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Correctness of a constrained control Mayer's problem for a class of singularly perturbed functional-differential systems*

by

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Abstract: A Mayer's problem for a singularly perturbed controlled system with the general type of a small state delay is considered. The control is subject to geometrical constraints. The cost functional is a function of the terminal value of the slow state variable. A simpler parameter-free optimal control problem (the reduced problem) is associated with the original problem. A convergence of the optimal value of the cost functional in the original problem to the optimal value of the cost functional in the reduced problem, as a parameter of singular perturbation tends to zero, is established. An asymptotic suboptimality of the optimal control of the reduced problem in the original problem is shown. These results are extended to some more general optimal control problems. An illustrative example is presented.

Keywords: functional-differential system, time delay, Mayer's problem, bounded control, singular perturbation, correctness, suboptimal control.

1. Introduction

Control problems for singularly perturbed systems are studied extensively for about four recent decades. Most of publications in the topic are devoted to analysis of undelayed dynamics control problems (see e.g. Artstein, 2005; Dmitriev and Kurina, 2006; Dontchev, 1983; Dontchev and Zolezzi, 1993; Gajic and Lim, 2001; Kokotovic, Khalil and O'Reilly, 1999; Naidu, 2002, and references therein). Control problems for singularly perturbed delayed systems are studied much less (see e.g. Dmitriev and Kurina, 2006; Fridman, 1990, 2006; Glizer, 1998, 2000, 2004, 2005, 2006, 2007; Kopeikina, 1989; Lin-Chen and Goodall, 2004; Reddy and Sannuti, 1974; Slavov, 1995, and references therein).

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One of the important issues, arising in control theory, is optimization of a controlled system with respect to a given performance index. The rich literature is devoted to studying this issue for singularly perturbed undelayed systems with unconstrained controls (see e.g. Kokotovic, Khalil and O'Reilly, 1999; Gajic and Lim, 2001, and references therein) and constrained controls (see e.g. Dontchev, 1983; Dontchev and Zolezzi, 1993, and references therein) by using the concept of separation of time scales and the order reduction approach. For the analysis of optimal control problems with singularly perturbed undelayed dynamics, not allowing for application of the order reduction approach, the method, based on the notion of limit distributions of control and fast state variable on the fast time scale, was developed in a number of works (see e.g. Artstein, 2005, and references therein).

Optimal control problems for singularly perturbed systems with delays were studied only in several works. The problems with unconstrained controls were considered in Fridman (1990), Glizer (1998, 2000, 2007), Reddy and Sannuti (1974) (the case of small delay) and in Glizer (2005, 2006) (the case of nonsmall delay). To our best knowledge, optimal control problems with constrained controls for singularly perturbed delayed systems were not studied in literature.

In this paper, we consider an optimal control problem with a prescribed duration for a singularly perturbed time-dependent system with the general type of delay in state variables. The system is linear with respect to the states. The initial conditions for this system are given. The control is constrained. The cost functional is a function of the terminal value of the slow state variable. The delay is small of order of the small parameter $\varepsilon > 0$ multiplying a part of the derivatives in the system.

Singularly perturbed systems with small delays of order of the small multiplier for a part of the derivatives are highly significant for the domain of functional-differential equations (see e.g. Artstein and Slemrod, 2001; Fridman, 1996; Glizer, 2003; Halanay, 1966; Mitropol'skii, Fodchuk and Klevchuk, 1986, and references therein), in control theory (see e.g. Fridman, 1990, 2006; Glizer, 1998, 2000, 2004, 2007; Reddy and Sannuti, 1974, and references therein), and in various applications (see e.g. Lange and Miura, 1994; Lizana, 1999; Reddy and Sannuti, 1975, and references therein). Singularly perturbed systems with nonsmall delays also have a considerable importance in theory and applications (see e.g. Cooke and Meyer, 1966; Donchev and Slavov, 1995, 1997; Glizer, 2005, 2006; Hale and Tanaka, 2000; Kopeikina, 1989; Magalhaes, 1984; Mallet-Paret and Nussbaum, 1989, and references therein). It should be noted that the methods of analysis of singularly perturbed systems with small and nonsmall delays essentially differ.

For the problem considered in the paper, an asymptotic behavior (as $\varepsilon \to +0$) of the optimal value of the cost functional is studied. This study is carried out by an extension of the order reduction approach to such a class of problems. Namely, first, a simpler ε-free optimal control problem (the reduced one) is associated with the original problem. Second, it is shown that the optimal value of the cost functional of the latter converges to the optimal value of the cost functional of the former, while $\varepsilon \to +0$. Such a convergence means the correctness of the singularly perturbed optimal control problem considered in the paper. An estimate of the difference between the optimal values of the cost functional in the original and reduced problems also is obtained. As a consequence of these results, it is shown that the optimal control of the reduced problem can serve as a suboptimal control in the original problem for all sufficiently small values of ε, i.e., robustly with respect to this parameter.

The paper is organized as follows. In the next section, the problem is formulated rigorously. Section 3 is devoted to some preliminary results. Main results are presented in Section 4. An illustrative example is considered in Section 5. In Section 6, some extensions of the main results are discussed. Sections 7 and 8 are devoted to proofs of the lemmas presented in Section 3.

The following notations are applied in the paper:

1. E^n is the n-dimensional real Euclidean space.
2. The Euclidean norm of either a matrix or a vector is denoted as $\|\cdot\|$.
3. I_n denotes the n-dimensional identity matrix.
4. The prime, as an upper index, denotes the transposition either of a matrix A, $(A^{'})$ or of a vector x, $(x^{'})$.
5. $\mathrm{col}(x, y)$, where $x \in E^n, y \in E^m$, denotes the column block-vector of the dimension $n + m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x, y) = (x^{'}, y^{'})^{'}$.

2. Problem formulation and main assumptions

2.1. Original problem

Consider the following system

$$
\begin{aligned}
dx(t)/dt &= \int_{-h}^{0}\Big[d_\eta A_1(t,\eta)\Big]x(t+\varepsilon\eta) + \int_{-h}^{0}\Big[d_\eta A_2(t,\eta)\Big]y(t+\varepsilon\eta) \\
&+ B_1(t,u(t)), \qquad t \in [0,T], \qquad (1) \\
\varepsilon dy(t)/dt &= \int_{-h}^{0}\Big[d_\eta A_3(t,\eta)\Big]x(t+\varepsilon\eta) + \int_{-h}^{0}\Big[d_\eta A_4(t,\eta)\Big]y(t+\varepsilon\eta) \\
&+ B_2(t,u(t)), \qquad t \in [0,T], \qquad (2)
\end{aligned}
$$

where $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^r$ (u is a control); $\varepsilon > 0$ is a small parameter; $A_1(t,\eta), A_2(t,\eta), A_3(t,\eta)$ and $A_4(t,\eta)$ are matrices of corresponding dimensions; $B_1(t,u)$ and $B_2(t,u)$ are vectors of corresponding dimensions; $h > 0$ is a given constant and $T > 0$ is a given time instant, both independent of ε.

Note that $x(\cdot)$ and $y(\cdot)$ are called the slow and fast state variables, respectively, of the system (1)-(2).

In the sequel, we assume:

A1. The matrix-valued functions $A_i(t,\eta)$, $(i=1,...,4)$ are given for $(t,\eta)\in[0,T]\times(-\infty,+\infty)$ and satisfy the conditions:
(a) $A_i(t,\eta)=0$, $(i=1,...,4)$, $\forall(t,\eta)\in[0,T]\times[0,+\infty)$;
(b) $A_i(t,\eta)=A_i(t,-h)$, $(i=1,...,4)$, $\forall(t,\eta)\in[0,T]\times(-\infty,-h]$;
(c) $A_k(t,\eta)$, $(k=1,2)$, are continuously differentiable with respect to $t\in[0,T]$ uniformly in $\eta\in(-\infty,+\infty)$;
(d) $A_l(t,\eta)$, $(l=3,4)$, are twice continuously differentiable with respect to $t\in[0,T]$ uniformly in $\eta\in(-\infty,+\infty)$;
(e) for each $t\in[0,T]$, $A_i(t,\eta)$, $(i=1,...,4)$, are continuous from the left with respect to $\eta\in(-h,0)$;
(f) $A_i(t,\eta)$ and $\partial A_i(t,\eta)/\partial t$, $(i=1,...,4)$, have bounded variations with respect to η on the interval $[-h,0]$ uniformly in $t\in[0,T]$;
(g) $\partial^2 A_l(t,\eta)/\partial t^2$, $(l=3,4)$, have bounded variations with respect to η on the interval $[-h,0]$ uniformly in $t\in[0,T]$.

