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Shape sensitivity analysis of eigenvalues revisited*

by

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Abstract: The paper can be considered as a complement to previous papers of the authors. An insight into applied asymptotic analysis of boundary value problems in singularly perturbed domains is presented. As a result, the asymptotic expansions of eigenvalues are obtained and discussed in terms of integral attributes of the geometrical perturbations including the virtual mass tensor, polarization tensor etc. The results are presented in such a way that can be easily employed in numerical methods for shape optimization and inverse problems.

Keywords: spectral problem, singular perturbation, eigenvalues of Laplacian, shape sensitivity analysis, topology optimization.

1. Introduction

Shape optimization problems for eigenvalues are among the most popular subjects of extended studies in applied PDE's, we refer the reader, e.g., to Bucur and Buttazzo (2005), Henrot (2006), Sokołowski and Zolésio (1992), Zolésio (1981), for a review of known results, and to Nazarov and Sokołowski (2008) for a list of references from the field of asymptotic analysis.

Recently, the asymptotic analysis in singularly perturbed geometrical domains (Mazja, Nazarov and Plamenevskii, 1991) is applied to shape optimization (Sokołowski and Żochowski, 1999) and the topological derivatives of shape functionals are obtained for elliptic boundary value problems with singularly perturbed boundaries. In the paper we present certain results on topological

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derivatives for the spectral problems with the Laplace operator. Namely, the asymptotic analysis of eigenvalues is performed with respect to singular perturbations of domains (see Fig. 1a, b, c). The results can be directly used in some applications, in particular, in the shape and topology sensitivity analysis of the Helmholtz equation. Compared to the existing results in the literature, the technical difficulties of the asymptotic procedures concern the variable coefficients of differential operators in limit problems that particularly arise from the curved boundaries. The known results are mainly given for singular perturbations of isolated points of the boundary (small holes in the domain, see Mazja, Nazarov and Plamenevskii, 1984; Kamotski and Nazarov, 1998; Campbell and Nazarov, 2001; Mazja, Nasarov and Plamenerskii, 1991; Ozawa, 1985, and others), perturbations of straight boundaries, including perturbations by changing the type of boundary conditions (see Gadyl'shin, 1986, and others), and the dependence on the curvature has been clarified only in Nazarov and Sokołowski (2008), where it was shown that the first order correction term for an eigenvalue is independent of the curvature, even if the appropriate change of curvilinear variables leads to differential expressions depending explicitly on the curvature. We revisit our results in Nazarov and Sokołowski (2008) with two goals. First, we correct all misprints which, unfortunately have appeared in Nazarov and Sokołowski (2008) (see the end of Section 2). Second, we elucidate and explicate here the integral characteristics of geometrical perturbations, which form the asymptotic expansions for eigenvalues, and, therefore, the topological derivative of the eigenvalues as the main correction term.

The description of shape optimisation problems for eigenvalues can be found e.g., in monographs Bucur and Buttazzo (2005), Henrot (2006), Sokołowski and Zolésio (1992), Zolésio (1981), and we recall that the method of boundary variations goes back to Hadamard, so the structure of the shape gradient of an differentiable shape functional is called the Hadamard formula (Sokołowski and Zolésio, 1992). There is a natural gap between the regularity of boundaries, from one side for the results on the existence of optimal domains, and the necessary optimality conditions, where stronger assumptions on the regularity of boundaries of admissible domains are necessary to compute the directional derivatives of eigenvalues with respect to domain perturbations.

We provide the analysis of non-smooth perturbations of boundaries which uses the same tools (Mazja, Nasarow and Plamenevskii, 1991) as the derivation of topological derivatives of shape functionals. In this way we extend the notion of *shape gradient* to the case of singular boundary perturbations. The obtained formulae can be employed to get information from optimality conditions about the decrease or increase of eigenvalues for the specific boundary perturbations in the form of caverns and knops. Such an information is interesting on its own for the analysis of optimal solutions to shape optimisation problems for eigenvalues.

The outline of the paper is the following. In Section 2 asymptotics of solutions to spectral problems are introduced. In Section 3 integral characteristics of small domains, which serve as perturbations, are defined by certain solutions

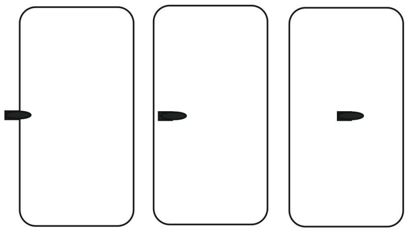

Figure 1. A bullet pierces a pillow.

to boundary value problems in unbounded domains with particular data. In Section 4 the min-max principle for eigenvalues is recalled and discussed for asymptotics in specific boundary value problems. In Section 5 the case of multiple eigenvalues is focused on. In Section 6 a control on eigenvalue increments and a simple example of singular boundary perturbations are presented.

