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On parametric Hurwitz stability margin of real polynomials*

by

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Abstract: The paper deals with the problem of determining Hurwitz stability of a ball of polynomials defined by a weighted l_p norm in the coefficient space where p is an arbitrary positive integer including infinity. The solution of the case when the weights are supposed to be the same for coefficient being above and below its nominal value corresponding to symmetric ball has been given by Tsypkin and Polyak. However, sometimes it seems to be useful to have a possibility to consider these weights as different, resulting in the asymmetric ball. This is, for example, the situation where the weights express our level of confidence that the real value of a coefficient lies in some interval. Such approach is used if the value of a coefficient is estimated by an expert.

Solution of the problem is based on frequency domain plot in the complex plane and on applying the Zero Exclusion Theorem. The main idea consists in separation of the original problem into four subproblems and using an appropriate coordinate transformation which makes the value set independent of frequency. This transformation makes it possible to move the relative value set into the origin of the complex plane and to easily formulate the necessary and sufficient condition of Hurwitz stability of asymmetric ball of polynomials with prescribed radius or determine the maximum radius preserving stability. The whole graphical procedure consists of four plots instead of one, needed in the symmetric case.

Keywords: robust stability, parametric uncertainty, continuous-time systems.

1. Introduction

Since the publication of the celebrated Kharitonov theorem, Kharitonov (1978), the area of robust stability analysis of linear systems with parametric uncertainty has been intensively developed. A comprehensive survey of results

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achieved in last decades can be found in Barmish (1994) and Bhattacharyya et al. (1995). Several results have been achieved especially for interval systems. A very simple proof of the Kharitonov theorem was pointed out by Dasgupta (1988). The simplification of the Kharitonov theorem for low order systems was given in Anderson et al. (1987), while Mansour et al. (1989) and Kraus et al. (1988) have given several counterpart results on robust Schur stability and strong Kharitonov theorems for Schur interval systems. A unifying frequency domain approach for robust stability analysis was presented in Dasgupta et al. (1991).

A more general case represents consideration of an uncertain polynomial where the coefficients are constrained by some l_p weighted norm, p being a positive integer. Hurwitz stability margin of such a ball of polynomials was determined by Tsypkin and Polyak (1991). The graphical method developed by them is based on a complex plane frequency domain plot. The main idea consists in transforming the coordinates of traditional frequency plot such that the value set becomes independent of frequency. This idea is stressed in Mansour (1994).

Based on that the generalization of Tsypkin-Polyak locus is given, in this paper we take into account the case of different weights, considered for the coefficient being above and below its nominal value. This consideration is useful if the weights reflect our level of confidence that the true value of a coefficient lies in some interval. The nominal value need not be necessarily equal to the center of the interval. Such approach is adopted e.g. by Bondia and Picó (2003a) in the concept of fuzzy linear systems where the uncertain parameters of a linear system are described by fuzzy numbers. In Bondia and Picó (2003b) fuzzy numbers are used to distinguish between the most-cases and the worst-cases behavior of a system.

2. The Zero Exclusion Principle

In this section a modification of the fundamental stability criterion in frequency domain will be presented.

Let A be a connected region in the $(n+1)$-dimensional space. Let us consider a family of polynomials

$$p(s, A) = a_0 + a_1 s + \cdots + a_n s^n, a_i \in \Re, \mathbf{a} = [a_0, \ldots, a_n], \mathbf{a} \in A. \quad (1)$$

DEFINITION 1 *Polynomial $p(s, \mathbf{a})$ is said to be D-stable if and only if all its roots lie in an open connected domain $D \subset C$.*

DEFINITION 2 *A family of polynomials $p(s, A)$ is said to be D-stable if and only if all its members are D-stable, i.e. $p(s, \mathbf{a})$ is D-stable polynomial $\forall \mathbf{a} \in A$.*

To derive the main result of this paper the well-known boundary crossing theorem will be used.

THEOREM 1 (BOUNDARY CROSSING THEOREM) *The family of polynomials $p(s, A)$ (1) of invariant degree is D-stable if and only if*
a) there exists a D-stable polynomial $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$,
b) $s^* \notin roots(p(s, A))$, $\forall s^* \in \partial D$
where ∂D stands for boundary of D.

This intuitive result simply states the fact that the first encounter of polynomial with fixed degree (i.e. coefficient a_n does not include zero) with instability has to be on the boundary of stability domain. Computationally more efficient version of the boundary crossing theorem is formulated by the zero exclusion principle.

THEOREM 2 (ZERO EXCLUSION PRINCIPLE) *The family of polynomials $p(s, A)$ (1) is D-stable if and only if*
a) there exists a D-stable polynomial $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$,
b) $0 \notin p(s^*, A)$, $\forall s^* \in \partial D$,
c) coefficient a_n does not include 0.

