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Shape optimization of control problems described by wave equations*

by

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Abstract: The control problem with multidimensional integral functional under wave type constraints for control is considered. Next a type of deformation with control of the domain is described and then we define suitable shape functional. Having defined trajectory and control of deformation dual dynamic programming tools are applied to derive optimality condition for the shape functional with respect to that deformation.

Keywords: sufficient optimality conditions, control with wave equation, shape optimization, dual dynamic programming.

1. Introduction

Our aim is to present a new approach to shape optimization. Shape functionals are difficult to study by dynamic programming methods. The reason is that shape means we are dealing with problems defined on multidimensional domains for which we have not the classical dynamic programming methods. The second difficulty is with choosing suitable deformation of the domain under consideration to get a new functional (shape functional) depending on some new quantities. They should allow to apply to that functional known mathematical tools to have possibilities to determine some optimality conditions with respect to chosen deformation. The most popular method is to remove from that domain some ball with small radius being a parameter in the deformation and then try to calculate a derivative - topological derivative (see, e.g., Nazarov and Sokołowski, 2003). In last few years we find in literature several notions of derivatives applied to shape functionals (see, e.g., Nazarov and Sokołowski,

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2003; Sokołowski and Żochowski, 1999a). They use the variation of the geometrical domain resulting in the change of the topological characteristic. Our approach is close to the classical control problem. We show that the dual dynamic programming approach is still applicable to shape optimization to be able to analyze the shape functional.

We consider the following optimal control problem (P):

$$\text{minimize } J(x,u)=\int_{[0,T]\times\Omega} L(t,z,x(t,z),u(t,z))dtdz+\int_{\Omega} l\left(x\left(T,z\right)\right)dz \quad (1)$$

subject to

$$x_{tt}\left(t,z\right)-\Delta_z x(t,z)=f(t,z,x(t,z),u(t,z)) \;\text{ a. e. on }\; (0,T)\times\Omega \quad (2)$$

$$x\left(0,z\right)=\varphi\left(0,z\right),\;\; x_t\left(0,z\right)=\psi\left(0,z\right) \;\text{ on }\; \Omega \quad (3)$$

$$x(t,z)=0 \;\text{ on }\; (0,T)\times\Gamma \quad (4)$$

$$u(t,z)\in U \;\text{ a. e. on }\; (0,T)\times\Omega \quad (5)$$

where Ω is a given bounded domain of R^n with boundary $\Gamma=\partial\Omega$ of C^2, $\Sigma=(0,T)\times\Gamma$, $U\subset R^m$ is given nonempty set, L, $f:[0,T]\times\bar{\Omega}\times R\times R^m\to R$, $l:R\to R$, and $\varphi,\psi:R^{n+1}\to R$ are given functions, $\varphi\left(0,\cdot\right)\in L^2(\Omega)$, $\psi\left(0,\cdot\right)\in H^{-1}(\Omega)$; $x:[0,T]\times\Omega\to R$, $x\in W^{2,2}((0,T)\times\Omega)\cap C([0,T];L^2(\Omega))$ and $u:[0,T]\times\Omega\to R^m$ is Lebesgue measurable function. We assume that the functions L,f,l are lower semicontinuous in their domains of definition. In the paper we assume that the system (2)-(5) admits at least one solution belonging to $W^{2,2}((0,T)\times\Omega)\cap C([0,T];L^2(\Omega))$. About the existence and regularity problems for that system see, e.g., Lasiecka, Lions and Triggiani (1986). We call a pair $x(t,z)$, $u(t,z)$ admissible if it satisfies (2)-(5) and $L(t,z,x(t,z),u(t,z))$ is summable; then the corresponding trajectory $x(t,z)$ is said to be admissible.

For that problem we apply our earlier result from Nowakowski (2008) (compare also Galewska, Nowakowski, 2006) concerning sufficient optimality conditions in terms of dual dynamic programming PDE (described in Section 2). Next we present a new type of deformation of the domain following Zolésio (see, e.g., Nazarov and Sokołowski, 2003) but adding to that deformation a control, allowing to control (to some extent) that deformation and then we define a suitable shape functional. Having defined trajectory and control of deformation we are able to apply dual dynamic programming tools (see Nowakowski, 1992) to derive optimality condition for our shape functional with respect to that deformation.