A2. The vector-valued functions $B_j(t,u)$, $(j=1,2)$, are defined for $(t,u)\in[0,T]\times D_B$, where $D_B\subseteq E^r$ is a given set.

For (1)-(2), the initial conditions are given as

$$x(\tau)=\varphi_x(\tau),\quad y(\tau)=\varphi_y(\tau),\ \tau\in[-\varepsilon h,0), \tag{3}$$

$$x(0)=x_0,\quad y(0)=y_0, \tag{4}$$

where $\varphi_x(\tau)$ and $\varphi_y(\tau)$ are given vector-valued functions; x_0 and y_0 are given vectors. In the sequel, we assume:

A3. There exists a positive constant ε_0, such that the vector-valued functions $\varphi_x(\tau)$ and $\varphi_y(\tau)$ are continuously differentiable for $\tau\in[-\varepsilon_0 h,0]$.

The admissible controls are measurable functions for $t\in[0,T]$ satisfying the inclusion

$$u(t)\in D_u,\quad t\in[0,T], \tag{5}$$

where $D_u\subseteq D_B$ is a given set. The set of all admissible controls $u(t)$ is denoted by U. The performance index, evaluating the control process, is

$$J(u(t))\triangleq F\Big(x(T)\Big)\to\min_{u(t)\in U}, \tag{6}$$

where $F(x)$ is a given scalar function defined for $x\in E^n$.

In the sequel, we assume:

A4. $\|B_j(t,u)\|\le c_B\ \ \forall(t,u)\in[0,T]\times D_u$, $\ (j=1,2)$, where $c_B>0$ is some constant.

A5. The function $F(x)$ is continuous and has continuous first-order partial derivatives for $x\in E^n$.

The problem (1)-(6) is called the Original Optimal Control Problem (OOCP).

2.2. Reduced problem

Setting formally $\varepsilon = 0$ in (1)-(2), and redenoting x by $\bar{x}$, y by $\bar{y}$ and u by $\bar{u}$, we obtain the system

$$d\bar{x}(t)/dt = \bar{A}_1(t)\bar{x}(t) + \bar{A}_2(t)\bar{y}(t) + B_1(t, \bar{u}(t)), \tag{7}$$

$$0 = \bar{A}_3(t)\bar{x}(t) + \bar{A}_4(t)\bar{y}(t) + B_2(t, \bar{u}(t)), \tag{8}$$

where

$$\bar{A}_i(t) = \int_{-h}^{0} d_\eta A(t, \eta), \quad i = 1, ..., 4. \tag{9}$$

Consider the following matrix depending on t and a complex parameter λ

$$W(t, \lambda) = \int_{-h}^{0} \exp(\lambda\eta) d_\eta A_4(t, \eta). \tag{10}$$

In the sequel, we assume:

A6. All roots λ of the equation $\det[W(t, \lambda) - \lambda I_m] = 0$ lie inside the left-hand half-plane for all $t \in [0, T]$.

Due to (9) and (10), $W(t, 0) = \bar{A}_4(t) \quad \forall t \in [0, T]$ yielding, by using the assumption A6,

$$\det \bar{A}_4(t) \neq 0 \quad \forall t \in [0, T]. \tag{11}$$

Thus, by using (11), we can eliminate y from (7)-(8), which yields the following system

$$d\bar{x}(t)/dt = \bar{A}_0(t)\bar{x}(t) + \bar{B}_0(t, \bar{u}(t)), \quad t \in [0, T], \tag{12}$$

where

$$\bar{A}_0(t) = \bar{A}_1(t) - \bar{A}_2(t)\bar{A}_4^{-1}(t)\bar{A}_3(t), \tag{13}$$

$$\bar{B}_0(t, \bar{u}) = B_1(t, \bar{u}) - \bar{A}_2(t)\bar{A}_4^{-1}(t)B_2(t, \bar{u}). \tag{14}$$

For (12), the initial condition and the performance index are, respectively,

$$\bar{x}(0) = x_0, \tag{15}$$

$$\bar{J}(\bar{u}(t)) \stackrel{\triangle}{=} F\Big(\bar{x}(T)\Big) \to \min_{\bar{u}(t) \in U}. \tag{16}$$

The problem (12)-(16) is called the Reduced Optimal Control Problem (ROCP). It can be seen that the ROCP is much simpler than the OOCP. The ROCP is of a lower dimension, it is delay-free and it is ε-free.

2.3. Objectives of the paper

The objectives of the paper are the following:
(1) to establish the convergence of the optimal value of the cost functional in the OOCP to the optimal value of the cost functional in the ROCP for $\varepsilon \to +0$, i.e., to establish the correctness of the OOCP;
(2) to estimate the difference between the optimal values of the cost functional in the OOCP and ROCP;
(3) to establish the suboptimality of the ROCP optimal control in the OOCP for all sufficiently small $\varepsilon > 0$, i.e., robustly with respect to this parameter.

3. Preliminary results

Consider the block matrix

$$A(t,\eta,\varepsilon) = \begin{pmatrix} A_1(t,\eta) & A_2(t,\eta) \\ (1/\varepsilon)A_3(t,\eta) & (1/\varepsilon)A_4(t,\eta) \end{pmatrix}. \tag{17}$$

Let, for a given $\varepsilon > 0$, the $\Big((n+m)\times(n+m)\Big)$-matrix $\Psi(t,s,\varepsilon)$ be the solution of the problem

$$\partial\Psi(t,s,\varepsilon)/\partial t = \int_{-h}^{0}\Big[d_\eta A(t,\eta,\varepsilon)\Big]\Psi(t+\varepsilon\eta,s,\varepsilon), \quad 0 \le s < t \le T, \tag{18}$$

$$\Psi(t,s,\varepsilon) = 0, \ \ t < s; \qquad \Psi(s,s,\varepsilon) = I_{n+m}. \tag{19}$$

It is clear that $\Psi(t,s,\varepsilon)$ is the fundamental matrix solution of the homogeneous version ($B_j(t,u) \equiv 0$, $(j = 1,2)$) of the system (1)-(2).

Let the $n \times n$-matrix $\bar\Psi(t,s)$ be the solution of the problem

$$\partial\bar\Psi(t,s)/\partial t = \bar A_0(t)\bar\Psi(t,s), \quad 0 \le s < t \le T, \tag{20}$$

$$\bar\Psi(s,s) = I_n. \tag{21}$$

The matrix $\bar\Psi(t,s)$ is the fundamental matrix solution of the homogeneous version ($\bar B_0(t,\bar u) \equiv 0$) of the system (12).