The set $p(s^*, A), s^* \in \partial D$ is called the value set. In the case of Hurwitz stability ∂D corresponds to the imaginary axis (semiaxis) of the complex plane.

Considering $p_1(\omega) = h(\omega) + j\omega g(\omega)$ instead of $p(j\omega) = h(\omega) + jg(\omega)$ where $h(s)$ and $sg(s)$ are the even and odd parts of the polynomial $p(s)$, respectively, the following theorem can be stated.

THEOREM 3 *The family of polynomials $p(s, A)$ (1) is Hurwitz stable if and only if*
a) there exists a Hurwitz stable polynomial $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$,
b) $0 \notin p_1(\omega, A)$, $\forall \omega \geq 0$,
c) the coefficient a_n does not include 0,
d) for $\omega = 0$ the value set $p_1(\omega, A)$ does not include points on the imaginary axis.

REMARK 1 From the monotonic phase increase property for Hurwitz polynomials follows that the frequency plot of $p(s, \mathbf{a}^*)$ in the complex plane goes through n quadrants in the counterclockwise direction.

REMARK 2 Part d) of Theorem 3 is equivalent to the condition that the coefficient a_0 does not include 0 because $h(0) = 0$ is equivalent to $a_0 = 0$.

Since dividing of the even and odd parts of a polynomial by some positive functions cannot affect zero exclusion or inclusion in the value set we can replace $p_1(\omega)$ by $p_2(\omega) = h(\omega)/S(\omega) + jg((\omega)/S(\omega))$ where $S(\omega)$ and $T(\omega)$ are positive functions of $\omega \geq 0$. Moreover, if $\lim_{\omega\to\infty} h(\omega)/S(\omega)$ and $\lim_{\omega\to\infty} g(\omega)/T(\omega)$ are finite, we can replace condition c) of Theorem 3 by the condition c) of the following theorem.

THEOREM 4 *The family of polynomials $p(s, A)$ (1) is Hurwitz stable if and only if*
a) there exists a Hurwitz stable polynomial $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$,
b) $0 \notin p_2(\omega, A) \forall \omega \geq 0$,
c) for $\omega = \infty$ the value set $p_2(\omega, A)$ does not include points on the imaginary axis for n even or points on the real axis for n odd,
d) for $\omega = 0$ the value set $p_2(\omega, A)$ does not include points on the imaginary axis.

The equivalence of condition c) of Theorem 4 with condition c) of Theorem 3 is based on the fact that if and only if a_n is zero then $h(\omega)/S(\omega)$ or $g(\omega)/T(\omega)$ vanishes for $\omega = \infty$.

3. The generalized Tsypkin-Polyak locus

Let us consider a family of polynomials (1) centered at a nominal point $\mathbf{a}^0 = [a_0^0, a_1^0, \ldots, a_n^0]$ with the coefficients lying in the asymmetric weighted l_p ball of radius ρ,

$$A := \left\{ \mathbf{a} : \left[\sum_{k=0}^{n} \left| \frac{a_k - a_k^0}{\alpha_k} \right|^p \right]^{\frac{1}{p}} \leq \rho \right\} \tag{2}$$

where $\alpha_k = \underline{\alpha_k}$ for $a_k < a_k^0$ and $\alpha_k = \overline{\alpha_k}$ for $a_k \geq a_k^0$.

In (2) $\underline{\alpha_k} > 0$ and $\overline{\alpha_k} > 0$ are given weights for coefficients being below and above their nominal values respectively, $1 \leq p \leq \infty$ is a fixed integer, $\rho \geq 0$ is the radius of the ball. The family of polynomials (1) associated with the set (2) is loosely referred to as the asymmetric ball of polynomials. The objective is to check if the asymmetric l_p ball of polynomials (2) with prescribed ρ is robustly Hurwitz stable or not and also to determine the maximal ρ preserving robust stability of (2).

Let us again decompose a member of family of polynomials (1) into its even and odd parts. For $s = j\omega$ we can write

$$p(j\omega, \mathbf{a}) = h(\omega, \mathbf{a}) + j\omega g(\omega, \mathbf{a}), \mathbf{a} \in A. \tag{3}$$

The nominal polynomial $p_0(s)$ evaluated in $s = j\omega$ can then be written as

$$p_0(j\omega) = p(j\omega, \mathbf{a}^0) = h_0(\omega) + j\omega g_0(\omega) \tag{4}$$

where

$$\begin{aligned} h_0(\omega) &= a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \cdots \\ g_0(\omega) &= a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \cdots . \end{aligned} \tag{5}$$

Denote by $\Delta a_k = a_k - a_k^0$ and $\mu_k = \Delta a_k/\alpha_k$. The deviations of even and odd parts of a polynomial then can be expressed as

$$\begin{aligned}\Delta h(\omega) &= h(\omega,\mathbf{a}) - h_0(\omega) = \sum_{k_{\text{even}}} (-1)^{k/2}\Delta a_k \omega^k \\ \Delta g(\omega) &= g(\omega,\mathbf{a}) - g_0(\omega) = \sum_{k_{\text{odd}}} (-1)^{(k-1)/2}\Delta a_k \omega^{k-1}\end{aligned} \tag{6}$$

respectively.

