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Jacobi type conditions for singular extremals*

by

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Abstract: We consider the class of optimal control problems linear in the control, and study a singular extremal. If the Lagrange multipliers are unique, the quadratic order optimality conditions have the form of sign definiteness of a quadratic functional (the second variation of Lagrange function) with totally zero Legendre coefficient. Using the Goh transformation, we convert it to a functional possibly satisfying the strengthened Legendre condition, involving also an additional parameter, and by applying the Hestenes approach, determine its sign definiteness in terms of the conjugate point, i.e. give Jacobi type conditions.

Keywords: singular extremal, Goh transformation, Legendre quadratic form, Euler–Jacobi equation, conjugate point

1. Quadratic order conditions of optimality

1.1. The problem under study

Consider the following optimal control problem on a fixed time interval $[t_0, t_1]$:

Problem A :

$$\begin{cases} \dot{x} = f_0(t,x) + F(t,x)\,u, & (1)\\ u \in U(t), & (2)\\ \eta_j(p) = 0, \quad j = 1,\ldots,\mu, & (3)\\ \varphi_i(p) \le 0, \quad i = 1,\ldots,\nu, & (4)\\ J = \varphi_0(p) \to \min, & (5)\end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $p = (x(t_0), x(t_1)) \in \mathbb{R}^{2n}$, the function $x(t)$ is absolutely continuous, $u(t)$ is measurable and essentially bounded. The data functions η_j, φ_i are assumed to be twice smooth; f_0, F are continuous and have jointly continuous first and second derivatives w.r.t. x. (The problem on

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a nonfixed time interval can be easily reduced to problem A by passing to a new time variable.)

Let be given an admissible process (x^0, u^0) with $u^0(t)$ taking values strictly inside $U(t)$, which means that

$$\exists\, \varepsilon > 0 \quad \text{such that} \quad u^0(t) + B_\varepsilon(0) \subset U(t) \quad \text{a.e. on } [t_0,\, t_1]\,. \tag{6}$$

Without loss of generality, assume also that $\varphi_i(p^0) = 0$ for all $i = 0, 1, \ldots, \nu$, hence all the inequality constraints are active. Assume that the process (x^0, u^0) satisfies the Maximum Principle (which is equivalent in this case to the Euler–Lagrange (EL) equation), which says that there exist multipliers $\alpha_0, \ldots, \alpha_\nu \geq 0$, β, $\psi(t)$, not all equal to zero and such that, composing the Pontryagin function $H = \psi\,(f_0(t,x) + F(t,x)\,u)$ and the terminal Lagrange function $l(p) = \sum_{i=0}^{\nu} \alpha_i\, \varphi_i(p) + \sum_{j=1}^{\mu} \beta_j\, \eta_j(p)$, we should obtain the following relations:

$$\dot\psi = -H_x(\psi, x^0, u^0), \qquad \psi(t_0) = l_{x_0}(p^0)\,, \qquad \psi(t_1) = -l_{x_1}(p^0)\,,$$

$$H_u(\psi, x^0, u^0) = \psi(t)\, F(t, x^0(t)) = 0.$$

The last relation means that we have a totally singular extremal. The question is: what are further (higher order) necessary and sufficient conditions for the given process (x^0, u^0) to be optimal in one or another sense? This question was studied by many authors for more than 40 years, since early 1960s (see Kelley, Kopp, Moyer, 1967, and later references in Dmitruk, 1997). For problem A it was completely solved by the author in a series of papers, Dmitruk (1977, 1978, 1983, 1987-88, 1992, 1994, 1997), for the two types of minimum: the weak and the so-called Pontryagin minima. In those papers the author obtained "adjoint pairs" of necessary and sufficient conditions of a special quadratic order typical for problem A. The necessary condition consists of nonnegativity, and the sufficient condition consists of positive definiteness of a quadratic functional (or of the maximum of a family of quadratic functionals) on a cone in the space of variations. In this paper we will show how one can pass from these "basic" quadratic order conditions to Jacobi type conditions in terms of conjugate points, thus will propose a procedure for verifying these "basic" conditions.