2. Duality and sufficient optimality conditions

Let us recall some facts concerning sufficient optimality conditions for problem (P) from Nowakowski (2008). Let $P\subset R^{n+3}$ be a set of variables $(t,z,p)=(t,z,y^0,y)$, $(t,z)\in[0,T]\times\bar{\Omega}$, $y^0\le 0$, $y\in R$, and let $\overline{c}=(c^0,c)\in R^2$ be fixed. We adopt the convention that $\overline{c}p = \left(c^0 y^0, cy\right)$ for $(t, z, p) \in P$. Let $\tilde{x} : P \to R$ be such a function of the variables (t, z, p) that for each admissible trajectory $x(t, z)$ there exists a function $p(t, z) = (y^0, y(t, z))$, $p \in W^{2,2}([0, T] \times \bar{\Omega}) \cap C([0, T]; L^2(\Omega))$, $(t, z, p(t, z)) \in P$ such that

$$x(t, z) = \tilde{x}(t, z, p(t, z)) \ \text{ for } \ (t, z) \in [0, T] \times \bar{\Omega}. \tag{6}$$

Existence of the function $\tilde{x}$ will be explained below - after the formulae (15). Now, let us introduce an auxiliary function $V(t, z, p) : P \to R$, of C^2, such that the following two conditions are satisfied:

$$\begin{aligned} V(t, z, \overline{c}p) &= c^0 y^0 V_{y^0}(t, z, \overline{c}p) + cy V_y(t, z, \overline{c}p) = \overline{c}p V_p(t, z, \overline{c}p), \\ \text{for } (t, z) &\in (0, T) \times \Omega, (t, z, \overline{c}p) \in P. \end{aligned} \tag{7}$$

The condition (7) is a generalization of transversality condition known in classical mechanics as orthogonality of momentum to the front of wave. Examples of such functions are given in Nowakowski (2008) in the section of Examples. Similarly as in classical dynamic programming, define at $(t, \tilde{p}(\cdot))$, where $\tilde{p}(\cdot) = (\tilde{y}^0, \tilde{y}(\cdot))$ is any function $\tilde{p} \in W^{2,2}(\Omega)$, $(t, z, \tilde{p}(z)) \in P$, $(t, z) \in [0, T] \times \Omega$, a dual value function S_D by the formula

$$S_D\left(t, \tilde{p}\left(\cdot\right)\right) := \inf \left\{ -c^0 \tilde{y}^0 \int_{[t,T] \times \Omega} L(\tau, z, x(\tau, z), u(\tau, z)) d\tau dz - c^0 \tilde{y}^0 \int_{\Omega} l\left(x\left(T, z\right)\right) dz \right\} \tag{8}$$

where the infimum is taken over all admissible pairs $x(\tau, \cdot)$, $u(\tau, \cdot)$, $\tau \in [t, T]$ such that

$$x(t, z) = \tilde{x}(t, z, \tilde{p}(z)), \quad \text{for } \ z \in \Omega, \tag{9}$$

$$\tilde{x}(t, z, \tilde{p}(z)) = 0 \ \text{ for } \ z \in \partial\Omega \tag{10}$$

i.e. whose trajectories start at $(t, \tilde{x}(t, \cdot, \tilde{p}(\cdot))$ and for which there exists such a function

$$p(\tau, z) = (\tilde{y}^0, y(\tau, z)), \ p \in W^{2,2}([t, T] \times \bar{\Omega}) \cap C([t, T]; L^2(\Omega)), \ (\tau, z, p(\tau, z)) \in P,$$

that $x(\tau, z) = \tilde{x}(\tau, z, p(\tau, z))$ for $(\tau, z) \in (t, T) \times \bar{\Omega}$ and

$$y(t, z) = \tilde{y}(z) \ \text{ for } \ z \in \bar{\Omega}. \tag{11}$$

Then, integrating (7) over Ω, for any function $\tilde{p}(\cdot) = (\tilde{y}^0, \tilde{y}(\cdot))$, $\tilde{p} \in W^{2,2}(\Omega)$, $(t, z, \tilde{p}(z)) \in P$, $(t, z, \overline{c}\tilde{p}(z)) \in P$, such that $x(\cdot, \cdot)$ satisfying $x(t, z) = \tilde{x}(t, z, \tilde{p}(z))$ for $z \in \overline{\Omega}$, is an admissible trajectory, we also have the equalities:

$$\int_{\Omega} V(t, z, \overline{c}\tilde{p}(z)) dz = -\int_{\Omega} \tilde{y}(z) x(t, z, \tilde{p}(z)) dz - S_D\left(t, \tilde{p}\left(\cdot\right)\right) \tag{12}$$

with

$$c^0 \int_\Omega \tilde{y}^0 V_{y^0}(t, z, \overline{c}\tilde{p}(z))dz = -S_D\left(t, \tilde{p}\left(\cdot\right)\right), \tag{13}$$

and assuming

$$\tilde{x}(t, z, \tilde{p}(z)) = -V_y(t, z, \overline{c}\tilde{p}(z)), \text{ for } (t, z) \in (0, T) \times \bar{\Omega}, (t, z, \overline{c}\tilde{p}(z)) \in P.$$