Let us discuss four different cases according to the signs of $\Delta h(\omega)$ and $\Delta g(\omega)$.

Case 1. $\Delta h(\omega) \geq 0, \Delta g(\omega) \geq 0$:
For $\Delta h(\omega) \geq 0$ we can write

$$\Delta h(\omega) \leq \sum_{k/2_{\text{even}}} \mu_k \overline{\alpha_k}\omega^k - \sum_{k/2_{\text{odd}}} \mu_k \underline{\alpha_k}\omega^k. \tag{7}$$

For its absolute value we have

$$\begin{aligned}|\Delta h(\omega)| &\leq \left|\sum_{k/2_{\text{even}}} \mu_k \overline{\alpha_k}\omega^k\right| + \left|\sum_{k/2_{\text{odd}}} -\mu_k \underline{\alpha_k}\omega^k\right| \\ &\leq \sum_{k/2_{\text{even}}} \left|\mu_k \overline{\alpha_k}\omega^k\right| + \sum_{k/2_{\text{odd}}} \left|-\mu_k \underline{\alpha_k}\omega^k\right|.\end{aligned} \tag{8}$$

By applying Hölders inequality one obtains

$$|\Delta h(\omega)| \leq \left(\sum_{k_{\text{even}}} |\mu_k|^p\right)^{\frac{1}{p}} \left(\sum_{k/2_{\text{even}}} (\overline{\alpha_k}\omega^k)^q + \sum_{k/2_{\text{odd}}} (\underline{\alpha_k}\omega^k)^q\right)^{\frac{1}{q}} \tag{9}$$

where q is the index conjugate to p:

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{10}$$

Analogically, for $\Delta g(\omega) \geq 0$ we have

$$\Delta g(\omega) \leq \sum_{(k-1)/2_{\text{even}}} \mu_k \overline{\alpha_k}\omega^{(k-1)} - \sum_{(k-1)/2_{\text{odd}}} \mu_k \underline{\alpha_k}\omega^{(k-1)} \tag{11}$$

and

$$|\Delta g(\omega)| \leq \left(\sum_{k_{\text{odd}}} |\mu_k|^p\right)^{\frac{1}{p}} \left(\sum_{(k-1)/2_{\text{even}}} (\overline{\alpha_k}\omega^{(k-1)})^q + \sum_{(k-1)/2_{\text{odd}}} (\underline{\alpha_k}\omega^{(k-1)})^q\right)^{\frac{1}{q}}. \tag{12}$$

Let us introduce

$$S_{p1}(\omega) = \left(\sum_{k/2_{\text{even}}} (\overline{\alpha_k}\omega^k)^q + \sum_{k/2_{\text{odd}}} (\underline{\alpha_k}\omega^k)^q \right)^{\frac{1}{q}} \tag{13}$$

and

$$T_{p1}(\omega) = \left(\sum_{(k-1)/2_{\text{even}}} (\overline{\alpha_k}\omega^{(k-1)})^q + \sum_{(k-1)/2_{\text{odd}}} (\underline{\alpha_k}\omega^{(k-1)})^q \right)^{\frac{1}{q}}. \tag{14}$$

Let us note that $S_{p1}(\omega)$ and $T_{p1}(\omega)$ are positive functions of $\omega \geq 0$. Moreover, $\lim_{\omega\to\infty} h(\omega, \mathbf{a})/S_{p1}(\omega)$ and $\lim_{\omega\to\infty} g(\omega, \mathbf{a})/T_{p1}(\omega)$ are finite for all $p(s, \mathbf{a}), \mathbf{a} \in A$ defined by (2).