Let the $m \times m$-matrix $\tilde\Psi(\xi,s)$ be the solution of the problem

$$\partial\tilde\Psi(\xi,s)/\partial\xi = \int_{-h}^{0}\Big[d_\eta A_4(s,\eta)\Big]\tilde\Psi(\xi+\eta,s), \quad 0 \le s \le T, \ \ \xi > 0, \tag{22}$$

$$\tilde\Psi(\xi,s) = 0, \ \ \xi < 0; \quad \tilde\Psi(0,s) = I_m, \qquad 0 \le s \le T. \tag{23}$$

Due to the assumption A6 and results of Hale and Lunel (1993), the matrix $\tilde\Psi(\xi,s)$ satisfies the inequality

$$\|\tilde\Psi(\xi,s)\| \le a\exp(-\beta\xi), \quad 0 \le s \le T, \ \ \xi \ge 0. \tag{24}$$

REMARK 1 *In (24) and in the sequel, a and β denote some positive constants independent of ε.*

LEMMA 1 *Let $\Psi_1(t,s,\varepsilon)$, $\Psi_2(t,s,\varepsilon)$, $\Psi_3(t,s,\varepsilon)$ and $\Psi_4(t,s,\varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand and lower right-hand blocks of the matrix $\Psi(t,s,\varepsilon)$ of the dimensions $n\times n$, $n\times m$, $m\times n$ and $m\times m$, respectively. Then, under the assumptions A1 and A6, there exists a number $\varepsilon_1>0$, such that these blocks satisfy the following inequalities for all $\varepsilon\in(0,\varepsilon_1]$ and $0\le s\le t\le T$:*

$$\|\Psi_1(t,s,\varepsilon)-\bar{\Psi}(t,s)\|\le a\varepsilon, \tag{25}$$

$$\|\Psi_2(t,s,\varepsilon)+\varepsilon\bar{\Psi}(t,s)\bar{A}_2(s)\bar{A}_4^{-1}(s)\|\le a\varepsilon\Big[\varepsilon+\exp\Big(-\beta(t-s)/\varepsilon\Big)\Big], \tag{26}$$

$$\|\Psi_3(t,s,\varepsilon)+\bar{A}_4^{-1}(t)\bar{A}_3(t)\bar{\Psi}(t,s)\|\le a\Big[\varepsilon+\exp\Big(-\beta(t-s)/\varepsilon\Big)\Big], \tag{27}$$

$$\Big\|\Psi_4(t,s,\varepsilon)-\tilde{\Psi}\Big((t-s)/\varepsilon,s\Big)\Big\|\le a\varepsilon. \tag{28}$$

The proof of the lemma is presented in Section 7

Let for any $\varepsilon\in(0,\varepsilon_0]$, $z(t,\varepsilon)=\mathrm{col}\Big(x(t,\varepsilon),y(t,\varepsilon)\Big)$, $t\in[0,T]$ be the solution of the original singularly perturbed system (1)-(2) with a given $u(t)\in U$, and the initial conditions (3)-(4). Due to the assumption A4, the definition of the set U of control functions and results of Hale and Lunel (1993), this solution exists and is unique. Let for the same $u(t)$ as in $z(t,\varepsilon)$, $\bar{x}(t)$, $t\in[0,T]$ be the solution of the reduced system (12) with $\bar{u}(t)$, replaced by $u(t)$, and the initial condition (15). This solution also exists and is unique.

LEMMA 2 *Under the assumptions A1-A4 and A6, the following inequalities are satisfied for all $\varepsilon\in(0,\varepsilon_2]$, ($\varepsilon_2\le\min\{\varepsilon_0,\varepsilon_1\}$), and all $u(t)\in U$:*

$$\|x(t,\varepsilon)\|\le a,\qquad \|y(t,\varepsilon)\|\le a,\quad t\in[0,T], \tag{29}$$

$$\|\bar{x}(t)\|\le a,\quad t\in[0,T], \tag{30}$$

$$\|x(t,\varepsilon)-\bar{x}(t)\|\le a\varepsilon,\qquad t\in[0,T], \tag{31}$$

where $a>0$ is some constant independent of both ε and $u(t)$.

The proof of the lemma is presented in Section 8.

4. Main results

Denote in the OOCP, for a given $\varepsilon\in(0,\varepsilon_0]$,

$$J_\varepsilon^*\stackrel{\triangle}{=}\inf_{u(t)\in U}J(u(t)), \tag{32}$$

and in the ROCP

$$\bar{J}^*\stackrel{\triangle}{=}\inf_{\bar{u}(t)\in U}\bar{J}(\bar{u}(t)). \tag{33}$$

THEOREM 1 *Let the assumptions A1-A6 be valid. Then,*

$$|J_\varepsilon^* - \bar{J}^*| \le a\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{34}$$

where the positive constant ε_2 is defined in Lemma 2.

Proof. First of all, let us note that, by Lemma 2 and the assumption A5,

$$J_\varepsilon^* > -\infty \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{35}$$
$$\bar{J}^* > -\infty. \tag{36}$$

Due to (32) and (35), for a given $\varepsilon \in (0, \varepsilon_2]$, there exist sequences $\{u_p(t)\}$, $(u_p(t) \in U,\ p = 1, 2, ...)$, and $\{\delta_p\}$, $(\delta_p \ge 0,\ p = 1, 2, ...)$,

$$\lim_{p \to +\infty} \delta_p = 0, \tag{37}$$

such that in the OOCP

$$J(u_p(t)) = J_\varepsilon^* + \delta_p, \qquad p = 1, 2, ... \ . \tag{38}$$

Similarly, due to (33) and (36), there exist sequences $\{\bar{u}_p(t)\}$, $(\bar{u}_p(t) \in U,\ p = 1, 2, ...)$, and $\{\bar{\delta}_p\}$, $(\bar{\delta}_p \ge 0,\ p{=}1, 2, ...)$,

$$\lim_{p \to +\infty} \bar{\delta}_p = 0, \tag{39}$$

such that in the ROCP

$$\bar{J}(\bar{u}_p(t)) = \bar{J}^* + \bar{\delta}_p, \qquad p = 1, 2, ... \ . \tag{40}$$

Let us fix any $\varepsilon \in (0, \varepsilon_2]$ and any $p \in \{1, 2, ...\}$. Then, using Lemma 2, the equations (6) and (16), and the assumption A5, one has directly

$$|J(u_p(t)) - \bar{J}(u_p(t))| \le a\varepsilon, \qquad |J(\bar{u}_p(t)) - \bar{J}(\bar{u}_p(t))| \le a\varepsilon, \tag{41}$$

where $a > 0$ is some constant independent of both $\varepsilon \in (0, \varepsilon_2]$ and $p \in \{1, 2, ...\}$.
Substitution of (38) and (40) into (41) yields, after a simple algebra

$$-a\varepsilon - \delta_p \le J_\varepsilon^* - \bar{J}(u_p(t)) \le a\varepsilon - \delta_p, \tag{42}$$
$$-a\varepsilon + \bar{\delta}_p \le -\bar{J}^* + J(\bar{u}_p(t)) \le a\varepsilon + \bar{\delta}_p. \tag{43}$$

By virtue of (32) and (33), we have the inequalities

$$J_\varepsilon^* - J(\bar{u}_p(t)) \le 0, \tag{44}$$
$$0 \le -\bar{J}^* + \bar{J}(u_p(t)). \tag{45}$$

By adding the left-hand inequality in (42) and the inequality (45) we get

$$-a\varepsilon - \delta_p \le J_\varepsilon^* - \bar{J}^*. \tag{46}$$

Similarly, by adding the right-hand inequality in (43) and inequality (44) we get

$$J_\varepsilon^* - \bar{J}^* \le a\varepsilon + \bar{\delta}_p. \tag{47}$$

The inequalities (46) and (47) directly lead to the inequality

$$|J_\varepsilon^* - \bar{J}^*| \le a\varepsilon + \nu_p, \qquad \nu_p = \max\{\delta_p, \bar{\delta}_p\}. \tag{48}$$

The inequality (48) is valid for any $\varepsilon \in (0, \varepsilon_2]$ and $p \in \{1, 2, ...\}$, while the constant a is independent of these ε and p. Moreover, due to (37) and (39), $\lim_{p\to+\infty} \nu_p = 0$. The latter, along with (48), leads to the hypothesis of the theorem. ∎

Following the definition of correctness of optimal control problems with singularly perturbed undelayed dynamics (see Dontchev, 1983), we say that the OOCP is posed correctly if $\lim_{\varepsilon\to+0} J_\varepsilon^* = \bar{J}^*$. As a direct consequence of Theorem 1, we have the following corollary.