It turns out that the function $V(t, z, p)$, defined by (12), (13), satisfies the second order partial differential system

$$\begin{aligned} & V_{tt}(t, z, \bar{c}p) - \Delta_z V(t, z, \bar{c}p) + H(t, z, -V_y(t, z, \bar{c}p), \bar{c}p) = 0, \\ & (t, z) \in (0, T) \times \Omega, (t, z, p) \in P, \end{aligned} \tag{14}$$

where

$$H(t, z, x, \bar{c}p) = c^0 y^0 L(t, z, x, u(t, z, p)) + cyf(t, z, x, u(t, z, p)),$$

and $u(t, z, p)$ is optimal dual feedback control on $(0, T) \times \Omega$, and the dual second order partial differential system of multidimensional dynamic programming (DSPDEMDP)

$$\begin{aligned} & \sup\left\{V_{tt}(t, z, \overline{c}p) - \Delta_z V(t, z, \overline{c}p) + c^0 y^0 L(t, z, -V_y(t, z, p), u)\right. \\ & \left.+cyf(t, z, -V_y(t, z, \overline{c}p), u) : u \in U\right\} = 0,\ (t, z) \in (0, T) \times \Omega, (t, z, \overline{c}p) \in P. \end{aligned} \tag{15}$$

Let us note that the function $\tilde{x}(t, z, p)$, introduced at the beginning of this section a little bit artificially, in fact is defined by $-V_y(t, z, p)$, where V is a solution to (15), i.e. knowing the set P and V_y we are able to know the set $\dot{X}$, where our original problem we need to consider.

The verification theorem provides sufficient optimality conditions for (P) in terms of a solution $V(t, z, p)$ of the dual second order partial differential equation of multidimensional dynamic programming.

THEOREM 1 *Let $\overline{x}(t, z)$, $\overline{u}(t, z)$, $(t, z) \in (0, T) \times \bar{\Omega}$, be an admissible pair. Assume that there exist $\overline{c} = (c^0, c) \in R^2$ and a C^2 solution $V(t, z, p)$ of DSPDEMDP (15) on P such that (7) holds. Let further $\overline{p}(t, z) = (\overline{y}^0, \overline{y}(t, z))$, $\overline{p} \in W^{2,2}([0, T] \times \Omega) \cap C([0, T]; L^2(\Omega))$, be such a function that $\overline{x}(t, z) = -V_y(t, z, \overline{cp}(t, z))$ for $(t, z) \in (0, T) \times \bar{\Omega}$. Suppose that $V(t, z, p)$ satisfies the boundary condition for $(T, z, \overline{c}p) \in P$,*

$$c^0 \overline{y}^0 \int_\Omega (d/dt)\, V_{y^0}(T, z, \overline{c}p)dz = c^0 \overline{y}^0 \int_\Omega l\left(-V_y\left(T, z, \overline{c}p\right)\right)dz. \tag{16}$$

Moreover, assume that

$$\begin{aligned} & V_{tt}(t, z, \overline{cp}(t, z)) - \Delta_z V(t, z, \overline{cp}(t, z)) + c^0 \overline{y}^0 L(t, z, -V_y(t, z, \overline{cp}(t, z)), \overline{u}(t, z)) \\ & + c\overline{y}(t, z) f(t, z, -V_y(t, z, \overline{cp}(t, z)), \overline{u}(t, z)) = 0, \text{ for } (t, z) \in (0, T) \times \Omega. \end{aligned} \tag{17}$$

Then $\overline{x}(t,z)$, $\overline{u}(t,z)$, $(t,z) \in (0,T) \times \Omega$, *is an optimal pair relative to all admissible pairs* $x(t,z)$, $u(t,z)$, $(t,z) \in (0,T) \times \Omega$, *for which there exists such a function* $p(t,z) = (\overline{y}^0, y(t,z))$, $p \in W^{2,2}([0,T] \times \Omega) \cap C([0,T]; L^2(\Omega))$, *that* $x(t,z) = -V_y(t,z,\overline{c}p(t,z))$ *for* $(t,z) \in (0,T) \times \Omega$, *and*

$$y(0,z) = \overline{y}(0,z) \quad \textit{for} \quad z \in \Omega. \tag{18}$$

The examples of how to use that theorem and the description of requirement that $V(t,z,p)$ be a C^2 solution of DSPDEMDP (15) on P can be found in Nowakowski (2008).