Our approach is similar to that in the classical calculus of variations (CCV) — obtaining conditions in terms of quadratic functionals and then analyzing their sign definiteness, and differs mainly by the fact that here the quadratic functionals have totally zero Legendre coefficient. After some transformation they can be converted into functionals possibly satisfying the strengthened Legendre condition, but involving additional parameters, and this specificity should be properly taken into account. (Another approach based on methods of differential geometry is pursued in Stefani, 2003, 2004, for particular cases of problem A with a scalar control, where conditions for a strong minimum are proposed.)

To simplify the exposition, we assume here that the collection of Lagrange multipliers for the given process (x^0, u^0) is unique, up to normalization, with

$\alpha_0 > 0$, so we can set $\alpha_0 = 1$. (The "basic" quadratic order conditions were obtained in the cited papers without this assumption.) This, in particular, implies that equality constraints (1), (3) near the process (x^0, u^0) are nondegenerate at the first order, or, in other words, the system (1), (3) is first-order controllable at (x^0, u^0).

1.2. Quadratic order conditions

To formulate the "basic" quadratic order conditions of optimality, we have to define the following objects: a) the quadratic functional (quadratic form), b) the cone of critical variations, and c) the estimating quadratic functional (order of minimum).

Denote by $W = AC^n \times L^r_\infty$ the space of all pairs $w = (x, u)$, $x \in AC^n$, $u \in L^r_\infty$ on the given time interval $[t_0, t_1]$.

a) The quadratic form. The existing collection (here unique) of Lagrange multipliers generates the corresponding Lagrange function

$$\mathcal{L}(x, u) = l(x_0, x_1) + \int_{t_0}^{t_1} \psi(t)\, (\dot{x} - f_0(t, x) - F(t, x)\, u))\, dt,$$

and its second variation at $(x^0, u^0) \in W$:

$$\Omega(\bar{x}, \bar{u}) = d^2\mathcal{L}[x^0, u^0]\,(\bar{x}, \bar{u}) \;=\; (l''\bar{p}, \bar{p}) - \int_{t_0}^{t_1} [(H_{xx}\bar{x}, \bar{x}) + 2(\bar{x}, H_{xu}\bar{u})]\, dt\,;$$

which is a quadratic functional w.r.t. $\bar{w} = (\bar{x}, \bar{u})$. Note that this quadratic form does not contain "the main", Legendre term with $\bar{u}^2$, which is directly caused by the linearity of the state equation in u and the assumption (6), and which immediately puts us out of the framework of the classical Jacobi theory, that essentially assumes the presence of this term with a strictly positive coefficient (the strengthened Legendre condition).

b) The cone of critical variations. The above quadratic functional should be considered not on the whole space W, but only on the so-called *cone of critical variations* K, which is given by linearization of all constraints and the cost functional of the problem at the reference process (x^0, u^0) :

$$\dot{\bar{x}} = f'_{0x}\,\bar{x} + F'_x\,\bar{x}\,u^0 + F\,\bar{u}, \tag{7}$$

$$\eta'_j\,\bar{p} = 0, \quad j = 1, \ldots, \mu, \qquad \varphi'_i\,\bar{p} \le 0, \quad i = 0, 1, \ldots, \nu. \tag{8}$$

For convenience in further study, let us simplify conditions (8) as much as possible. First, for any i with $\alpha_i > 0$ we can replace here the inequality $\varphi'_i\,\bar{p} \le 0$ by the equality $\varphi'_i\,\bar{p} = 0$. In particular, we can take $\varphi'_0\,\bar{p} = 0$, since we assume $\alpha_0 = 1$. Moreover, then we can delete this equation altogether, because (due to the EL equation) it is a linear combination of all other obtained equations in (8) and system (7). If, after all such replacements, only one inequality in (8) remains, we can also delete it. This last trick is justified by the fact that the sign definiteness of a quadratic form on a half-space is equivalent to that on the whole space. However, in the general case, a finite number of inequalities $\varphi_i' \bar{p} \le 0$ corresponding to $\alpha_i = 0$ may remain.