Substituting (13) and (14) into (9), (12) and (2) gives

$$\left(\frac{|\Delta h(\omega)|}{S_{p1}(\omega)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_{p1}(\omega)}\right)^p \leq \sum_{k_{\text{even}}} |\mu_k|^p + \sum_{k_{\text{odd}}} |\mu_k|^p = \sum_{k=0}^{n} |\mu_k|^p \leq \rho^p \tag{15}$$

or equivalently

$$\left[\left(\frac{|\Delta h(\omega)|}{S_{p1}(\omega)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_{p1}(\omega)}\right)^p\right]^{\frac{1}{p}} \leq \rho. \tag{16}$$

It means that the value set of the polynomial (1) specified by (2) with $\Delta h(\omega) \geq 0$, $\Delta g(\omega) \geq 0$ evaluated in the coordinates $(h(\omega)/S_{p1}(\omega), g(\omega)/T_{p1}(\omega))$ is the upper right quarter of l_p-disc centered at the nominal polynomial

$$(h_0(\omega)/S_{p1}(\omega), g_0(\omega)/T_{p1}(\omega))$$

with radius ρ for any $\omega \geq 0$. Since the parameters in $\Delta h(\omega)$ and $\Delta g(\omega)$ are independent, any point of the upper right quarter of l_p-disc (including those on its boundary) can be reached. By applying Theorem 4 we get the following result.

Denote by $\mathcal{D}_{p3}(\rho)$ the quarter of l_p disc with radius (ρ) in the third quadrant in the complex plane:

$$\mathcal{D}_{p3}(\rho) := \left\{ (x, y) : x \leq 0, y \leq 0; [|x|^p + |y|^p]^{\frac{1}{p}} \leq \rho \right\}. \tag{17}$$

THEOREM 5 *The family of polynomials (1) specified by (2) with $\Delta h(\omega) \geq 0$ and $\Delta g(\omega) \geq 0$ is Hurwitz stable if and only if the frequency plot of the nominal polynomial $p(s, \mathbf{a}^0)$ in the complex plane $h(\omega)/S_{p1}(\omega) + j(g(\omega)/T_{p1})$*
a) goes through n quadrants in the counterclockwise direction,
b) does not intersect the quarter of l_p disc with radius ρ in the 3rd quadrant, $\mathcal{D}_{p3}(\rho)$,
c) $a_n^0 > \rho\underline{\alpha_n}$ for $n = 4i + 2, 4i + 3, i = 0, 1, 2, \ldots$.

REMARK 3 The condition 5c) comes from 4c), the condition 4d) is always satisfied.

Case 2. $\Delta h(\omega) \le 0, \Delta g(\omega) \le 0$:
For $\Delta h(\omega) \le 0$ we have

$$\Delta h(\omega) \ge \sum_{k/2_{\text{even}}} \mu_k \underline{\alpha_k} \omega^k - \sum_{k/2_{\text{odd}}} \mu_k \overline{\alpha_k} \omega^k \tag{18}$$

or equivalently, for its absolute value

$$\begin{aligned} |\Delta h(\omega)| &\le \left| \sum_{k/2_{\text{even}}} \mu_k \underline{\alpha_k} \omega^k \right| + \left| \sum_{k/2_{\text{odd}}} -\mu_k \overline{\alpha_k} \omega^k \right| \\ &\le \sum_{k/2_{\text{even}}} \left| \mu_k \underline{\alpha_k} \omega^k \right| + \sum_{k/2_{\text{odd}}} \left| -\mu_k \overline{\alpha_k} \omega^k \right|. \end{aligned} \tag{19}$$

Using Hölders inequality we obtain

$$|\Delta h(\omega)| \le \left(\sum_{k_{\text{even}}} |\mu_k|^p \right)^{\frac{1}{p}} \left(\sum_{k/2_{\text{even}}} (\underline{\alpha_k} \omega^k)^q + \sum_{k/2_{\text{odd}}} (\overline{\alpha_k} \omega^k)^q \right)^{\frac{1}{q}} \tag{20}$$

where q is the index conjugate to p.
Analogically, for $\Delta g(\omega)$ we have

$$\Delta g(\omega) \ge \sum_{(k-1)/2_{\text{even}}} \mu_k \underline{\alpha_k} \omega^{(k-1)} + \sum_{(k-1)/2_{\text{odd}}} -\mu_k \overline{\alpha_k} \omega^{(k-1)} \tag{21}$$

and for the absolute value

$$|\Delta g(\omega)| \le \left(\sum_{k_{\text{odd}}} |\mu_k|^p \right)^{\frac{1}{p}} \left(\sum_{(k-1)/2_{\text{even}}} (\underline{\alpha_k} \omega^{(k-1)})^q + \sum_{(k-1)/2_{\text{odd}}} (\overline{\alpha_k} \omega^{(k-1)})^q \right)^{\frac{1}{q}}. \tag{22}$$