We give the notion of an optimal dual feedback control and sufficient optimality conditions in its terms.

Definition 1 *A function* $u = \tilde{u}(t,z,p)$ *from* P *of the points* $(t,z,p) = (t,z,y^0,y)$, $(t,z) \in (0,T) \times \Omega$, $y^0 \le 0$, $y \in R$, *into* U *is called a dual feedback control, if there is any solution* $\tilde{x}(t,z,p)$, P, *of the partial differential equation*

$$\tilde{x}_{tt}(t,z,p) - \Delta_z \tilde{x}(t,z,p) = f(t,z,\tilde{x}(t,z,p),\tilde{u}(t,z,p)) \tag{19}$$

satisfying boundary condition

$$\tilde{x}(t,z,p) = 0 \quad \textit{on} \quad (0,T) \times \Gamma, (t,z,p) \in P$$

such that for each admissible trajectory $x(t,z)$, $(t,z) \in [0,T] \times \Omega$, *there exists such a function* $p(t,z) = (y^0, y(t,z))$, $p \in W^{2,2}([0,T] \times \Omega) \cap C([0,T]; L^2(\Omega))$, *that (6) holds.*

Definition 2 *A dual feedback control* $\overline{u}(t,z,p)$ *is called an optimal dual feedback control, if there exist a function* $\overline{x}(t,z,p)$, $(t,z,p) \in P$, *corresponding to* $\overline{u}(t,z,p)$ *as in Definition 1, and a function* $\overline{p}(t,z) = (\overline{y}^0, \overline{y}(t,z))$, $\overline{p} \in W^{2,2}([0,T] \times \Omega) \cap C([0,T]; L^2(\Omega))$, *such that dual value function* S_D *(see (8)) is defined at* $(t, \overline{p}(t,\cdot))$ *by* $\overline{u}(\tau,z,p)$ *and corresponding to them* $\overline{x}(\tau,z,p)$, $(\tau,z,p) \in P$, $\tau \in [t,T]$, *i.e.*

$$S_D(t, \overline{p}(t,\cdot)) = -c^0 \overline{y}^0 \int_{[t,T]\times\Omega} L(\tau, z, \overline{x}(\tau,z,\overline{p}(\tau,z)), \overline{u}(\tau,z,\overline{p}(\tau,z))) d\tau dz \\ -c^0 \overline{y}^0 \int_\Omega l\left(\overline{x}(T,z,\overline{p}(T,z))\right) dz \tag{20}$$

and, moreover, there is $V(t,z,p)$ *satisfying (7) for which* V_{y^0} *satisfies the equality*

$$c^0 \int_\Omega y^0 V_{y^0}(t,z,\overline{cp}(t,z)) dz = -S_D\left(t, \overline{p}(t,\cdot)\right)$$

and V_y *satisfies*

$$V_y(t,z,\overline{c}p) = -\overline{x}(t,z,p) \quad \textit{for} \quad (t,z) \in (0,T) \times \bar{\Omega}, (t,z,\overline{c}p) \in P. \tag{21}$$

THEOREM 2 *Let* $(\overline{u}(t,z,p)$, *be a dual feedback control in* P. *Suppose that there exist* $\overline{c} = (c^0, c) \in R^2$ *and a* C^2 *solution* $V(t,z,p)$ *of (15) on* P *such that (7) hold. Let* $\overline{p}(t,z) = (\overline{y}^0, \overline{y}(t,z))$, $\overline{p} \in W^{2,2}([0,T]\times\Omega) \cap C([0,T]; L^2(\Omega))$, $(t,z,\overline{p}(t,z)) \in P$, $(t,z,\overline{cp}(t,z)) \in P$, *be such a function that* $\overline{x}(t,z) = \overline{x}(t,z,\overline{p}(t,z))$, $\overline{u}(t,z) = \overline{u}(t,z,\overline{p}(t,z))$,$(t,z) \in (0,T)\times\Omega$, *is an admissible pair, where* $\overline{x}(t,z,p)$, $(t,z,p) \in P$, *is corresponding to* $\overline{u}(t,z,p)$ *as in Definition 1. Assume further that* V_y *and* V_{y^0} *satisfy:*

$$V_y(t,z,\overline{c}p) = -\overline{x}(t,z,p) \text{ for } (t,z) \in [0,T]\times\Omega,\ (t,z,p) \in P,\ (t,z,\overline{c}p) \in P, \quad (22)$$

$$\begin{aligned} & c^0\overline{y}^0 \int_\Omega V_{y^0}(t,z,\overline{cp}(t,z))dz \\ & = -c^0\overline{y}^0 \int_{[t,T]\times\Omega} L(s,z,\overline{x}(s,z,\overline{p}(s,z)),\overline{u}(s,z,\overline{p}(s,z)))dsdz \\ & \qquad -c^0\overline{y}^0 \int_\Omega l\left(\overline{x}(T,z,\overline{p}(T,z))\right)dz. \end{aligned} \quad (23)$$