c) The order of minimum. Now, define the following estimating quadratic functional, that we regard as a quadratic order of minimum:

$$\gamma(\bar{x}, \bar{u}) = |\bar{x}(t_0)|^2 + |\bar{y}(t_1)|^2 + \int_{t_0}^{t_1} |\bar{y}(t)|^2\, dt, \tag{9}$$

$$\text{where} \qquad \dot{\bar{y}} = \bar{u}\,, \qquad \bar{y}(t_0) = 0. \tag{10}$$

One can see that this estimating functional includes an additional "artificial" state variable $\bar{y}$ (to be more exact, the variation of an artificial state variable y, satisfying the equation $\dot{y} = u$, $y(t_0) = 0$, which is not explicitly introduced, since the variable y itself will not be used in what follows), and does not explicitly include the control variation $\bar{u}$; it includes the last only implicitly, through $\bar{y}$.

Now we are ready to formulate the quadratic order conditions of optimality for problem A. Let us start with the weak minimality, by which we mean the minimality w.r.t. the norm $||w||' = ||x||_C + ||u||_\infty$. In this case, since $u^0(t)$ lies strictly inside $U(t)$, the inclusion constraint $u \in U$ is inessential, so we can neglect it.

THEOREM 1 *a) Let $w^0 = (x^0, u^0)$ provide a weak minimum in problem A. Then*

$$\Omega(\bar{w}) \ \ge\ 0 \quad \textit{for all } \bar{w} \in K. \tag{11}$$

b) Suppose that for some $a > 0$

$$\Omega(\bar{w}) \ \ge\ a\,\gamma(\bar{w}) \quad \textit{for all } \bar{w} \in K \tag{12}$$

(i.e., Ω is positive definite on K with respect to γ). Then $w^0 = (x^0, u^0)$ provides a weak minimum in problem A.

As one can see, these necessary and sufficient conditions constitute a pair of conditions with a minimal gap between them; we call them an *adjoint pair* of conditions. In this sense, these conditions are quite similar to those in the finite-dimensional analysis and CCV.

Part (a) of this theorem is a particular case of the necessary conditions in the optimal control problem with general nonlinear state equation $\dot{x} = f(t, x, u)$ (actually, in the general problem of CCV with additional inequality constraints (4)). These general necessary conditions were obtained in Levitin, Milyutin, Osmolovskii (1978). Part (b) was proved in Dmitruk (1977, 1978), solving thus the question of obtaining sufficient conditions for this nonclassical case.

REMARK 1 *If the collection of Lagrange multipliers is not unique, then, for any such collection and the corresponding second variation of the Lagrange function, part b) of Theorem 1 still holds true (see Dmitruk, 1977, 1978).*

Along with the notion of weak minimum, we consider also the notion of *Pontryagin minimum,* proposed by A.Ya.Dubovitskii and A.A.Milyutin. An admissible process $w^0 = (x^0, u^0)$ is said to provide a Pontryagin minimum in problem A if for any number N it provides a local minimum w.r.t. the norm $||x||_C + ||u||_1$ in problem A with additional constraint $|u(t)| \leq N$. This type of minimum lies obviously between the weak and strong minima, and it turned out to be very convenient in the study of higher order conditions (see, e.g., Milyutin, Osmolovskii, 1998).

The conditions for a Pontryagin minimum of order (9) are obtained in Dmitruk (1983, 1987-88, 1992, 1994) under the additional assumption that the functions f_0, F have jointly continuous third derivatives w.r.t. x. These conditions have the same form (11), (12) with an additional requirement on coefficients of *the third variation* of the Lagrange function with reference to the constraint $u \in U$, which cannot be neglected in this case. Here we do not write out this requirement; the details and proofs see in the above papers. (If $\dim u = 1$, this additional requirement holds trivially, hence the conditions for a Pontryagin minimum coincide with those for a weak minimum.)

Anyway, for both types of minimum we arrive at conditions (11) and (12). Our goal in this paper is to propose a procedure for verifying these conditions.