2. Asymptotics of eigenvalues under singular boundary perturbations

Let $\Omega, \omega \subset \mathbb{R}^2$ be domains with the boundaries $\partial\Omega, \partial\omega$ and the compact closures $\overline{\Omega}, \overline{\omega}$, respectively. $\partial\Omega$ is assumed to be of class C^∞ for simplicity. Given a small parameter $\varepsilon > 0$, we introduce the sets

$$\Omega(\varepsilon) = \Omega \setminus \overline{\omega_\varepsilon} \ , \quad \omega_\varepsilon = \{\xi \in \mathbb{R}^2 \ : \ \xi := \varepsilon^{-1} x \in \omega\} \ . \tag{1}$$

We further have to distinguish between several situations drawn in Fig. 1, where a bullet pierces a pillow. If the coordinate origin $\mathcal{O}$ is located on $\partial\Omega$ and in the interior of ω, we observe the boundary perturbation by the cavity $\theta_\varepsilon = \Omega \cap \omega_\varepsilon$ (Fig. 1a). Otherwise, we find a small hole (opening) $\theta_\varepsilon = \omega_\varepsilon$ which is situated near the boundary $\partial\Omega$ (Fig. 1b) or far from the boundary in the interior of the domain Ω (Fig. 1c). We emphasize that the analysis of the first two geometrical situations is performed in the same way, while for the third one it is performed in a slightly different way.

We proceed with the Neumann spectral problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega(\varepsilon), \quad \partial_n u^\varepsilon(x) = 0, \quad x \in \partial\Omega(\varepsilon), \tag{2}$$

where ∂_n stands for the outward normal derivative defined almost everywhere on the Lipschitz (by the assumption) boundary $\partial\Omega(\varepsilon)$. The problem (2) admits the eigenvalue sequence

$$\lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le \cdots \le \lambda_j^\varepsilon \le \cdots \to \infty \ , \tag{3}$$

where the eigenvalues are listed according to multiplicity and the first eigenvalue $\lambda_1^\varepsilon = 0$ is simple.

The first limit ($\varepsilon = 0$) problem in the entire domain Ω

$$-\Delta_x u^0(x) = \lambda^0 u^0(x), \quad x \in \Omega, \quad \partial_n u^0(x) = 0, \quad x \in \partial\Omega, \tag{4}$$

has the eigenvalue sequence

$$\lambda_1^0 < \lambda_2^0 \le \lambda_3^0 \le \cdots \le \lambda_j^0 \le \cdots \to \infty \ , \tag{5}$$

with the same properties, while the corresponding eigenfunctions $u_1^0, u_2^0, \ldots,$ $u_n^0, \ldots$ are subject to the normalization and orthogonality conditions

$$(u_j^0, u_k^0)_\Omega = \delta_{j,k} \ , \quad j, k \in \mathbb{N} := \{1, 2, \ldots\}, \tag{6}$$

where $(\cdot,\cdot)_\Omega$ stands for the scalar product in the Lebesque space $L^2(\Omega)$, and $\delta_{j,k}$ is the Kronecker symbol. In particular, the first eigenfunction is constant and the first eigenvalue λ_1^0 stays unperturbed. The remaining eigenvalues in (5) get certain perturbations in (3) and, we refer for the proof to Mazja, Nazarov and Plamenevskii (1984), Kamotski and Nazarov (1998), Nazarov and Sokołowski (2008), and Mazja, Nasarov and Plamenevskii (1991; Ch. 9) that the eigenvalues take the asymptotic form

$$\lambda_j^\varepsilon = \lambda_j^0 + \varepsilon^2 \left(\nabla_x u_j^0(\mathcal{O})^\top M(\theta) \nabla_x u_j^0(\mathcal{O}) + \lambda_j^0 |u_j^0(\mathcal{O})|^2 \mathrm{mes}_2 \theta\right) + O\left(\varepsilon^{5/2}\right) \tag{7}$$

in the case of a simple eigenvalue λ_j^0 (see Section 5 for the multiple case). In (7), the gradient $\nabla_x u_j^0(\mathcal{O})$ is a column vector in $\mathbb{R}^2$, $\nabla_x u_j^0(\mathcal{O})^\top$ is the transposed line vector and $M(\theta)$ is a matrix of size 2×2.

We emphasize that $\theta = \omega$ in the case of $\mathcal{O} \in \Omega$ but θ must be reconstructed by dilatation from θ_ε in the case of $\mathcal{O} \in \partial\Omega$.