Corollary 1 *Under the assumptions A1-A6, the OOCP is posed correctly.*

In the sequel of this section, we assume:

A7. The ROCP has a solution, and $\bar{u}^*(t)$ is its optimal control.

Under the assumption A7, we obtain that in the ROCP

$$\bar{J}^* = \bar{J}(\bar{u}^*(t)). \tag{49}$$

Let $J_\varepsilon(\bar{u}^*(t))$ be the value of the cost functional in the OOCP for $u(t) = \bar{u}^*(t)$.

Theorem 2 *Let the assumptions A1-A7 be valid. Then,*

$$0 \le J_\varepsilon(\bar{u}^*(t)) - J_\varepsilon^* \le a\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{50}$$

where the positive constant ε_2 is the same as in Lemma 2 and Theorem 1.

Proof. From Lemma 2, the equations (6),(16),(49) and the assumption A5, one obtains the following inequality for all $\varepsilon \in (0, \varepsilon_2]$: $|J_\varepsilon(\bar{u}^*(t)) - \bar{J}^*| \le a\varepsilon$. This inequality, along with (34), yields for all $\varepsilon \in (0, \varepsilon_2]$: $|J_\varepsilon(\bar{u}^*(t)) - J_\varepsilon^*| \le a\varepsilon$. The latter and the inequality $J_\varepsilon^* \le J_\varepsilon(\bar{u}^*(t))$ lead to the statement of theorem. ∎

Remark 2 *Theorem 2 implies that the control $u(t) = \bar{u}^*(t)$ is asymptotically suboptimal in the OOCP, i.e., it is suboptimal for all sufficiently small $\varepsilon > 0$.*

Remark 3 *The proofs of Theorems 1 and 2 are based neither on necessary nor on sufficient optimality conditions for both the OOCP and the ROCP. Moreover, these proofs do not use any assumption on the OOCP solution existence.*

5. Example

Consider the following particular case of the OOCP with scalar slow and fast state variables, and with a scalar control:

$$dx(t)/dt = -x(t) - 3x(t-\varepsilon) + 2y(t) + y(t-\varepsilon) + (t-5)u(t),\ t \in [0,2],\ (51)$$

$$\varepsilon dy(t)/dt = 3x(t) - x(t-\varepsilon) - 4y(t) + 2y(t-\varepsilon) + (t+2)u(t),\ t \in [0,2],\ (52)$$

$$x(\tau) = \tau, \quad y(\tau) = \tau^2, \quad \tau \in [-\varepsilon, 0), \qquad (53)$$

$$x(0) = 1, \quad y(0) = 2, \qquad (54)$$

$$|u(t)| \leq 1, \quad t \in [0,2], \qquad (55)$$

$$J(u(t)) \stackrel{\triangle}{=} -x(2) \to \min_{u(t):|u(t)|\leq 1}. \qquad (56)$$

Comparing (51)-(52) with (1)-(2), one obtains that for (51)-(52) the matrix-valued functions $A_i(t,\eta)$, $(i = 1, ..., 4)$ become scalar ones independent of t, i.e., $A_i(t,\eta) \equiv A_i(\eta)$, $t \in [0,2]$, $(i = 1, ..., 4)$, and these functions have the form

$$A_1(\eta) = \begin{cases} 2, & -\infty < \eta \leq -1, \\ -1, & -1 < \eta < 0, \\ 0, & \eta \geq 0, \end{cases} \qquad A_2(\eta) = \begin{cases} 1, & -\infty < \eta \leq -1, \\ 2, & -1 < \eta < 0, \\ 0, & \eta \geq 0, \end{cases} \qquad (57)$$

$$A_3(\eta) = \begin{cases} 4, & -\infty < \eta \leq -1, \\ 3, & -1 < \eta < 0, \\ 0, & \eta \geq 0, \end{cases} \qquad A_4(\eta) = \begin{cases} -6, & -\infty < \eta \leq -1, \\ -4, & -1 < \eta < 0, \\ 0, & \eta \geq 0. \end{cases} \qquad (58)$$

The ROCP, associated with the OOCP (51)-(56) has the form

$$d\bar{x}(t)/dt = -\bar{x}(t) + (2.5t - 2)\bar{u}(t), \quad t \in [0,2]; \quad \bar{x}(0) = 1, \qquad (59)$$

$$|\bar{u}(t)| \leq 1, \quad t \in [0,2], \qquad (60)$$

$$\bar{J}(\bar{u}(t)) \stackrel{\triangle}{=} -\bar{x}(2) \to \min_{\bar{u}(t):|\bar{u}(t)|\leq 1}. \qquad (61)$$

It is obtained directly that the ROCP has the unique solution. The optimal control and the optimal value of the cost functional are $\bar{u}^*(t) = \mathrm{sign}(2.5t - 2)$ and $\bar{J}^* = \bar{J}(\bar{u}^*(t)) = -1.532$. The OOCP also has the unique solution. In Table 1, the optimal value J_ε^* of the cost functional (56) in the OOCP, as well as the ratio $\Delta J_\varepsilon^* = |J_\varepsilon^* - \bar{J}^*|/\varepsilon$, are presented for various values of ε. It is seen that, for decreasing ε, the ratio ΔJ_ε^* increases. However, this increase slows down. Numerical calculations have shown that $\max_{\varepsilon\in(0,0.1]} \Delta J_\varepsilon^* = 1.987$, meaning that in this example, $\varepsilon_2 = 0.1$ and $a = 2$ provide for the fulfillment of Theorem 1. In Table 2, the value of $J_\varepsilon(\bar{u}^*(t))$, as well as the ratio $\Delta J_\varepsilon(\bar{u}^*(t)) = (J_\varepsilon(\bar{u}^*(t)) - J_\varepsilon^*)/\varepsilon$, are presented for various values of ε. It can be seen that, for decreasing ε, the ratio $\Delta J_\varepsilon(\bar{u}^*(t))$ decreases. Numerical calculations have shown that $\max_{\varepsilon\in(0,0.1]} \Delta J_\varepsilon(\bar{u}^*(t)) = 4.104$, which means that in this example, $\varepsilon_2 = 0.1$ and $a = 4.11$ provide for the fulfillment of Theorem 2. The results of both tables also show that in this example, $\bar{J}^*$ is a better approximation of J_ε^* than $J_\varepsilon(\bar{u}^*(t))$.

Table 1. Values of J_ε^* and ΔJ_ε^*

ε	0.1	0.08	0.06	0.04	0.02
J_ε^*	- 1.427	- 1.429	- 1.440	- 1.462	- 1.493
ΔJ_ε^*	1.050	1.294	1.536	1.768	1.959

Table 2. Values of $J_\varepsilon(\bar{u}^*(t))$ and $\Delta J_\varepsilon(\bar{u}^*(t))$

ε	0.1	0.08	0.06	0.04	0.02
$J_\varepsilon(\bar{u}^*(t))$	- 1.017	- 1.111	- 1.210	- 1.314	- 1.422
$\Delta J_\varepsilon(\bar{u}^*(t))$	4.104	3.975	3.833	3.686	3.571

6. Some extensions

In this section, we consider some extensions of the results of Section 4.

6.1. Mayer's problem with an intermediary cost functional

Consider the optimal control problem consisting of the dynamics equations (1)-(2), initial conditions (3)-(4), control constraint (5) and the performance index

$$J_I(u(t)) \stackrel{\triangle}{=} G\Big(x(t_1), x(t_2), ..., x(t_N)\Big) \to \min_{u(t)\in U}, \tag{62}$$

where $G(x_1, x_2, ..., x_N)$ is a given scalar function for $x_1 \in E^n$, $x_2 \in E^n$, ..., $x_N \in E^n$; $0 < t_1 < t_2 < ... < t_N = T$ are given time instants; U is the set of admissible controls defined in Section 2.1. Similarly to Section 2.1, this problem is called original.