Introduce

$$S_{p2}(\omega) = \left(\sum_{k/2_{\text{even}}} (\underline{\alpha_k} \omega^k)^q + \sum_{k/2_{\text{odd}}} (\overline{\alpha_k} \omega^k)^q \right)^{\frac{1}{q}} \tag{23}$$

and

$$T_{p2}(\omega) = \left(\sum_{(k-1)/2_{\text{even}}} (\underline{\alpha_k} \omega^{(k-1)})^q + \sum_{(k-1)/2_{\text{odd}}} (\overline{\alpha_k} \omega^{(k-1)})^q \right)^{\frac{1}{q}}. \tag{24}$$

Similarly to previous case, $S_{p2}(\omega)$ and $T_{p2}(\omega)$ are positive functions of $\omega \geq 0$ and both $\lim_{\omega\to\infty} h(\omega, \mathbf{a})/S_{p2}(\omega)$ and $\lim_{\omega\to\infty} g(\omega, \mathbf{a})/T_{p2}(\omega)$ are finite for all $p(s, \mathbf{a}), \mathbf{a} \in A$ defined by (2).

Substituting (23) and (24) into (20), (22) and (2) gives

$$\left(\frac{|\Delta h(\omega)|}{S_{p2}(\omega)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_{p2}(\omega)}\right)^p \leq \sum_{k_{\text{even}}} |\mu_k|^p + \sum_{k_{\text{odd}}} |\mu_k|^p = \sum_{k=0}^{n} |\mu_k|^p \leq \rho^p \tag{25}$$

or equivalently

$$\left[\left(\frac{|\Delta h(\omega)|}{S_{p2}(\omega)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_{p2}(\omega)}\right)^p\right]^{\frac{1}{p}} \leq \rho. \tag{26}$$

It means that the value set of the polynomial (1) specified by (2) with $\Delta h(\omega) \leq 0$, $\Delta g(\omega) \leq 0$ evaluated in the coordinates $(h(\omega)/S_{p2}(\omega), g(\omega)/T_{p2}(\omega))$ is the lower left quarter of l_p-disc centered at the nominal polynomial $(h_0(\omega)/S_{p2}(\omega), g_0(\omega)/T_{p2}(\omega))$ with radius ρ for any $\omega \geq 0$. Using the same arguments as above, application of Theorem 4 leads to the following result.

Denote by $\mathcal{D}_{p1}(\rho)$ the quarter of l_p disc with radius (ρ) in the first quadrant in the complex plane:

$$\mathcal{D}_{p1}(\rho) := \left\{(x, y) : x \geq 0, y \geq 0; [|x|^p + |y|^p]^{\frac{1}{p}} \leq \rho\right\}. \tag{27}$$

THEOREM 6 *The family of polynomials (1) specified by (2) with* $\Delta h(\omega) \leq 0$ *and* $\Delta g(\omega) \leq 0$ *is Hurwitz stable if and only if the frequency plot of the nominal polynomial* $p(s, \mathbf{a}^0)$ *in the complex plane* $h(\omega)/S_{p2}(\omega) + j(g(\omega)/T_{p2})$
a) goes through n quadrants in the counterclockwise direction,
b) does not intersect the quarter of l_p *disc with radius* ρ *in the 1st quadrant,* $\mathcal{D}_{p1}(\rho)$,
c) $a_n^0 > \rho\underline{\alpha_n}$ *for* $n = 4i, 4i+1, i = 0, 1, 2, \ldots,$
d) $a_0^0 > \rho\underline{\alpha_0}$.

REMARK 4 Conditions 6c) and 6d) are equivalent to conditions 4c) and 4d), respectively.

Using similar reasoning for the cases $\Delta h(\omega) \leq 0$, $\Delta g(\omega) \geq 0$ and $\Delta h(\omega) \geq 0$, $\Delta g(\omega) \leq 0$ the following theorems can be derived.

Denote by $\mathcal{D}_{p2}(\rho)$ and $\mathcal{D}_{p4}(\rho)$ the quarter of l_p disc with radius (ρ) in the second and fourth quadrant in the complex plane, respectively:

$$\begin{aligned} \mathcal{D}_{p2}(\rho) &:= \left\{(x, y) : x \leq 0, y \geq 0; [|x|^p + |y|^p]^{\frac{1}{p}} \leq \rho\right\} \\ \mathcal{D}_{p4}(\rho) &:= \left\{(x, y) : x \geq 0, y \leq 0; [|x|^p + |y|^p]^{\frac{1}{p}} \leq \rho\right\}. \end{aligned} \tag{28}$$

THEOREM 7 *The family of polynomials (1) specified by (2) with* $\Delta h(\omega) \leq 0$ *and* $\Delta g(\omega) \geq 0$ *is Hurwitz stable if and only if the frequency plot of the nominal polynomial* $p(s, \mathbf{a}^0)$ *in the complex plane* $h(\omega)/S_{p2}(\omega) + j(g(\omega)/T_{p1}(\omega))$
a) goes through n quadrants in the counterclockwise direction,
b) does not intersect the quarter of l_p *disc with radius* ρ *in the 4th quadrant,* $\mathcal{D}_{p4}(\rho)$*,*
c) $a_n^0 > \rho\underline{\alpha_n}$ *for* $n = 4i, 4i+1, i = 0, 1, 2, \ldots,$
d) $a_0^0 > \rho\underline{\alpha_0}$.