Remark 1 *Actually, the majorant for the asymptotic remainder in* (7) *is* $C\varepsilon^3$ *and, furthermore, the whole asymptotic expansions in powers of* ε *are available although coefficients in the expansions may become polynomial in* $|\ln \varepsilon|$ *(see Mazja, Nazarov and Plamenevskii, 1984; Kamotski and Nazarov, 1998). In the paper we formulate the relation* (7) *in the same way as the new results given in Nazarov and Sokołowski (2008) for the perturbations of spectral problem* (2) *in Fig. 1a and b. The main result in Nazarov and Sokołowski (2008) reads: For the spectral problem* (4) *the first correction term* $\varepsilon^2 \lambda_j'$ *is independent of* $|\ln \varepsilon|$ *and of the curvature of the contour* $\partial\Omega$ *at the point* $\mathcal{O}$.

In the case of $\mathcal{O} \in \partial\Omega$, the asymptotic formula (7) keeps its validity for the mixed boundary value problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \Omega(\varepsilon),\ u^\varepsilon(x) = 0,\ x \in \partial\Omega \setminus \overline{\omega_\varepsilon},\ \partial_n u^\varepsilon(x) = 0,$$
$$x \in \partial\omega_\varepsilon \cap \Omega, \tag{8}$$

and for the Dirichlet problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \Omega(\varepsilon),$$
$$u^\varepsilon(x) = 0,\ x \in \partial\Omega(\varepsilon) = \partial\Omega \cup \partial\omega_\varepsilon \cap \Omega. \tag{9}$$

We point out that the matrix $M(\theta)$ depends on the particular problem, i.e. as it can be expected, on the shape of the void and on the boundary conditions on the void as well as on the unperturbed boundary next to the void. Moreover, the first limit problem in Ω provides $\partial_n u^0(\mathcal{O}) = 0$ for (2) and $u_j^0(\mathcal{O}) = \partial_s u_j^0(\mathcal{O}) = 0$ for (9). In other words, the asymptotic formula (7) simplifies and involves only scalar characteristics for the boundary perturbations (Fig. 1a, b).

Remark 2 *For some specific cases, e.g., the Dirichlet problem with $\mathcal{O} \in \Omega$ and the mixed boundary value problem with $\mathcal{O} \in \partial\Omega$ and the Dirichlet and Neumann conditions on $\partial\omega_\varepsilon \cap \Omega$ and $\partial\Omega \setminus \overline{\omega_\varepsilon}$, respectively, the asymptotic expansions of eigenvalues (Mazja, Nazarov and Plamenevskii, 1984; Nazarov and Sokołowski, 2008) are much more elaborate. In particular, the main correction term is of order $|\ln\varepsilon|^{-1}$ with the unsatisfactory remainder $O(|\ln\varepsilon|^{-2})$, while the main term with the remainder $O(\varepsilon)$ becomes a holomorphic function in $|\ln\varepsilon|^{-1}$. A serious complication of the asymptotic procedure for systems of differential equations, e.g., in elasticity, provokes mistakes (see Movchan and Movchan, 1995, and the requisite correction in Campbell and Nazarov, 2001).*

In Figs. 2 and 3 we outline disposition of the Dirichlet and Neumann boundary conditions which lead to the eigenvalue perturbation of order ε^2 and $|\ln\varepsilon|^{-1}$, respectively.

The asymptotic formula (7) first of all, needs an appropriate description of the matrix $M(\theta)$ as an integral characteristics of the perturbation set θ. Unfortunately, the authors had chosen in Nazarov and Sokołowski (2008) a *lame way* to introduce $M(\theta)$ due to the wrong sign of the Poisson kernel (see formula (18) below) that has distorted the final asymptotic formulae in Nazarov and Sokołowski (2008) although after returning the sign minus to the kernel all calculations get valid. Our immediate objective is to introduce $M(\theta)$ properly.

3. Integral characteristics

Let us assume for simplicity that λ_j^0 is a simple eigenvalue (see Section 5 for the multiple eigenvalues). According to the asymptotic procedures developed in Mazja, Nasarow and Plamenevskii (1991, Chapter 9) and Nazarov and Sokołowski (2008), the asymptotic ansatz for the eigenfunction u_j^ε reads

$$u_j^\varepsilon(x) = u_j^0(x) + \varepsilon w_j(\varepsilon^{-1}x) + \varepsilon^2 u_j'(x) + \dots. \tag{10}$$

Here w_j is the boundary layer term in the form

$$w_j(\xi) = \sum_{p=1}^{2} W_p(\xi)\frac{\partial u_j^0}{\partial y_p}(\mathcal{O}) \tag{11}$$

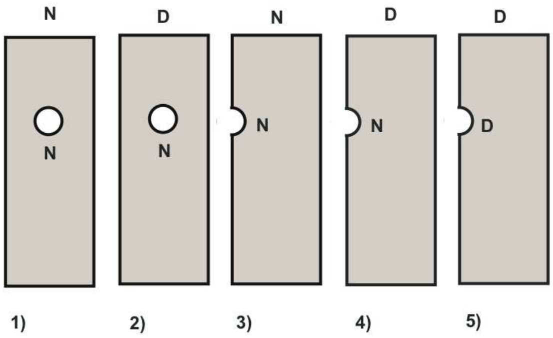

Figure 2. The boundary conditions provide the eigenvalue perturbation of order ε^2.