REMARK 4 *A cost functional of type (62) is called intermediary (see e.g. Bernhard, 1979). Optimal control problems and differential games with intermediary cost functionals were studied in a number of works (see e.g. Bernhard, 1979; Hagenaars, Imura and Nijmeijer, 2004; Lukoyanov and Reshetova, 1998; Turetsky, 1999).*

In the sequel, we assume:

A8. The function $G(x_1, x_2, ..., x_N)$ is continuous and has continuous partial derivatives for $x_1 \in E^n$, $x_2 \in E^n$, ..., $x_N \in E^n$.

Setting formally $\varepsilon = 0$ in the original problem (1)-(5),(62), one obtains, similarly to Section 2.2, the optimal control problem consisting of the dynamics equation (12), initial condition (15) and the performance index

$$\bar{J}_I(\bar{u}(t)) \stackrel{\triangle}{=} G\Big(\bar{x}(t_1), \bar{x}(t_2), ..., \bar{x}(t_N)\Big) \to \min_{\bar{u}(t)\in U}. \tag{63}$$

Similarly to Section 2.2, this problem is called reduced.

Denote in the original problem (1)-(5),(62), for a given $\varepsilon \in (0, \varepsilon_0]$,

$$J^*_{I,\varepsilon} \triangleq \inf_{u(t)\in U} J_I(u(t)), \tag{64}$$

and in the reduced problem (12),(15),(63)

$$\bar{J}^*_I \triangleq \inf_{\bar{u}(t)\in U} \bar{J}_I(\bar{u}(t)). \tag{65}$$

THEOREM 3 *Let the assumptions A1-A4,A6,A8 be valid. Then,*

$$|J^*_{I,\varepsilon} - \bar{J}^*_I| \le a\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{66}$$

where the positive constant ε_2 is defined in Lemma 2.

Proof. The theorem is proved similarly to Theorem 1. ∎

Like in Section 4, one obtains the correctness of the problem (1)-(5),(62). Now, we assume:

A9. The reduced problem (12),(15),(63) has a solution, and $\bar{u}^*_I(t)$ is its optimal control.

Let $J_{I,\varepsilon}(\bar{u}^*_I(t))$ be the value of the cost functional in the original problem (1)-(5),(62) for $u(t) = \bar{u}^*_I(t)$.

THEOREM 4 *Let the assumptions A1-A4,A6,A8,A9 be valid. Then,*

$$0 \le J_{I,\varepsilon}(\bar{u}^*_I(t)) - J^*_{I,\varepsilon} \le a\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{67}$$

where the positive constant ε_2 is the same as in Lemma 2 and Theorem 3.

Proof. The theorem is proved similarly to Theorem 2. ∎

Note that Theorem 4 implies the asymptotic suboptimality of the control $\bar{u}^*_I(t)$ in the original problem (1)-(5),(62).

6.2. Bolza's problem with an intermediary cost functional

Consider the optimal control problem consisting of the dynamics equations (1)-(2), initial conditions (3)-(4), control constraint (5) and the performance index

$$\begin{aligned} J_{BI}(u(t)) &\triangleq G\Big(x(t_1), x(t_2), ..., x(t_N)\Big) \\ &+ \int_0^T \Big(f^{'}(t)x(t) + g^{'}(t)y(t) + h(t, u(t))\Big)dt \to \min_{u(t)\in U}, \end{aligned} \tag{68}$$

where $f(t)$ and $g(t)$ are given vector-valued functions; $h(t, u)$ is a given scalar function. Like in Section 6.1, this problem is called original.

In the sequel, we assume:

A10. The vector-valued functions $f(t)$ and $g(t)$ are continuously differentiable for $t \in [0, T]$.

A11. The function $h(t, u)$ is defined and bounded for $(t, u) \in [0, T] \times D_u$, where the set D_u is introduced in Section 2.1 (see (5)).

Let us introduce the new state variable

$$v(t) = \int_0^t \Big(f^{'}(t)x(t) + g^{'}(t)y(t) + h(t, u(t))\Big)dt, \quad t \in [0, T]. \tag{69}$$

This state variable satisfies the differential equation

$$dv(t)/dt = f^{'}(t)x(t) + g^{'}(t)y(t) + h(t, u(t)), \quad t \in [0, T] \tag{70}$$

and the initial condition

$$v(0) = 0. \tag{71}$$

Note that the equation (70) can be rewritten in the equivalent form

$$dv(t)/dt = \int_{-h}^{0} \Big[d_\eta f_1^{'}(t, \eta)\Big]x(t + \varepsilon\eta) + \int_{-h}^{0} \Big[d_\eta g_1^{'}(t, \eta)\Big]y(t + \varepsilon\eta) + h(t, u(t)), \tag{72}$$

where $f_1(t, \eta) = f(t)\theta(\eta)$, $g_1(t, \eta) = g(t)\theta(\eta)$, $(t, \eta) \in [0, T] \times (-\infty, +\infty)$, and

$$\theta(\eta) = \begin{cases} 1, & -\infty < \eta < 0, \\ 0, & \eta \geq 0. \end{cases} \tag{73}$$

Thus, by using (69),(71) and (72), one can transform the optimal control problem (1)-(5),(68) to an equivalent one. This new optimal control problem consists of the dynamics equations (1)-(2) and (72), the initial conditions (3)-(4) and (71), the control constraint (5) and the performance index

$$\mathcal{J}_I(u(t)) \stackrel{\triangle}{=} G\Big(x(t_1), x(t_2), ..., x(t_N)\Big) + v(t_N) \to \min_{u(t) \in U}. \tag{74}$$

It can be seen directly that the problem (1)-(5),(71),(72),(74) is the Mayer's problem with an intermediary cost functional and singularly perturbed dynamics, i.e., it is of the type considered in Section 6.1.

Setting formally $\varepsilon = 0$ in the problem (1)-(5),(68) yields a reduced problem, associated with (1)-(5),(68). This reduced problem consists of the dynamics equation (12), initial condition (15) and the following performance index

$$\bar{J}_{BI} \stackrel{\triangle}{=} G\Big(\bar{x}(t_1), \bar{x}(t_2), ..., \bar{x}(t_N)\Big) + \int_0^T \Big(f_0^{'}(t)\bar{x}(t) + h_0(t, \bar{u}(t))\Big)dt \to \min_{\bar{u}(t) \in U}, \tag{75}$$

where $f_0(t) = f(t) - \bar{A}_3'(t)\Big(\bar{A}_4^{-1}(t)\Big)' g(t)$, $h_0(t,\bar{u}) = h(t,\bar{u}) - g'(t)\bar{A}_4^{-1}(t)B_2(t,\bar{u})$.

Similarly, by setting formally $\varepsilon = 0$ in the problem (1)-(5),(71),(72),(74), one obtains a reduced problem, associated with (1)-(5),(71),(72),(74). This reduced problem consists of the dynamics equations (12) and

$$d\bar{v}(t)/dt = f_0^{'}(t)\bar{x}(t) + h_0(t,\bar{u}(t)), \quad t \in [0,T], \tag{76}$$

the initial conditions (15) and

$$\bar{v}(0) = 0, \tag{77}$$

and the performance index

$$\bar{\mathcal{J}}_I(\bar{u}(t)) \triangleq G\Big(\bar{x}(t_1), \bar{x}(t_2), ..., \bar{x}(t_N)\Big) + \bar{v}(t_N) \to \min_{\bar{u}(t)\in U}. \tag{78}$$

By introducing the new state variable

$$\bar{v}(t) = \int_0^t \Big(f_0^{'}(t)\bar{x}(t) + h_0(t,\bar{u}(t))\Big)dt, \quad t \in [0,T], \tag{79}$$

we obtain the equivalence of the problems (12),(15),(75) and (12),(15),(76)-(78).