THEOREM 8 *The family of polynomials (1) specified by (2) with* $\Delta h(\omega) \geq 0$ *and* $\Delta g(\omega) \leq 0$ *is Hurwitz stable if and only if the frequency plot of the nominal polynomial* $p(s, \mathbf{a}^0)$ *in the complex plane* $h(\omega)/S_{p1}(\omega) + j(g(\omega)/T_{p2}(\omega))$
a) goes through n quadrants in the counterclockwise direction,
b) does not intersect the quarter of l_p *disc with radius* ρ *in the 2nd quadrant,* $\mathcal{D}_{p2}(\rho)$*,*
c) $a_n^0 > \rho\underline{\alpha_n}$ *for* $n = 4i+2, 4i+3, i = 0, 1, 2, \ldots$.

Since the frequency plot of any member of the asymmetric ball of polynomials (2) jumps between four cases, mentioned above, depending on frequency, it is obvious that the asymmetric ball is robustly Hurwitz stable if and only if the conditions of Theorems 5–8 are met all at once. From those theorems it also directly follows that the maximum ρ preserving stability of (2) is equal to the maximum ρ satisfying the conditions of Theorems 5–8.

THEOREM 9 *Let us denote the particular stability margins satisfying conditions of Theorems 5–8 by* ρ_{p1}*,* ρ_{p2}*,* ρ_{p3} *and* ρ_{p4} *respectively. Then the maximum radius preserving stability of asymmetric ball of polynomials (2) is* $\rho_{p\max} = \min\{\rho_{p1}, \rho_{p2}, \rho_{p3}, \rho_{p4}\}$.

Let us illustrate the derived result on an example.

4. Example

Consider the family of polynomials

$$p(s, A) = 433.5 + 667.5s + 502.6s^2 + 251.7s^3 + 80.3s^4 + 14.2s^5 + s^6$$

with

$$\begin{aligned} \underline{\alpha} &= [43.8, 29.6, 25.1, 15.0, 5.6, 1.4, 0.1], \\ \overline{\alpha} &= [48.2, 26.5, 29.1, 12.6, 4.3, 2.2, 0.4]. \end{aligned}$$

Let us determine the maximum stability radius for $p = 2$ and $p = \infty$ (see (2)).

For $p = 2$ the four plots corresponding to Theorems 5–8 are shown in Figs. 1–4, respectively. The particular stability margins are $\rho_{21} = 4.01$, $\rho_{22} = 2.68$, $\rho_{23} = 2.65$ and $\rho_{24} = 3.68$. The maximum radius preserving stability is $\rho_{2\max} = \min\{\rho_{21}, \rho_{22}, \rho_{23}, \rho_{24}\} = 2.65$.

For $p = \infty$ the four plots corresponding to Theorems 5–8 are shown in Figs. 5–8, respectively. The particular stability margins are $\rho_{\infty 1} = 1.23$, $\rho_{\infty 2} = 2.26$, $\rho_{\infty 3} = 1.44$ and $\rho_{\infty 4} = 2.17$. The maximum radius preserving stability is $\rho_{\infty\,\max} = \min\{\rho_{\infty 1}, \rho_{\infty 2}, \rho_{\infty 3}, \rho_{\infty 4}\} = 1.44$.