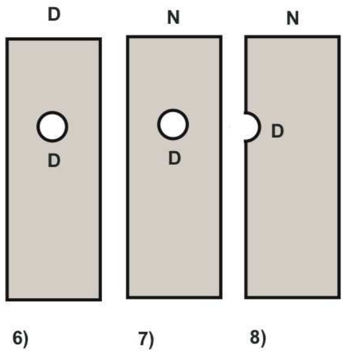

Figure 3. The boundary conditions provide the eigenvalue perturbation of order $|\ln \varepsilon|^{-1}$.

and u' implies the main regular correction. The function (11) is written in the stretched coordinates ξ (see formula (1)) and it is a solution of a boundary value problem in an unbounded domain with a proper decay as $|\xi| \to \infty$.

First, we consider the Neumann problem (1) in the case of $\mathcal{O} \in \omega$ (Fig. 1c). The second limit problem, obtained by stretching the coordinates and setting $\varepsilon = 0$, is but the exterior Neumann problem

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \mathbb{R}^2 \setminus \overline{\omega}, \quad \partial_n W(\xi) = G(\xi), \quad \xi \in \partial\omega \; . \tag{12}$$

Due to the Taylor formula

$$\begin{aligned} u_j^0(x) =& u_j^0(\mathcal{O}) + x^\top \nabla_x u_j^0(\mathcal{O}) + O(|x|^2) = \\ =& u_j^0(\mathcal{O}) + \varepsilon \xi^\top \nabla_x u_j^0(\mathcal{O}) + O(\varepsilon^2), \quad \xi \in \partial\omega_\varepsilon \; , \end{aligned} \tag{13}$$

the main discrepancy of the eigenfunction u_j^0 of the problem (4) in the Neumann boundary condition on $\partial\omega_\varepsilon$

$$G(\xi) = -n(\xi)^\top \nabla_x u_j^0(\mathcal{O}) \tag{14}$$

is compensated by the linear combination (11), where W_p is the decaying solution of (12) with the specific right-hand side $G_p(\xi) = -n_p(\xi)$. Since components of the unit normal $n(\xi) = (n_1(\xi), n_2(\xi))^\top$ are of mean zero on the contour $\partial\omega$, the solutions exist and take the form

$$W_p(\xi) = \sum_{q=1}^{2} M_{pq}(\omega)\frac{\partial\Phi}{\partial\xi_q}(\xi) + O(|\xi|^{-2}) = \tag{15}$$

$$= -\sum_{q=1}^{2} M_{pq}(\omega)\frac{\xi_q}{2\pi|\xi|^2} + O(|\xi|^{-2}), \quad |\xi| \to \infty\ .$$

Here $\Phi(\xi) = -(2\pi)^{-1}\ln|\xi|$ is the fundamental solution of the operator $-\Delta_\xi$ in $\mathbb{R}^2$.

The matrix $M(\omega)$ composed of the coefficients in (15) is called (Polya and Szegö, 1951, Appendix G) the matrix associated with the virtual mass form of the set $\overline{\omega}$. The representation

$$M_{pq}(\omega) = -\int\limits_{\mathbb{R}^2\setminus\overline{\omega}} \nabla_\xi W_p(\xi)^\top \nabla_\xi W_q(\xi) d\xi - \delta_{p,q}\text{mes}_2\omega \tag{16}$$

is known (see Polya and Szegö, 1951). Thus, $M(\omega)$ is a symmetric and negative definite matrix if the area $\text{mes}_2\omega$ of ω is positive.

Remark 3 *If $\overline{\omega} = \{\xi\ :\ |\xi_1| \le \ell, \xi_2 = 0\}$ is a crack of length $2\ell > 0$, the function G_1 and, therefore, the solution W_1 vanish so that the matrix $M(\overline{\omega})$ is degenerate. However, all asymptotic formulae remain valid (see Nazarov and Sokołowski, 2008).*

Finally, we refer to paper of Mazja, Nazarov and Plamenevskii (1984) and book of Mazja, Nasarov and Plamenevskii (1991, Ch. 9) for the asymptotic procedure to compose the Neumann problem in the punctured domain $\Omega\setminus\mathcal{O}$ in order to find out the correction term u' in (10). We emphasize that the compatibility condition in this problem provides the explicit formula for the correction term $\varepsilon^2\lambda'$ in (7).

In the case of $\mathcal{O} \in \partial\Omega$ we assume that Ω is located on the right of the x_2-axis, and that x_2-axis is tangent to the contour $\partial\Omega$ at the point $\mathcal{O}$ (see Fig. 1a).