Denote in the original problem (1)-(5),(68), for a given $\varepsilon \in (0,\varepsilon_0]$,

$$J_{BI,\varepsilon}^* \triangleq \inf_{u(t)\in U} J_{BI}(u(t)), \tag{80}$$

and in the reduced problem (12),(15),(75)

$$\bar{J}_{BI}^* \triangleq \inf_{\bar{u}(t)\in U} \bar{J}_{BI}(\bar{u}(t)). \tag{81}$$

THEOREM 5 *Let the assumptions A1-A4,A6,A8,A10,A11 be valid. Then,*

$$|J_{BI,\varepsilon}^* - \bar{J}_{BI}^*| \le a\varepsilon \quad \forall \varepsilon \in (0,\varepsilon_2], \tag{82}$$

where the positive constant ε_2 is defined in Lemma 2.

Proof. The statement of the theorem follows directly from the equivalence of the problems (1)-(5),(68) and (1)-(5),(71),(72),(74), the equivalence of the reduced problems (12),(15),(75) and (12),(15),(76)-(78), and Theorem 3. ∎

Theorem 5 implies that the original problem (1)-(5),(68) is posed correctly.

Now, we assume:

A12. The reduced problem (12),(15),(75) has a solution, and $\bar{u}_{BI}^*(t)$ is its optimal control.

Let $J_{BI,\varepsilon}(\bar{u}_{BI}^*(t))$ be the value of the cost functional in the original problem (1)-(5),(68) for $u(t) = \bar{u}_{BI}^*(t)$. Similarly to Theorems 2 and 4, we obtain the following theorem.

THEOREM 6 *Let the assumptions A1-A4,A6,A8,A10-A12 be valid. Then,*

$$0 \leq J_{BI,\varepsilon}(\bar{u}_{BI}^*(t)) - J_{BI,\varepsilon}^* \leq a\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_2], \tag{83}$$

where the positive constant ε_2 is the same as in Lemma 2 and Theorem 5.

Theorem 6 implies the asymptotic suboptimality of the control $\bar{u}_{BI}^*(t)$ in the original problem (1)-(5),(68).

7. Proof of Lemma 1

First, we prove the inequalities (25) and (27). By using the initial value problem (18)-(19) for the matrix $\Psi(t,s,\varepsilon)$, as well as the block form of this matrix and the block form (17) of the matrix $A(t,\eta,\varepsilon)$, one can write down the initial value problem for the matrices $\Psi_1(t,s,\varepsilon)$ and $\Psi_3(t,s,\varepsilon)$ as follows

$$\varepsilon^{(k-1)/2}\partial\Psi_k(t,s,\varepsilon)/\partial t = \int_{-h}^{0} \Big[d_\eta A_k(t,\eta)\Big]\Psi_1(t+\varepsilon\eta,s,\varepsilon)$$

$$+ \int_{-h}^{0} \Big[d_\eta A_{k+1}(t,\eta)\Big]\Psi_3(t+\varepsilon\eta,s,\varepsilon), \quad k=1,3, \quad 0 \leq s < t \leq T, \tag{84}$$

$$\Psi_1(t,s,\varepsilon) = 0, \quad \Psi_3(t,s,\varepsilon) = 0, \quad t < s, \tag{85}$$

$$\Psi_1(s,s,\varepsilon) = I_n, \quad \Psi_3(s,s,\varepsilon) = 0. \tag{86}$$

Since ε is a small positive parameter, (84)-(86) is an initial value problem for a singularly perturbed differential system with the general type of delay. The delay is small of order of the small multiplier ε for a part of the derivatives in the system. A problem, limited to (84)-(86), was considered in Glizer (2003) where its asymptotic solution has been constructed and justified. The difference between the problem in Glizer (2003) and (84)-(86) is that the initial conditions of the former are continuous, while the initial conditions of the latter have a break at $t = s$. Nevertheless, the results of Glizer (2003) are directly extended to (84)-(86). By virtue of these results, there exists a positive constant ε_{11} such that $\forall \varepsilon \in (0, \varepsilon_{11}]$, the matrices $\Psi_k(t,s,\varepsilon)$, $(k=1,3)$ can be represented as

$$\Psi_k(t,s,\varepsilon) = \bar{\Psi}_{k0}(t,s) + \Psi_{k0}^b(\xi_s,s) + O_k(t,s,\varepsilon), \quad 0 \leq s \leq t \leq T, \tag{87}$$

where $\xi_s = (t-s)/\varepsilon$; the matrices $\bar{\Psi}_{10}(t,s)$ and $\bar{\Psi}_{30}(t,s)$ have the form

$$\bar{\Psi}_{10}(t,s) = \bar{\Psi}(t,s), \quad \bar{\Psi}_{30}(t,s) = -\bar{A}_4^{-1}(t)\bar{A}_3(t)\bar{\Psi}(t,s), \; 0 \leq s \leq t \leq T, \tag{88}$$

the matrix $\Psi_{10}^b(\xi_s,s) \equiv 0$, $\xi_s \geq 0$, $0 \leq s \leq t \leq T$, while the matrix $\Psi_{30}^b(\xi_s,s)$, for $0 \leq s \leq t \leq T$, satisfies the initial value problem

$$\partial\Psi_{30}^b(\xi_s,s)/\partial\xi_s = \int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big]\Psi_{30}^b(\xi_s+\eta,s), \quad \xi_s > 0, \tag{89}$$

$$\Psi_{30}^b(\xi_s,s) = -\bar{\Psi}_{30}(s,s), \quad \xi_s \leq 0; \tag{90}$$

$O_k(t,s,\varepsilon)$, $(k=1,3)$, are known matrices satisfying the inequalities

$$\|O_k(t,s,\varepsilon)\| \le a\varepsilon, \qquad 0 \le s \le t \le T. \tag{91}$$

By virtue of the assumption A6 and results of Hale and Lunel (1993), one obtains that the solution of (89)-(90) exists, is unique and satisfies the inequality

$$\|\Psi_{30}^b(\xi_s,s)\| \le a\exp(-\beta\xi_s), \qquad 0 \le s \le t \le T, \quad \xi_s \ge 0. \tag{92}$$

Finally, the equations (87) and (88), and the inequalities (91) and (92) yield directly the inequalities (25) and (27) for all $\varepsilon \in (0,\varepsilon_{11}]$.

Now, let proceed to the proof of the inequalities (26) and (28). Similarly to (84)-(86), we have the initial value problem for the matrices $\Psi_l(t,s,\varepsilon)$, $(l=2,4)$

$$\varepsilon^{(l-2)/2}\partial\Psi_l(t,s,\varepsilon)/\partial t = \int_{-h}^{0}\Big[d_\eta A_{l-1}(t,\eta)\Big]\Psi_2(t+\varepsilon\eta,s,\varepsilon)$$

$$+\int_{-h}^{0}\Big[d_\eta A_l(t,\eta)\Big]\Psi_4(t+\varepsilon\eta,s,\varepsilon), \qquad l=2,4, \qquad 0 \le s < t \le T, \tag{93}$$

$$\Psi_2(t,s,\varepsilon)=0, \qquad \Psi_4(t,s,\varepsilon)=0, \quad t<s, \tag{94}$$

$$\Psi_2(s,s,\varepsilon)=0, \qquad \Psi_4(s,s,\varepsilon)=I_m. \tag{95}$$

Similarly to (87), one can obtain the following representations of the matrices $\Psi_l(t,s,\varepsilon)$, $(l=2,4)$ for all $\varepsilon \in (0,\varepsilon_{12}]$ with some positive ε_{12}:

$$\Psi_l(t,s,\varepsilon) = \Psi_{l0}^b(\xi_s,s) + \varepsilon\Big(\bar\Psi_{l1}(t,s) + \Psi_{l1}^b(\xi_s,s)\Big) + O_l(t,s,\varepsilon), \tag{96}$$

where $0 \le s \le t \le T$; the matrices $\Psi_{l0}^b(\xi_s,s)$, $(l=1,2)$ have the form

$$\Psi_{20}^b(\xi_s,s)=0, \qquad \Psi_{40}^b(\xi_s,s)=\tilde\Psi(\xi_s,s), \qquad 0 \le s \le t \le T, \quad \xi_s \ge 0; \tag{97}$$