Then $\overline{u}(t,z,p)$ *is an optimal dual feedback control.*

3. Shape optimization problem

Let $\tilde{\Omega}$ be a given simply connected domain in $\mathbb{R}^n$ with C^2 boundary Γ. Take any fixed point z_0 lying in the interior of $\tilde{\Omega}$. Let $\mathbf{U}$ be a given nonempty, compact set in $(R^m)^+$ i.e. all $u \in \mathbf{U}$ have coordinates $u_i \geq 0$. We intend to construct the deformation of $\tilde{\Omega}$ by removing from it the point z_0 and next a certain simply connected domain of C^2 with a small diameter, containing z_0. To this effect we extend the method of Sokołowski and Zolésio (1992), p. 43, and the following boundary value problem is constructed.

Given a Hoelder continuous function $v(z) : \tilde{\Omega}\backslash z_0 \to \mathbf{U}$ (a control function), find $w \in C^2(\tilde{\Omega}\backslash z_0)$ such that

$$\begin{cases} \Delta w = v & \text{in} \quad \tilde{\Omega}\backslash z_0, \\ w = 1 & \text{at} \quad z_0, \\ w = 0 & \text{on} \quad \Gamma. \end{cases} \quad (24)$$

Solution to (24) is $w(z,v)$ (we underline dependence of w on the control v) and it belongs to $C^2(\tilde{\Omega}\setminus z_0)$ (as $\mathbf{U}$ – bounded, $\tilde{\Omega}\setminus z_0$ is domain of C^2 and v a Hoelder continuous function, see Gilbarg and Trudinger, 1983). Thus, boundary of $\tilde{\Omega}\backslash\Omega_\rho$ is of C^2. Let us examine the family of level curves

$$w^{-1}(\rho) = \{z \in \tilde{\Omega}\setminus z_0 : w(z,v) = \rho\}, \quad 0 < \rho_0 \leq \rho \leq 1.$$

Put $\Gamma_{\rho_0} = w^{-1}(\rho_0)$ and $\Omega_\rho = \{z \in \tilde{\Omega}\backslash z_0 : \rho < w(z,v) < 1\}$. We assume $\Omega_1 = \emptyset$ and that $[\rho_0, 1] \ni \rho \to \Omega_\rho$ is monotone decreasing family of sets. We have $\|\nabla w(z,v)\| \neq 0$ in Ω_ρ, $\rho_0 \leq \rho < 1$ (see Sokołowski and Zolésio, 1992, pp. 43,44).

Following Sokołowski and Zolésio (1992) we introduce the field

$$W(z,v) = \|\nabla w(z,v)\|^{-2}\nabla w(z,v)$$

and the flow associated with this field:

$$T_\rho(z_0,v) = e^{\rho W}(z_0),$$

i.e. $T_\rho(z_0,v) = z(\rho,z_0,v)$, $z_0 \in \Gamma$, $\rho \in [\rho_0,1]$, where $z(\cdot,\cdot,\cdot)$ is a solution to the system of ordinary differential equations

$$\frac{d}{d\rho}z(\rho,z_0,v) = W(z(\rho,z_0,v),v), \tag{25}$$

with the initial condition $z(1,z_0,v) = z_0$. The transformations T_ρ, $\rho \in [\rho_0,1]$, have the following properties (see Sokołowski and Zolésio, 1992, p. 44):

$$T_1 = \mathcal{I} \text{ (the identity mapping on } \Gamma),$$

T_ρ mapps Γ onto $w^{-1}(\rho)$, i.e.

$$T_\rho(\Gamma,v) = w^{-1}(\rho), \text{ for } \rho \in [\rho_0,1).$$

We can deform $\tilde\Omega\backslash\Omega_\rho$ changing $\rho \in [\rho_0,1]$. Thus, we get the deformed domains:

$$\tilde\Omega\backslash\Omega_\rho = \tilde\Omega\backslash\{z \in \tilde\Omega\backslash z_0 : \rho < w(z,v) < 1\}, \text{ for } \rho \in [\rho_0,1).$$

We should stress that our deformation Ω_ρ depends on the control $v(z)$, too, i.e. we have to write $\Omega_\rho(v)$. We note that in (24) the control $v(z)$ is defined in $\tilde\Omega\backslash z_0$. However, if we consider (24) in $\tilde\Omega\backslash\Omega_\rho(v)$ then the solution $w(z,v)$ exists in $\tilde\Omega\backslash\Omega_\rho(v)$ and it will coincide with that of (24) if we consider it with new boundary conditions, i.e. $w = \rho$ on $\partial\Omega_\rho$, $w(z_0) = 1$. Of course, v is now also considered only in $\tilde\Omega\backslash\Omega_\rho(v)$ and to underline that we shall write in the next part of the paper that v is defined in $[\rho_0,1]$, i.e. $v(\tau)$, $\tau \in [\rho_0,1]$.