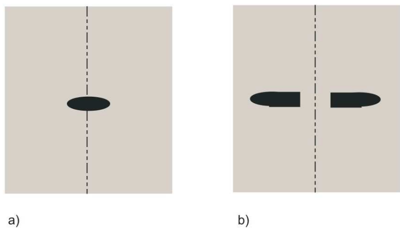

Figure 4. Sets Θ obtained as a union of θ with its mirror reflection.

Then[1] the second limit problem reads:

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \mathbb{R}^2_+ \setminus \overline{\omega}, \quad \partial_n W(\xi) = G(\xi), \quad \xi \in \partial\omega \cap \mathbb{R}^2_+, \tag{17}$$
$$\frac{\partial W}{\partial \xi_1}(\xi) = 0, \quad \xi \in \partial\mathbb{R}^2_+ \setminus \overline{\omega},$$

where $\mathbb{R}^2_+ = \{\xi \ : \ \xi_1 > 0\}$ is the half-plane. Owing to the Neumann condition $\partial_n u^0_j(\mathcal{O}) = 0$, the discrepancy (14) takes the form

$$G(\xi) = -n_2(\xi)\frac{\partial u^0_j}{\partial x_2}(\mathcal{O}).$$

Hence, $W_1 = 0$ in (11) while W_2 solves problem (17) with $G(\xi) = -n_2(\xi)$ and admits the asymptotic form

$$W_2(\xi) = -M_N(\theta)\frac{\xi_2}{\pi|\xi|^2} + O(|\xi|^{-2}), \quad \xi \to +\infty \ , \tag{18}$$

where $\theta = \omega \cap \mathbb{R}^2_+$. We point out that the factor of $M_N(\theta)$ implies the Poisson kernel and differs by $1/2$ compared to the derivative of the fundamental solution Φ in (15).

Let Θ be the union of the set θ and its mirror reflection (see Fig. 4a, b, with Fig. 1a, b, respectively), that is

$$\Theta = \theta \cup \{\xi \ : \ (-\xi_1, \xi_2) \in \theta\} \ . \tag{19}$$

We observe that there is a simple relation between the virtual mass matrix of the set Θ and the matrix $M(\theta)$ in the eigenvalue asymptotics. To this end, let us note that the restriction to $\mathbb{R}^2_+ \setminus \overline{\theta}$ of the decaying solution of the exterior problem (12) in the domain $\mathbb{R}^2_+ \setminus \overline{\Theta}$ and with the right-hand side $G_2(\xi) = -n_2(\xi)$ coincides with the solution $W_2(\xi)$ of the problem (17). Hence,

$$M_N(\theta) = \frac{1}{2}M_{22}(\Theta) \ . \tag{20}$$

[1] According to the calculation applied in Nazarov and Sokołowski (2008) to the Laplacian in curvilinear coordinates the same second limit problem (17) occurs in the case of curved boundary $\partial\Omega$.

In other words, the matrix $M(\theta)$ in the eigenvalue asymptotics for the Neumann problem (2) takes the form

$$M(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & M_N(\theta) \end{pmatrix}, \tag{21}$$

with the only nontrivial entry (20).

For the mixed boundary value problem (2), a similar argument can be used. Namely, the decaying solution of the corresponding second limit problem

$$-\Delta_\xi W_1(\xi) = 0, \quad \xi \in \mathbb{R}^2_+ \setminus \overline{\omega}, \quad W_1(\xi) = 0, \quad \xi \in \partial\mathbb{R}^2_+ \setminus \overline{\omega}, \tag{22}$$

$$\partial_n W_1(\xi) = -n_1(\xi), \quad \xi \in \partial\omega \cap \mathbb{R}^2_+, \tag{23}$$

is but the restriction to $\mathbb{R}^2_+ \setminus \overline{\omega}$ of the odd in the variable x_1 solution of the exterior problem (12) in $\mathbb{R}^2 \setminus \overline{\Theta}$ with the same right-hand side $-n_1(\xi)$ as in (23). Therefore, formulae (18), (20) and (21) can be replaced by

$$W_1(\xi) = -\frac{1}{2} M_{11}(\Theta) \frac{\xi_1}{\pi|\xi|^2} + O(|\xi|^{-2}), \quad M(\theta) = \frac{1}{2} \begin{pmatrix} M_{11}(\Theta) & 0 \\ 0 & 0 \end{pmatrix}. \tag{24}$$

The Dirichlet problem (9) gives rise to the second limit problem

$$\begin{aligned} &-\Delta_\xi W(\xi) = 0, \quad \xi \in \mathbb{R}^2_+ \setminus \overline{\omega}, \quad W(\xi) = 0, \quad \xi \in \partial\mathbb{R}^2_+ \setminus \overline{\omega}, \\ &W(\xi) = -\xi_1, \quad \xi \in \partial\omega \cap \mathbb{R}^2_+ . \end{aligned} \tag{25}$$