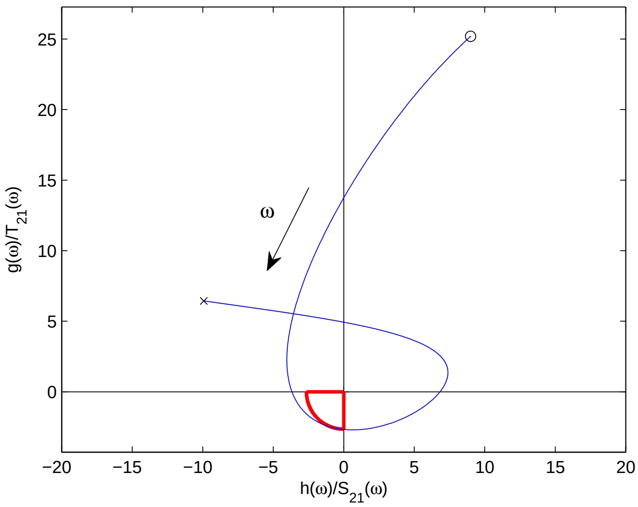

Figure 1. Frequency plot for $\Delta h(\omega) \geq 0, \Delta g(\omega) \geq 0$

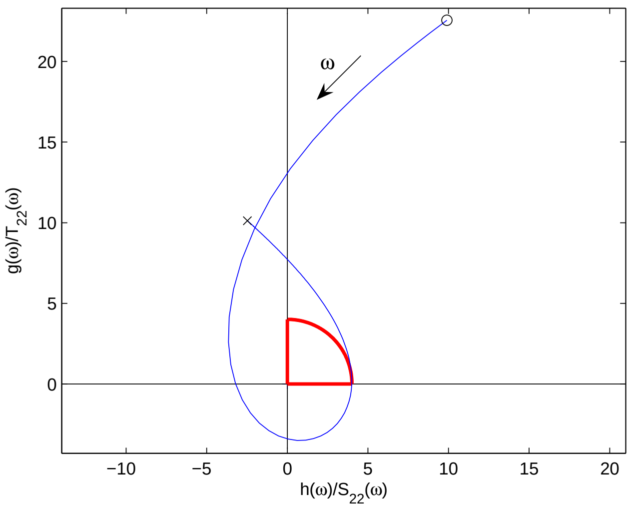

Figure 2. Frequency plot for $\Delta h(\omega) \leq 0, \Delta g(\omega) \leq 0$

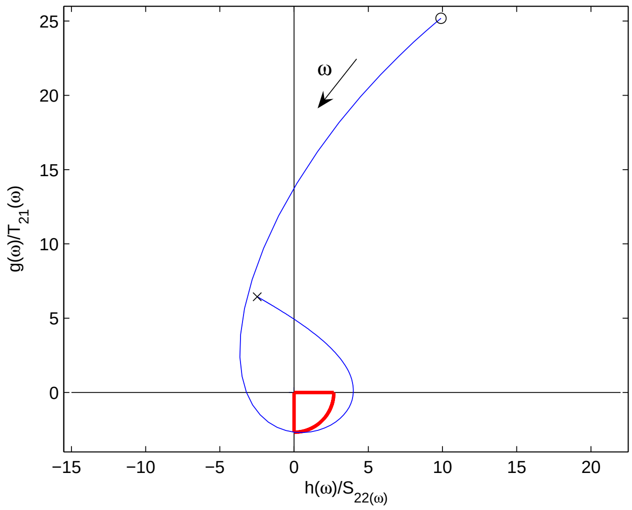

Figure 3. Frequency plot for $\Delta h(\omega) \leq 0, \Delta g(\omega) \geq 0$

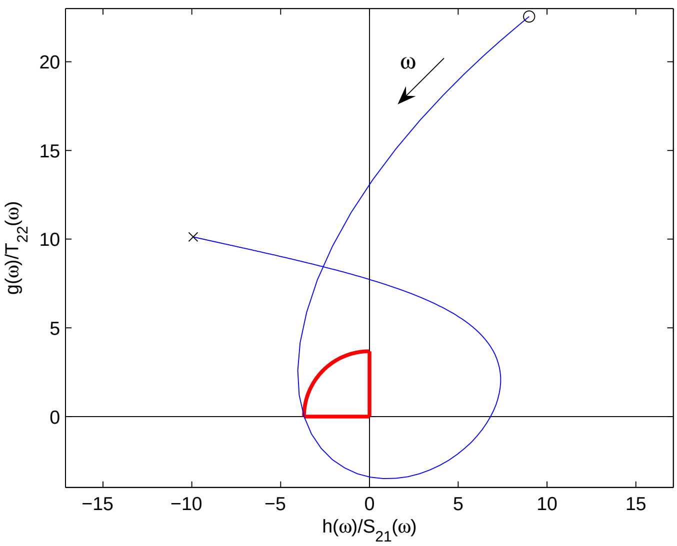

Figure 4. Frequency plot for $\Delta h(\omega) \geq 0, \Delta g(\omega) \leq 0$

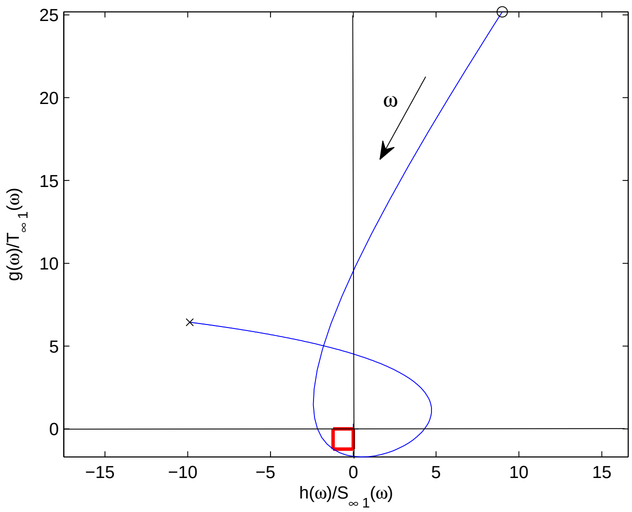

Figure 5. Frequency plot for $\Delta h(\omega) \geq 0, \Delta g(\omega) \geq 0$

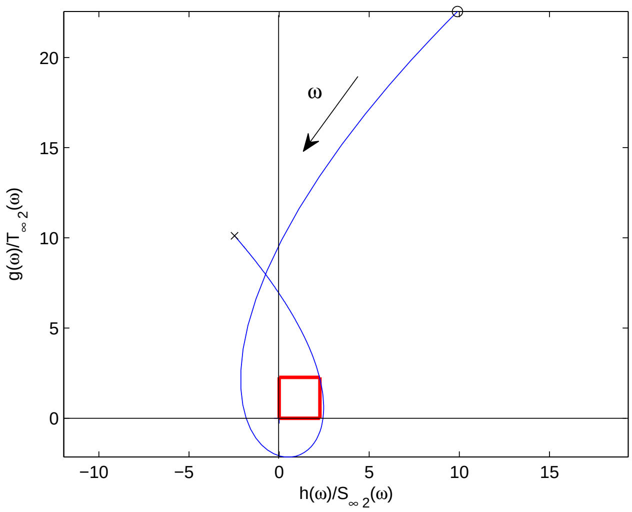

Figure 6. Frequency plot for $\Delta h(\omega) \leq 0, \Delta g(\omega) \leq 0$

