum_{i=1}^{N-1} \beta_i \tau_i \\
&\Rightarrow 1 = |\nabla_\Omega(\hat{\xi})|^2 = \sum_{i=1}^{N-1} \beta_i \nabla_\Omega(\hat{\xi}) \cdot \tau_i = 0
\end{aligned}$$

and we get a contradiction. Therefore assumption (A-3) is also verified. ■

Remark 6.3 When ω has a relative boundary γ, then it is necessary to use the assumptions of Theorem 2.2 in order to cover the boundary γ with triangles since some of the vertices may lie outside of ω. In that case the patch $p_\omega(\Delta)$ and the triangle Δ should be replaced by the smaller patch $p_\omega(\Delta) \cap \overline{\omega}$ and the smaller domain $U_\Delta \stackrel{\text{def}}{=} p_\omega^{-1}(p_\omega(\Delta) \cap \omega)$ in Δ on which assumption (H_3) must now be imposed in the absence of a specific assumption on the boundary γ.

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