Let us consider again the symmetrized set (19) and replace (22), (25) by the exterior Dirichlet problem

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \mathbb{R}^2_+ \setminus \overline{\Theta}, \quad W(\xi) = G(\xi), \quad \xi \in \partial\Theta. \tag{26}$$

Let W_p be a bounded solution of (26) for $G(\xi) = -\xi_p, p = 1, 2$. Such a solution is unique and admits the asymptotic expansion

$$W_p(\xi) = c_p - \sum_{q=1}^{2} P_{pq}(\Theta) \frac{\xi_q}{2\pi|\xi|^2} + O(|\xi|^{-2}), \quad |\xi| \to \infty ,$$

where c_p is a constant, $c_1 = 0$ by the symmetry, and the coefficients $P_{pq}(\Theta)$ form the matrix $P(\Theta)$ associated with the polarization tensor of $\overline{\Theta}$ (see Polya and Szegö, 1951, Appendix G). It is known that

$$P_{pq}(\Theta) = \int\limits_{\mathbb{R}^2 \setminus \overline{\Theta}} \nabla_\xi W_p(\xi)^\top \nabla_\xi W_q(\xi) d\xi + \delta_{p,q} \mathrm{mes}_2 \Theta \tag{27}$$

(compare (16)) and, therefore, $P(\Theta)$ is a symmetric positive definite 2×2-matrix. The restriction of W_1 onto $\mathbb{R}^2 \setminus \overline{\omega}$ solves the problem (22), (25) and it follows that

$$M(\theta) = \frac{1}{2} \begin{pmatrix} P_{11}(\Theta) & 0 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

We refer to Nazarov and Sokołowski (2008) for the arguments completing the asymptotic ansatz (10) and the derivation of an expression for the correction term $\varepsilon^2 \lambda'_j$ in the eigenvalue asymptotics (7).

4. Min-max principle for eigenvalues

The operator theory in Hilbert spaces furnishes the representation of eigenvalues for the Dirichlet problem (9),

$$\lambda_j^\varepsilon = \max_{\mathcal{E}_j} \inf_{v \in \mathcal{E}_j \setminus \{0\}} \frac{\|\nabla_x v; L^2(\Omega(\varepsilon))\|^2}{\|v; L^2(\Omega(\varepsilon))\|^2}, \quad j \in \mathbb{N}, \tag{29}$$

(see, e.g., Birman and Solomyak, 1987, Section 10.2) where $\mathcal{E}_j$ is an arbitrary subspace in $H_0^1(\Omega(\varepsilon); \partial\Omega(\varepsilon))$ of codimension $j-1$, i.e., $\mathcal{E}_1 = H_0^1(\Omega(\varepsilon); \partial\Omega(\varepsilon))$ is a subspace of the Sobolev space $H^1(\Omega(\varepsilon))$ of functions which vanish on the boundary $\partial\Omega(\varepsilon)$.

Since by construction $\Omega(\varepsilon) = \Omega \setminus \overline{\omega}_\varepsilon \subset \Omega$, it follows that $H_0^1(\Omega(\varepsilon); \partial\Omega(\varepsilon)) \subset H_0^1(\Omega; \partial\Omega)$ and, thus, the formulas (29) with $\varepsilon > 0$ and (29) with $\varepsilon = 0$ provide the relationship

$$\lambda_j^\varepsilon > \lambda_j^0, \quad j \in \mathbb{N}, \tag{30}$$

which is in accord with the asymptotic expansion (7) taking, in view of (28), the form

$$\lambda_j^\varepsilon = \lambda_j^0 + \frac{\varepsilon^2}{2} P_{11}(\Theta) \left| \frac{\partial u_j^0}{\partial x_1}(\mathcal{O}) \right|^2 + O(\varepsilon^{5/2}). \tag{31}$$

We emphasize that $P_{11}(\Theta) > 0$ by (27) and the equalities $u_j^0(\mathcal{O}) = 0$, $\frac{\partial u_j^0}{\partial x_2}(\mathcal{O}) = 0$, which simplify (7), follow from the Dirichlet condition in the first limit problem

$$-\Delta_x u^0(x) = \lambda^0 u^0(x), \quad x \in \Omega, \quad u^0(x) = 0, \quad x \in \partial\Omega. \tag{32}$$

Note that the spectral problem (32) admits the eigenvalues (5) where $\lambda_1^0 > 0$ is simple by the maximum principle.

If $\mathcal{O} \in \partial\Omega$, one may consider the domain $\Omega(\varepsilon) = \Omega \cup \omega_\varepsilon$ perturbed by a knoll. All asymptotic formulae are preserved, however, by the same argument as above the inequality (30) changes for $\lambda_j^\varepsilon < \lambda_j^0$ while, simultaneously the factor $P_{11}(\Theta)$ becomes negative (see Nazarov and Sokołowski, 2008, Lemma 5.1).

