Control and Cybernetics

vol. **37** (2008) No. 3

Book review:

LAWS OF SMALL NUMBERS: EXTREME AND RARE EVENTS

by

Michael Falk, Jürg Hüsler and Rolf-Dieter Reiß

The reviewed volume is the second edition of the seminal book, which was first published in 1994. Because of increasing interest and many new advances in the theory and applications of extreme and rare events, the new edition is greatly expanded and carefully revised. The entire new part about theory of multivariate extremes is added. Additionally, a lot of new results from various sources in statistical literature are incorporated into other parts. The main aim of the first edition was to focus on a mathematically oriented development of the theory underlying all applications. In the second edition this attitude is strengthened, e.g. the section on the statistical software Xtremes from first edition was transferred to another book. In almost all chapters the authors develop a general theory first and then they apply this theory to specific subfields and present examples of applications.

The law of small numbers is the well known result in the probability theory. It constitutes the approximation of binomial distribution with a small probability of success in each run, by appropriate Poisson distribution. Famous example is the number of cavalrymen killed by friendly horse-kick, which was given in the fundamental book "*The law of small numbers*" by Ladislaus von Bortkiewicz. In this example the frequencies of number of cavalrymen killed in ten particular regiments of the Royal Prussian army over the twenty year period fit quite well to some Poisson distribution.

The authors start the Part I of their book with the history mentioned above in order to develop a particular extension of the law of small numbers, called by them the *functional laws of small numbers.* This extension links some different fields, like nonparametric regression analysis and extreme value theory. The concept is based on the following reasoning, provided by the authors: suppose that we are interested in a random sample of independent, identically distributed (iid) elements which fall into a given subset of the sample space. This sample may be described by the truncated empirical point process and if the probability of falling into the subset is small enough, then we could approximate this truncated point process by a Poisson point process. If the mentioned subset is located in the centre of the distribution, the link to regression analysis may be established. Otherwise, if the subset is at the border, the extreme value theory is more appropriate. In the next chapter authors present results in the extreme value theory, which are based on the condition that the upper tail of the underlying density function is in the δ-neighbourhood of a generalized Pareto distribution. Then, starting from the example of regression analysis, where conditional distribution function is established, the relation between Poisson approximation of truncated empirical point processes and reduction of conditional statistical problems to unconditional ones is discussed.

Part II was added in the second edition of the book. In this part the developments in multivariate extreme value theory based on the Pickands representations of extreme value distributions is discussed. The authors start from the basic theory of multivariate maxima. They study the limiting distributions of componentwise defined maxima of independent identically distributed d-variate random vectors. These considerations lead them to so called max-stable distribution functions which could be represented as the de Haan-Resnick and the Pickands representations. Similarly to the univariate case, the multivariate, generalized Pareto density functions are also introduced. Such density functions are among the most innovative and fruitful topics during the last years. The Pickands approach is discussed in detail in next chapters. Based on the Pickands representation of multivariate extreme value density functions, the spectral decomposition of multivariate density functions into univariate ones using the Pickands coordinates like distance and (pseudo)angle is presented. Then, conditions and results from the univariate setting to the multivariate one are directly transferred. In the next chapter, the considerations are restricted to bivariate random vectors. Such approach allows to study the properties of distributions in more detailed way. For example, the tail dependence parameter, which measures the tail dependence between two random variables, and the test for tail independence are introduced. Also for these problems the Pickands approach turns out to be very fruitful. This part of the book finishes with supplementary discussion on point process approach used for exceedances and upper order statistics.

Part III is devoted to development of the rare events theory for non iid observations. Applications of this theory may be found in risk analysis, telecommunication modelling and finance investigations. The authors start from some examples which provide motivation extension of the classical extreme value theory for iid sequences to a theory of non iid sequences. They introduce some classes of various non iid sequences. Then the general theory of extremes and exceedances of high boundaries by nonstationary random sequences is introduced. In the following chapter the results for stationary and locally stationary Gaussian processes are discussed. Especially interesting in case of simulations and approximations may be the relation between the continuous Gaussian process and its discrete approximation. Then the discussion about some extensions to general rare events is provided. In such case the authors use triangular arrays of the rare events approach. In the last chapter a few nonstandard applications of the previously developed theory are summarized, like the statistical estimation of the cluster distribution and of the extremal index in a stationary situation.

The reviewed book provides a very good survey of theory and applications of extremes and rare events. The readers should be very familiar with basic theory of probability. Additional knowledge in point processes and Gaussian processes may be also useful. The theory presented in this book is very detailed, developments discussed are quite diverse and the described results are very various. The mathematical rigour of notation is worth stressing, like the mathematical strictness of the language. However, some titles of sections are more "popularly" written (like "Why the Hellinger Distance?", for example). Almost every chapter consists of introductory theory, review of mathematical results, like proven lemmas and theorems, and presentation of some applications where the discussed theory is used.

The book, as a monograph of some sort, is useful for both students and researchers. Taking into account the wide range of the results and the theory itself presented in this book, it should be recommended for readers interested in various flavours of the theory and applications of extremes and rare events.

Maciej Romaniuk

Michael Falk, Jürg Hüsler and Rolf-Dieter Reiß, *Laws of Small Numbers: Extreme and Rare Events.* Second, revised and extended edition. Birkhäuser Verlag, Basel–Boston–Berlin, 2004. XIII + 376 pages. ISBN 978-3-7643-2416-2. Price: 39.90 EUR + VAT (softcover).