the matrices $\bar\Psi_{21}(t,s)$ and $\bar\Psi_{41}(t,s)$ satisfy the system

$$\partial\bar\Psi_{21}(t,s)/\partial t = \bar A_1(t)\bar\Psi_{21}(t,s) + \bar A_2(t)\bar\Psi_{41}(t,s), \quad 0 \le s < t \le T, \tag{98}$$

$$\bar\Psi_{21}(s,s) = \int_0^{+\infty}\left[\int_{-h}^{0}\Big[d_\eta A_2(s,\eta)\Big]\tilde\Psi(\sigma+\eta)\right]d\sigma, \tag{99}$$

$$0 = \bar A_3(t)\bar\Psi_{21}(t,s) + \bar A_4(t)\bar\Psi_{41}(t,s), \quad 0 \le s \le t \le T; \tag{100}$$

the matrix $\Psi_{21}^b(\xi_s,s)$, for $0 \le s \le t \le T$, satisfies the initial value problem

$$\partial\Psi_{21}^b(\xi_s,s)/\partial\xi_s = \int_{-h}^{0}\Big[d_\eta A_2(s,\eta)\Big]\tilde\Psi(\xi_s+\eta,s), \quad \xi_s>0, \tag{101}$$

$$\Psi_{21}^b(0,s) = -\int_0^{+\infty}\left[\int_{-h}^{0}\Big[d_\eta A_2(s,\eta)\Big]\tilde\Psi(\sigma+\eta)\right]d\sigma; \tag{102}$$

the matrix $\Psi_{41}^{b}(\xi_s, s)$, for $0 \le s \le t \le T$, satisfies the initial value problem

$$\partial \Psi_{41}^{b}(\xi_s, s)/\partial \xi_s = \int_{-h}^{0} \Big[d_\eta A_3(s,\eta)\Big] \Psi_{21}^{b}(\xi_s+\eta) + \int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big] \Psi_{41}^{b}(\xi_s+\eta)$$

$$+ \xi_s \int_{-h}^{0} \Big[d_\eta \Big(\partial A_4(s,\eta)/\partial s\Big)\Big] \tilde{\Psi}(\xi_s+\eta), \quad \xi_s > 0, \tag{103}$$

$$\Psi_{41}^{b}(\xi_s, s) = -\bar{\Psi}_{41}(s,s), \quad \xi_s \le 0; \tag{104}$$

$O_l(t,s,\varepsilon)$, $(l=2,4)$, are known matrices satisfying the inequalities

$$\|O_l(t,s,\varepsilon)\| \le a\varepsilon^2, \qquad 0 \le s \le t \le T. \tag{105}$$

Consider the system (98)-(100). Note that, due to (24) and (97), the integral in (99) converges. Now, using (11),(13) and (20)-(21) we get for $0 \le s \le t \le T$

$$\bar{\Psi}_{21}(t,s) = \bar{\Psi}(t,s)\bar{\Psi}_{21}(s,s), \qquad \bar{\Psi}_{41}(t,s) = -\bar{A}_4^{-1}(t)\bar{A}_3(t)\bar{\Psi}_{21}(t,s), \tag{106}$$

meaning the boundedness of $\bar{\Psi}_{41}(t,s)$ for $0 \le s \le t \le T$.

Proceed to the problem (101)-(102). For $0 \le s \le t \le T$, this problem has the unique solution

$$\Psi_{21}^{b}(\xi_s, s) = -\int_{\xi_s}^{+\infty} \Big[\int_{-h}^{0} \Big[d_\eta A_2(s,\eta)\Big] \tilde{\Psi}(\sigma+\eta, s)\Big] d\sigma, \qquad \xi_s \ge 0, \tag{107}$$

yielding, by using (24) and (97), the estimate

$$\|\Psi_{21}^{b}(\xi_s, s)\| \le a \exp(-\beta\xi_s), \qquad 0 \le s \le t \le T, \quad \xi_s \ge 0. \tag{108}$$

By virtue of the assumption A6, the equation (97), the inequalities (24),(108) and the results of Hale and Lunel (1993), there exists a unique solution of the problem (103)-(104), and this solution satisfies the inequality

$$\|\Psi_{41}^{b}(\xi_s, s)\| \le a \exp(-\beta\xi_s), \qquad 0 \le s \le t \le T, \quad \xi_s \ge 0. \tag{109}$$

Now, using the expression $\xi_s = (t-s)/\varepsilon$, the equations (96) for $l=4$ and (97), the inequalities (105) for $l=4$ and (109), as well as the boundedness of $\bar{\Psi}_{41}(t,s)$, we obtain directly the inequality (28) for all $\varepsilon \in (0, \varepsilon_{12}]$. Similarly, by using the expression for ξ_s, the equations (96) for $l=2$ and (106), the inequalities (105) for $l=2$ and (108), as well as the fact that $\Psi_{20}^{b}(\xi_s, s) \equiv 0$, we obtain the following inequality for all $\varepsilon \in (0, \varepsilon_{12}]$ and $0 \le s \le t \le T$:

$$\|\Psi_2(t,s,\varepsilon) - \varepsilon\bar{\Psi}(t,s)\bar{\Psi}_{21}(s,s)\| \le a\varepsilon\Big[\varepsilon + \exp\Big(-\beta(t-s)/\varepsilon\Big)\Big]. \tag{110}$$

To complete the proof of (26), we transform equivalently the matrix $\bar{\Psi}_{21}(s,s)$ given by (99). For this purpose, we transform the equation (22). Its integration with respect to ξ on the interval $[0, +\infty)$, and using (23),(24) and (97), yield

$$-I_m = \int_{0}^{+\infty} \Big[\int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big] \tilde{\Psi}(\xi+\eta, s)\Big] d\xi. \tag{111}$$

Due to (24) and (97), the integral in (111) converges absolutely. Hence, by the Fubini Theorem, we can change the order of integration in this integral. Thus,

$$-I_m = \int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big] \int_{0}^{+\infty} \tilde{\Psi}(\xi+\eta,s)d\xi. \tag{112}$$

The transformation of variables $\xi = \sigma - \eta$ in the improper integral leads to

$$-I_m = \int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big] \int_{\eta}^{+\infty} \tilde{\Psi}(\sigma,s)d\sigma. \tag{113}$$

Now, (23) and the fact that $\eta \in [-h,0]$ imply

$$-I_m = \int_{-h}^{0} \Big[d_\eta A_4(s,\eta)\Big] \int_{0}^{+\infty} \tilde{\Psi}(\sigma,s)d\sigma, \tag{114}$$

yielding, by virtue of (9) and (11),

$$\int_{0}^{+\infty} \tilde{\Psi}(\sigma,s)d\sigma = -\bar{A}_4^{-1}(s). \tag{115}$$

Similarly to the transformation of the right-hand side in (111) to the right-hand side in (114), one can transform the expression (99) for $\bar{\Psi}_{21}(s,s)$ as follows

$$\bar{\Psi}_{21}(s,s) = \bar{A}_2(s) \int_{0}^{+\infty} \tilde{\Psi}(\sigma,s)d\sigma. \tag{116}$$

Finally, substitution of (115) into (116), and then of the resulting expression into (110) yields directly the inequality (26) for all $\varepsilon \in (0,\varepsilon_{12}]$. Setting $\varepsilon_1 = \min\{\varepsilon_{11},\varepsilon_{12}\}$ completes the proof of the lemma.