For the Neumann problem, the max-min principle (29) applies in the same manner but for a crack $\overline{\omega}$ only (see Remark 3). Clearly, $H^1(\Omega) \subset H^1(\Omega(\varepsilon))$ because functions in the domain $\Omega(\varepsilon)$ with the cut $\overline{\omega}$ can have a jump over the crack lips. Thus, the relation $\lambda_j^\varepsilon \le \lambda_j^0$ is valid, which in the case of a selvage microcrack is consistent with the asymptotic formula

$$\lambda_j^\varepsilon = \lambda_j^0 + \varepsilon^2 \left(\frac{1}{2} M_{11}(\Theta) \left| \frac{\partial u_j^0}{\partial x_1}(\mathcal{O}) \right|^2 + \lambda_j^0 |u_j^0(\mathcal{O})|^2 \mathrm{mes}_2 \theta \right) + O(\varepsilon^{3/2}) \ , \tag{33}$$

with the simple observations: $M_{11}(\Theta) < 0$ and $\mathrm{mes}_2\theta = 0$.

The above examination of asymptotic formulae for eigenvalues is an obvious indirect way to check the signs of the second terms of the asymptotic ansatz (7). Sadly enough, this simple step was not taken into account in Nazarov and Sokołowski (2008).

5. Perturbation of a multiple eigenvalue

Let us consider the Neumann spectral problem (2) in the particular case of Fig. 1a, we refer to Nazarov and Sokołowski (2008) for the justification of our asymptotic procedure. Assume, that λ_j^0 is an eigenvalue of multiplicity $\varkappa_j > 1$, i.e.,

$$\lambda_{j-1}^0 < \lambda_j^0 = \cdots = \lambda_{j+\varkappa_j-1}^0 < \lambda_{j+\varkappa_j}^0 . \tag{34}$$

In such a case the asymptotic ansätze (10) and

$$\lambda_p^\varepsilon = \lambda_j^0 + \varepsilon^2 \lambda_p' + O(\varepsilon^{5/2}) \tag{35}$$

are still valid for $p = j, \ldots, j + \varkappa_j - 1$, however, the principal term takes the form of the linear combinations

$$u^{p0} = a_1^p u_j^0 + \cdots + a_{\varkappa_j}^p u_{j+\varkappa_j-1}^0 \tag{36}$$

of eigenfunctions corresponding to the eigenvalue λ_j^0. Coefficients of the columns $a^p = (a_1^p, \ldots, a_{\varkappa_j}^p)$ in (36) are to be determined such that

$$a^p \cdot a^q = \delta_{p,q}, \quad p, q = j, \ldots, j + \varkappa_j - 1 \ . \tag{37}$$

Since λ_j^0 is an eigenvalue of multiplicity $\varkappa_j$, each of the problem for the regular correction terms $u_j', \ldots, u_{j+\varkappa_j-1}'$ in (10) gets $\varkappa_j$ compability conditions, which can be written in the form of the following linear system of $\varkappa_j$ algebraic equations

$$\lambda_p' a^p = \mathbf{M} a^p \tag{38}$$

with the matrix $\mathbf{M} = (\mathbf{M}_{mk})_{m,k=0}^{\varkappa_j - 1}$ of the size $\varkappa_j \times \varkappa_j$,

$$\mathbf{M}_{mk} = M(\theta)\partial_s u^0_{j+k}(\mathcal{O})\partial_s u^0_{j+m}(\mathcal{O}) + \lambda^0_j u^0_{j+k}(\mathcal{O}) u^0_{j+m}(\mathcal{O}) mes_2(\omega). \quad (39)$$

Formula (39) is derived in exactly the same way as it is for the term $\varepsilon^2 \lambda'_j$ in (7) (see Mazja, Nazarov and Plamenevskii, 1984; Nazarov and Sokołowski, 2008; and Mazja, Nasarov and Plamenevskii, 1991, Ch. 9, for details).

The matrix $\mathbf{M}$ is symmetric, and its real eigenvalues $\lambda^{j\prime}, \dots, \lambda^{j+\varkappa_j-1\prime}$ correspond to the eigenvectors $a^j, \dots, a^{j+\varkappa_j-1}$, satisfying the orthogonality and normalization conditions (37). Actually, just these attributes of the matrix $\mathbf{M}$ with the elements (39) are included in the asymptotic ansätze (10) and (35) for the eigenvalues λ^ε_p and the eigenfunctions u^ε_p of the problem (2) for $p = j, \dots, j + \varkappa_j - 1$ in case (34). An estimate of the asymptotic remainder in the eigenvalue expansion (35) is obtained in Nazarov and Sokołowski (2008).