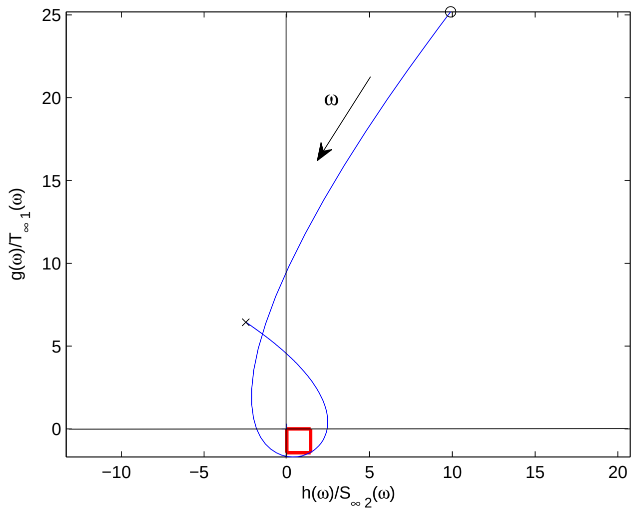

Figure 7. Frequency plot for $\Delta h(\omega) \leq 0, \Delta g(\omega) \geq 0$

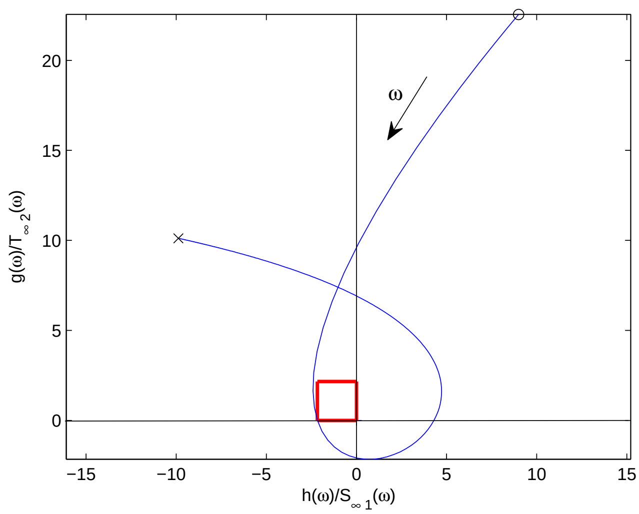

Figure 8. Frequency plot for $\Delta h(\omega) \geq 0, \Delta g(\omega) \leq 0$

5. Conclusion

Extension of the Tsypkin-Polyak locus to the case of different weights considered for coefficient being above and below its nominal value was presented in this paper. It was shown that four plots have to be drawn instead of one in order to determine the maximum radius of the asymmetric ball of polynomials preserving Hurwitz stability. The result was demonstrated on an illustrative example.

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