In control theory we write (25) as:

$$\frac{d}{d\tau}z(\tau,z_0,v) = W(z(\tau,z_0,v),v), \qquad \tau \in [\rho_0,1] \tag{26}$$

$$z(1,z_0,v) = z_0$$

and the solutions of (26) are denoted by the pair $(z(\tau),v(\tau))$, $\tau \in [\rho_0,1]$ or simply (z,v) if there is no misunderstanding. Any trajectory $z(\tau)$ under control $v(\tau)$, $\tau \in [\rho_0,1]$, satisfying (26) shall be called admissible and the pair (z,v) an admissible pair. The set of graphs of all admissible trajectories we denote by Z.

Let us take now any closed subinterval I of $[0,T]$ (I may reduce to a point $t_0 \in (0,T)$) and consider a diffeomorfism $h : [0,T] \times [\rho_0,1] \to [0,T]$ such that

$h([0,T]\times\ \rho_0) = I$ and $h([0,T]\times\ 1) = [0,T]$. For $\rho \in [\rho_0, 1]$ denote $h([0,T]\times \rho) = I_\rho$ and by $t(\rho)$ the begining of the interval I_ρ. Next consider the sets $I_\rho \times \tilde{\Omega}\backslash\Omega_\rho(v)$, $\rho \in [\rho_0, 1]$. For each domain $I_\rho \times \tilde{\Omega}\backslash\Omega_\rho(v)$, $\rho \in [\rho_0, 1]$ we can consider the optimal control problem (P). For each that problem with the domain $I_\rho \times \tilde{\Omega}\backslash\Omega_\rho(v)$, according to the sufficiency Theorem 1 there exists an optimal dual value depending on ρ and trajectory $z(\tau)$ corresponding to the control $v(\tau)$ defined in $[\rho, 1]$ (see also definition (2))

$$J(\rho, v) = -S_D(t(\rho), \overline{p}(t(\rho), \cdot), v) = c^0 \bar{y}^0 \int_{\tilde{\Omega}\backslash\Omega_\rho(v)} V_{y^0}(t(\rho), s, \overline{p}(t(\rho), s))ds,$$

where $\overline{p}(t(\rho), \cdot)$ is the dual optimal trajectory for the problem (P) with the domain $I_\rho \times \tilde{\Omega}\backslash\Omega_\rho(v)$. In this way we get the new functional $J(\rho, v)$ depending only on ρ, $\rho \in [\rho_0, 1]$ and control $v(\tau)$, $\tau \in [\rho, 1]$, which determine trajectory $z(\tau)$. We can treat $J(\rho, v)$ as a terminal functional at $(\rho, z(\rho))$ with state $z(\tau)$ and control $v(\tau)$ defined in $[\rho, 1]$. However, the starting point is for us now $(1, z(1))$ and the terminal point $(\rho, z(\rho))$. We want to minimize $J(\rho, v)$ with respect to $\rho \in [\rho_0, 1]$ and all admissible controls $v(\tau)$, $\tau \in [\rho, 1]$. To this effect we shall consider first the problem:

minimize $J(\rho_0, v)$

among all admissible controls $v(s)$, $s \in [\rho_0, 1]$. If the minimizing control exists, denote it by $\bar{v}(s)$, $s \in [\rho_0, 1]$. Then we prove that along $\bar{v}(s)$, $s \in [\rho_0, 1]$ the value $J(s, \bar{v}(s))$ is constant. In order to be in agreement with control theory we denote $J(\rho, v)$ as $J(\rho, z(\rho))$. Therefore now, by the boundary condition for dynamic programming we will assume the value