6. Control of eigenvalues

The asymptotic expansion (7) for the first eigenvalue λ^ε_1 of the Dirichlet problem (9) in the domain $\Omega(\varepsilon)$ with the small cavity θ_ε (Fig. 1a and b) takes the form (31) where the coefficient $P_{11}(\Theta)$ is positive (see (27), (28)). Thus, the eigenvalue increment $\triangle\lambda^\varepsilon_1 = \lambda^\varepsilon_1 - \lambda^0_1 > 0$ (see (30)) becomes maximal (is maximized) provided that the absolute maximum of the function $\partial\Omega \ni x \mapsto \partial_n u^0_1(x)$ is attained at the point $\mathcal{O} \in \partial\Omega$.

For the Neumann problem (2), the first eigenvalue $\lambda^\varepsilon_1 = 0$ is stable and in the case of the simple eigenvalue λ^0_1 the increment $\triangle\,\lambda^\varepsilon_j$ is given by (33) with the negative coefficient $M_{11}(\Theta)$ while $\triangle\,\lambda^\varepsilon_j$ can be of any sign. Indeed, if $\mathcal{O}$ constitutes a local maximum of the function $\partial\Omega \ni x \mapsto |u^0_j(x)|$, then $\nabla_x u^0_j(\mathcal{O}) = 0$ and $\triangle\lambda^\varepsilon_j \geq 0$, however, in the case $u^0_j(\mathcal{O}) = 0$, $\nabla_x u^0_j(\mathcal{O}) \neq 0$ we have $\triangle\lambda^\varepsilon_j < 0$ because the coefficient $M_{22}(\Theta)$ is negative.

If θ_ε is a selvage micro-crack, i.e., a cut of length ε on the boundary $\partial\Omega$ (see Remark 2 and the end of Section 4) then $\mathrm{mes}_2\theta = 0$ and, therefore, $\triangle\,\lambda^\varepsilon_j \leq 0$. The asymptotic expansion can be also employed for solving one more shape optimization problem, namely to maximize the difference $\lambda^\varepsilon_3 - \lambda^\varepsilon_2$ in the case of simple eigenvalues $\lambda^\varepsilon_3 > \lambda^\varepsilon_2 > 0$. From formulae (7) and (20), (21) it follows that the difference becomes maximal, provided at the point $\mathcal{O}$ the absolute maximum of the function $\partial\Omega \ni x \mapsto |\nabla_x u^0_3(x)|^2 - |\nabla_x u^0_2(x)|^2$ is attained.

Example: Dirichlet problem with a Neumann hole We consider $\Omega = (0,\pi)^2$ and the Dirichlet spectral problem in Ω. In such a case we can determine all eigenvalues and eigenfunctions, namely

$$\lambda_n = p^2 + q^2, \quad p, q = 1, 2, \dots,$$

$$u_n = \sqrt{\frac{2}{\pi}} \sin px_1 \sin qx_2$$

and therefore, ought to follow the formulae in Section 5.

For a simple eigenvalue, e.g., for the case of $p = q$, we have the following formula for the topological derivative at a point $\mathcal{O} \in \Omega$,

$$\lambda_n^\varepsilon - \lambda_n = \varepsilon^2[-2\pi|\nabla u_n(\mathcal{O})|^2 + \pi\Lambda_n|u_n(\mathcal{O})|^2] + \dots$$

When we can exchange $p \neq q$, we have a double eigenvalue $\lambda_n = \lambda_{n+1} = p^2 + q^2$, with the eigenfunctions of the form

$$u_n = \sqrt{\frac{2}{\pi}} \sin px_1 \sin qx_2 \tag{40}$$

$$u_{n+1} = \sqrt{\frac{2}{\pi}} \sin qx_1 \sin px_2. \tag{41}$$

Our procedure applies also in such a case, namely we construct the 2×2-matrix $M = (M_{jk})$, and the coefficients of M are given by

$$M_{jk} = -2\pi\nabla u_j(\mathcal{O})^\top \nabla u_k(\mathcal{O}) + \pi\lambda_n u_j(\mathcal{O})u_k(\mathcal{O}) \ , \tag{42}$$

where we denote $u_j = \sqrt{\frac{2}{\pi}} \sin px_1 \sin qx_2, u_k = \sqrt{\frac{2}{\pi}} \sin qx_1 \sin px_2$. The eigenvalues of matrix M are denoted by $\gamma_1 \leq \gamma_2$, respectively, and determined from the equation $Mz = \gamma z$, and the formula for the topological derivative of the double eigenvalue λ_n takes the form

$$\lambda_n^\varepsilon - \lambda_n = \varepsilon^2\gamma_1 + \dots \tag{43}$$

$$\lambda_{n+1}^\varepsilon - \lambda_n = \varepsilon^2\gamma_2 + \dots \tag{44}$$

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