8. Proof of Lemma 2

We begin with the proof of (29). We introduce the block vectors

$$B(t,u) = \begin{pmatrix} B_1(t,u) \\ B_2(t,u) \end{pmatrix}, \quad \varphi(\tau) = \begin{pmatrix} \varphi_x(\tau) \\ \varphi_y(\tau) \end{pmatrix}, \quad z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{117}$$

and block matrices

$$E_\varepsilon = \begin{pmatrix} I_n & 0 \\ 0 & (1/\varepsilon)I_m \end{pmatrix}, \qquad A(t,\eta) = \begin{pmatrix} A_1(t,\eta) & A_2(t,\eta) \\ A_3(t,\eta) & A_4(t,\eta) \end{pmatrix}. \tag{118}$$

Then, using the variation of constant formula (see Hale and Lunel, 1993) we get

$$z(t,\varepsilon) = \Psi(t,0,\varepsilon)z_0 + \int_{0}^{\varepsilon h} \Psi(t,\omega,\varepsilon)E_\varepsilon \left(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A(\omega,\eta)\Big]\varphi(\omega+\varepsilon\eta)\right)d\omega$$
$$+ \int_{0}^{t} \Psi(t,s,\varepsilon)E_\varepsilon B(s,u(s))ds, \quad t \in [0,T]. \tag{119}$$

Using the block form of $\Psi(t,s,\varepsilon)$ and (117)-(119), one can write down the blocks $x(t,\varepsilon)$ and $y(t,\varepsilon)$ of the vector $z(t,\varepsilon)$ as follows

$$
\begin{aligned}
&x(t,\varepsilon) = \Psi_1(t,0,\varepsilon)x_0 + \Psi_2(t,0,\varepsilon)y_0 \\
&+ \int_0^{\varepsilon h} \Bigg\{ \Psi_1(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_1(\omega,\eta) \Big] \varphi_x(\omega+\varepsilon\eta) \\
&+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_2(\omega,\eta) \Big] \varphi_y(\omega+\varepsilon\eta) \Bigg) \\
&+ (1/\varepsilon)\Psi_2(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_3(\omega,\eta) \Big] \varphi_x(\omega+\varepsilon\eta) \\
&+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_4(\omega,\eta) \Big] \varphi_y(\omega+\varepsilon\eta) \Bigg) \Bigg\} d\omega \\
&+ \int_0^t \Big(\Psi_1(t,s,\varepsilon)B_1(s,u(s)) + (1/\varepsilon)\Psi_2(t,s,\varepsilon)B_2(s,u(s)) \Big) ds, \qquad (120) \\
&y(t,\varepsilon) = \Psi_3(t,0,\varepsilon)x_0 + \Psi_4(t,0,\varepsilon)y_0 \\
&+ \int_0^{\varepsilon h} \Bigg\{ \Psi_3(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_1(\omega,\eta) \Big] \varphi_x(\omega+\varepsilon\eta) \\
&+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_2(\omega,\eta) \Big] \varphi_y(\omega+\varepsilon\eta) \Bigg) \\
&+ (1/\varepsilon)\Psi_4(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_3(\omega,\eta) \Big] \varphi_x(\omega+\varepsilon\eta) \\
&+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_4(\omega,\eta) \Big] \varphi_y(\omega+\varepsilon\eta) \Bigg) \Bigg\} d\omega \\
&+ \int_0^t \Big(\Psi_3(t,s,\varepsilon)B_1(s,u(s)) + (1/\varepsilon)\Psi_4(t,s,\varepsilon)B_2(s,u(s)) \Big) ds. \qquad (121)
\end{aligned}
$$

The inequalities (24) and (25)-(28) yield the following estimates of the matrices $\Psi_i(t,s,\varepsilon)$, $(i=1,...,4)$ for all $\varepsilon \in (0,\varepsilon_1]$ and $0 \le s \le t \le T$:

$$
\|\Psi_k(t,s,\varepsilon)\| \le a, \qquad k = 1,3, \tag{122}
$$

$$
\|\Psi_2(t,s,\varepsilon)\| \le a\varepsilon, \qquad \|\Psi_4(t,s,\varepsilon)\| \le a\Big[\varepsilon + \exp\Big(-\beta(t-s)/\varepsilon \Big)\Big]. \tag{123}
$$

Now, using the assumptions A1-A4, the definition of the set U, the equations (85) and (94), and the inequalities (122) and (123) we obtain the inequalities (29). The inequality (30) is proved similarly by using the following expression for $\bar{x}(t)$:

$$\bar{x}(t) = \bar{\Psi}(t,0)x_0 + \int_0^t \bar{\Psi}(t,s)\bar{B}_0(s,u(s))ds, \quad t \in [0,T]. \tag{124}$$

We proceed to the proof of (31), based on some analysis of (120). First, note that, for $\omega \in [0, \varepsilon h]$, the term ω/ε varies from 0 to h. For $\omega \in [0, \varepsilon h]$ and $\eta \in [-h, -\omega/\varepsilon]$, the term $\omega + \varepsilon\eta$ varies in the interval $[-h, 0]$. Hence, due to the assumptions A1 and A3, the following inequalities hold for all $\varepsilon \in (0, \varepsilon_0]$:

$$\left\| \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_k(\omega,\eta) \Big] \varphi_x(\omega + \varepsilon\eta) \right\| \le a, \quad k = 1,3, \quad \omega \in [0, \varepsilon h], \tag{125}$$

$$\left\| \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_l(\omega,\eta) \Big] \varphi_y(\omega + \varepsilon\eta) \right\| \le a, \quad l = 2,4, \quad \omega \in [0, \varepsilon h]. \tag{126}$$

Using the inequality (25), the first inequalities in (122),(123) and the inequalities (125),(126), we obtain the following inequalities for all $\varepsilon \in (0, \varepsilon_2]$:

$$\left\| \Psi_1(t,0,\varepsilon)x_0 - \bar{\Psi}(t,0)x_0 \right\| \le a\varepsilon, \quad \left\| \Psi_2(t,0,\varepsilon)y_0 \right\| \le a\varepsilon, \quad t \in [0,T], \tag{127}$$

$$\Bigg\| \int_0^{\varepsilon h} \Psi_1(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_1(\omega,\eta) \Big] \varphi_x(\omega + \varepsilon\eta)$$
$$+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_2(\omega,\eta) \Big] \varphi_y(\omega + \varepsilon\eta) \Bigg) d\omega \Bigg\| \le a\varepsilon, \quad t \in [0,T], \tag{128}$$

$$\Bigg\| \int_0^{\varepsilon h} (1/\varepsilon)\Psi_2(t,\omega,\varepsilon) \Bigg(\int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_3(\omega,\eta) \Big] \varphi_x(\omega + \varepsilon\eta)$$
$$+ \int_{-h}^{-\omega/\varepsilon} \Big[d_\eta A_4(\omega,\eta) \Big] \varphi_y(\omega + \varepsilon\eta) \Bigg) d\omega \Bigg\| \le a\varepsilon, \quad t \in [0,T]. \tag{129}$$

Now, we analyze the last integral term in (120). By subtracting the integral part in the expression (124) for $\bar{x}(t)$ from this term and using (14), we obtain

$$G(t,\varepsilon) \triangleq \int_0^t \Big(\Psi_1(t,s,\varepsilon)B_1(s,u(s)) + (1/\varepsilon)\Psi_2(t,s,\varepsilon)B_2(s,u(s)) \Big) ds$$
$$- \int_0^t \bar{\Psi}(t,s)\bar{B}_0(s,u(s))ds = \int_0^t \Big(\Psi_1(t,s,\varepsilon) - \bar{\Psi}(t,s) \Big) B_1(s,u(s))ds$$
$$+ \int_0^t \Big((1/\varepsilon)\Psi_2(t,s,\varepsilon) + \bar{\Psi}(t,s)\bar{A}_2(s)\bar{A}_4^{-1}(s) \Big) B_2(s,u(s))ds. \tag{130}$$

Using the assumptions A2 and A4, the definition of the set U, and the inequalities (25) and (26), we get the following estimate of $G(t,\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_2]$:

$$\|G(t,\varepsilon)\| \leq a\varepsilon, \qquad t \in [0,T], \tag{131}$$

where $a > 0$ is some constant independent of both ε and $u(t) \in U$. Finally, the equations (120),(130) and the inequalities (127)-(129),(131) yield (31).

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