$$J(\rho_0, z(\rho_0)). \tag{27}$$

For the functional $J(\rho, z(\rho))$ we can form dual dynamic construction as in Nowakowski (1992), where now our functional does not depend explicitly on a state $z(s)$, $s \in [\rho_0, 1]$. Thus, we should treat that problem so as to minimize $J(\rho_0, z(\rho_0))$ among all controls $v : [\rho_0, 1] \to \mathbf{U}$ and states $z(s)$, $s \in [\rho_0, 1]$. Having that in mind and applying construction from Nowakowski (1992), let $Y(s,p)$ be a function defined on a set $\mathbf{P} \subset [\rho_0, 1] \times R^{n+1}$, $(s,p) = (s, y^0, y)$, $y^0 \le 0$ and satisfying there

$$Y(s,p) = y^0 Y_{y^0}(s,p) + y Y_y(s,p) = p Y_p(s,p). \tag{28}$$

We require (restricting eventually the set Z) that for each admissible trajectory $z(s)$, $s \in [\rho_0, 1]$ satisfying (26) there exist a $p(s) = (y^0, y(s))$ – absolutely continuous such that $z(s) = -Y_y(s, p(s))$, $s \in [\rho_0, 1]$, $p(1) = p_0$; p_0 is fixed for all admissible trajectory $z(s)$, $s \in [\rho_0, 1]$.

If

$$y^0 Y_{y^0}(\rho_0, p) = -J_D(\rho_0, p) \tag{29}$$

with dual value function J_D defined by

$$J_D(\rho_0, p) = \inf\{-y^0 S_D(\rho_0, \overline{p}(t(\rho_0), \cdot), v)\} = \inf\{-y^0 J(\rho_0, z(\rho_0))\} \qquad (30)$$

where infimum is taken over all admissible pairs $(z(s), v(s))$, $s \in [\rho_0, 1]$, whose trajectories are starting at $(1, -Y_y(1, p_0))$, then by (27)

$$y^0 Y_{y^0}(\rho_0, p) = y^0 J(\rho_0, -Y_y(\rho_0, p)), \ \ (\rho_0, p) \in \mathbf{P} \qquad (31)$$

and $Y(s, p)$ satisfies

$$Y_s(s, p) + \overline{H}(s, -Y_y(s, p), p) = 0, \ (s, p) \in \mathbf{P},$$

where $\overline{H}(s, z, p) = yW(z, v(s, p))$ and $v(s, p)$ is an optimal dual feedback control and the partial differential equation of dynamic programming:

$$\min \{Y_s(s, p) + yW(-Y_y(s, p), v) : \ v \in \mathbf{U}\} = 0, \ (s, p) \in \mathbf{P}. \qquad (32)$$

REMARK 1 *Let us discuss differences between the approach described above and that described in Sokołowski and Żochowski (1999a) using topological derivative. Both in Sokołowski and Żochowski (1999a) and here we remove from $\tilde{\Omega}$ some small ball with center z_0 (in Sokołowski and Żochowski, 1999a) and simply connected domain of C^2 with small diameter containing z_0 (here). In both case we get the function $J(\rho)$ (in Sokołowski and Żochowski, 1999a) and $J(\rho, z(\rho))$ (here) depending on parameter ρ. In both cases our objective is to analyze the behavior of $J(\rho)$, $J(\rho, z(\rho))$, as $\rho \to 0^+$, $\rho \to 1^-$, respectively. In Sokołowski and Żochowski (1999a) it is needed to evaluate the limits of derivatives $J'(\rho)$, $J''(\rho)$ for $\rho \to 0^+$ using suitable (of their form) shape derivative and concluding with explicit formula (in linear quadratic problem) for the topological derivative for that problem. Its importance is described in Sokołowski and Żochowski (1999b). In our paper we propose a different approach, which is based on dual dynamic method. We do not need to calculate the derivatives $\frac{d}{d\rho} J(\rho, z(\rho))$ or $\frac{d^2}{d\rho^2} J(\rho, z(\rho))$ to make the analysis mentioned (in general - especially in nonlinear problems - they do not exist). Instead of that, we form a dual function $J_D(\rho, p)$ and a new function $Y(\rho, p)$, subject to conditions (29) and (30), which satisfies (32). Thus, if we are able to find a solution to (32) then we know $J_D(\rho, p)$ and so $J(\rho, \bar{z}(\rho))$ for all $\rho \in [\rho_0, 1]$ ($\bar{z}(\rho)$, $\rho \in [\rho_0, 1]$ is an optimal state - see next section), in particular, at 1.Therefore we have then the optimal value $J(\rho, \bar{z}(\rho))$ with respect to all small deformations described above i.e. $z(\tau)$ $\tau \in [\rho, 1]$. We would like to stress that in our case we do not need differentiability of $\rho \to J(\rho, z(\rho))$ and even it need not be continuous, only the auxiliary function $Y(\rho, p)$ has to be differentiable - see the next section.*

4. Verification theorem

We prove the verification theorem which allows us to give sufficient optimality conditions to determine the optimal state $\bar{z}(s)$ under control $\overline{v}(s)$, $s \in [\rho_0, 1]$.

THEOREM 3 *Let $Y(s,p)$ be a C^1 solution of the above partial differential equation (32) on $\mathbf{P}$ and such that (28) and (31) hold. Let $(\bar{z}(s), \overline{v}(s))$ be an admissible pair, $s \in [\rho_0, 1]$ and let $\overline{p}(s)$, $s \in [\rho_0, 1]$, $\overline{p}(1) = p_0$ – absolutely continuous be such that $\bar{z}(s) = -Y_y(s, \overline{p}(s))$, $s \in [\rho_0, 1]$ and satisfying*

$$Y_s(s, \overline{p}(s)) + \overline{y}(s)W(-Y_y(s, \overline{p}(s)), \overline{v}(s)) = 0, \qquad in\ [\rho_0, 1]. \tag{33}$$

Then, $\bar{z}(\tau), \overline{v}(\tau)$, $\tau \in [\rho, 1]$ is an optimal pair relative to all admissible pairs $z(\tau)$, $v(\tau)$, $\tau \in [\rho, 1]$ for which there exists such a function $p(\tau) = (\bar{y}^0, y(\tau))$, $p(1) = p_0$, p – absolutely continuous that $z(\tau) = -Y_y(\tau, p(\tau))$, $\tau \in [\rho, 1]$. The function $[\rho_0, 1] \ni s \to \overline{y}^0 Y_{y^0}(s, \overline{p}(s)) = \overline{y}^0 J(s, -Y_y(s, \overline{p}(s)))$ is constant and equal $\overline{y}^0 Y_{y^0}(1, p_0))$.

Proof. Take any admissible pair $z(s)$, $v(s)$, $s \in [\rho_0, 1]$, whose graph of trajectory is contained in Z and for which there exists an absolutely continuous function $p(s) = \left(\overline{y}^0, y(s)\right)$, $p(1) = p_0$ lying in P such that $z(s) = -Y_y(s, p(s))$ for $s \in [\rho_0, 1]$. Then, from (28), we have , for almost all $s \in [\rho_0, 1]$

$$Y_s(s, p(s)) = \overline{y}^0\,(d/ds)\,Y_{y^0}(s, p(s)) + y(s)\,(d/ds)\,Y_y(s, p(s)) \tag{34}$$

Let $X(s, p(s))$ be a function defined in $\mathbf{P}$ by the formula

$$X\,(s, p\,(s)) := -\overline{y}^0 Y_{y^0}(s, p(s)). \tag{35}$$

Since $\overline{y}^0\,(d/ds)\,Y_{y^0}(s, p(s)) = -\,(d/ds)\,X\,(s, p\,(s))$ and

$$(d/ds)\,Y_y(s, p(s)) = -W(-Y_y(s, p(s)), v(s))\ \textit{a.e.}\ \text{on}\ [\rho_0, 1], \tag{36}$$

it follows, by (34) and (35), that for almost all $s \in [\rho_0, 1]$,

$$(d/ds)\,X\,(s, p\,(s)) = -Y_s(s, p(s)) - y(s)W(-Y_y(s, p(s)), v(s)) \quad . \tag{37}$$

Thus, by (37) and (32), we get

$$(d/ds)\,X\,(s, p\,(s)) \le 0 \ \text{ a. e. on } [\rho_0, 1]. \tag{38}$$

Similarly, by (37) and (33), we obtain

$$(d/ds)\,X(s, \overline{p}(s)) = 0 \ \text{ a. e. on } [\rho_0, 1]. \tag{39}$$

The above inequality and equality mean that function $X(s, p\,(s))$ is a nonincreasing function of s and $X(s, \overline{p}(s))$ is constant on $[\rho_0, 1]$ and equals $-\overline{y}^0 Y_{y^0}(\rho_0, \overline{p}(\rho_0))$ $= -\overline{y}^0 Y_{y^0}(1, p_0))$. Thus, by (35) and $-\overline{y}^0 Y_{y^0}(1, p(1)) = -\overline{y}^0 Y_{y^0}(1, p_0))$, we get for $s \in [\rho_0, 1]$

$$-\overline{y}^0 Y_{y^0}(s, \overline{p}(s)) \le -\overline{y}^0 Y_{y^0}(s, p(s)), \tag{40}$$

$$-\overline{y}^0 J(s, \bar{z}(s)) \le -\overline{y}^0 J(s, z(s)), \tag{41}$$

which proves the assertion of the theorem. ∎

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