1.3. The linear-quadratic situation

Having obtained conditions (11), (12), we face a natural question: how one can verify them? First of all, let us state the situation we arrive at.

We have a quadratic form of the type

$$\Omega(\bar{w}) = g(\bar{x}(t_0), \bar{x}(t_1)) + \int_{t_0}^{t_1} (Q\bar{x}, \bar{x}) + 2(C\bar{x}, \bar{u})\, dt, \tag{13}$$

where g is a finite-dimensional quadratic form in $\mathbb{R}^{2n}$ and the matrices Q, C are of appropriate dimensions, and we have a cone $K \subset W$ given by constraints of the type

$$\dot{\bar{x}} = A(t)\bar{x} + F(t)\bar{u}, \tag{14}$$

$$a_i\, \bar{x}(t_0) + b_i\, \bar{x}(t_1) = 0, \qquad i = 1, \ldots, \mu, \tag{15}$$

$$a'_j\, \bar{x}(t_0) + b'_j\, \bar{x}(t_1) \leq 0, \qquad j = 1, \ldots, \nu, \tag{16}$$

where a_i, b_i, a'_j, b'_j $\in \mathbb{R}^n$ are some vectors, the matrices $Q(t)$, $A(t)$ have measurable bounded entries, and $C(t)$, $F(t)$ have Lipschitz continuous entries. (In fact, $Q(t) = -H_{xx}$, $C(t) = -2H_{ux}$, $A(t)$ is the coefficient at $\bar{x}$ in (7), all calculated along $w^0(t)$, $F(t) = F(t, x^0(t))$, but from now on, we do not need these particular expressions.) Our aim is to determine whether the estimate (12) holds with some $a > 0$ or $a \geq 0$.

This is a linear-quadratic situation, which results from the quadratic order study of the reference extremal process in problem A. Having stated this situation, we can forget the initial problem A from now on, leaving only specific features of Ω and K. Recall that in CCV the question of sign definiteness of a quadratic form is solved by the well-known Jacobi conditions about the conjugate (or focal) points. However, this approach does not work in our case, because the key assumption of the Jacobi theory, the strengthened Legendre condition, is not satisfied. So, what to do then?

To overcome this obstacle, we use a simple transformation, probably first proposed in Goh (1966), which is the following change of state variables:

$$\bar{x} \mapsto (\bar{\xi}, \bar{y}), \qquad \bar{x} = \bar{\xi} + F\bar{y}.$$

So, the state variable $\bar{x}$ is now replaced by two state variables, $\bar{\xi}$ and $\bar{y}$.

In view of (10) and (14), $\bar{\xi}$ obeys the dynamics

$$\dot{\bar{\xi}} = A\bar{\xi} + B\bar{y}, \qquad \text{where } B = AF - \dot{F}, \tag{17}$$

and the initial condition $\bar{\xi}(t_0) = \bar{x}(t_0)$. An important feature is that equation (17) does not contain $\bar{u}$! The control variation $\bar{u}$ now comes, in the simplest way, only into Eq. (10) for $\bar{y}$. These two facts make it possible to obtain some nice properties of the functional Ω.

First of all, we see that after the above transformation, Ω can be reduced to the form:

$$\Omega = g(\bar{\xi}(t_0),\, \bar{\xi}(t_1) + F(t_1)\bar{y}(t_1)) + (C(t_1)\bar{\xi}(t_1), \bar{y}(t_1)) + \\ + \int_{t_0}^{t_1} (Q\bar{\xi}, \bar{\xi}) + 2(P\bar{\xi}, \bar{y}) + (R\bar{y}, \bar{y}) + (CF\,\bar{y}, \bar{u})\, dt,$$

where $P(t)$, $R(t)$ are some matrices with measurable bounded entries. The term $(C\bar{\xi}, \bar{u}) = (C\bar{\xi}, \dot{\bar{y}})$ was integrated by parts in order to exclude $\bar{u}$, leaving thus only one term in Ω containing $\bar{u}$, i.e., $(CF\,\bar{y}, \bar{u})$. A remarkable fact discovered by Goh is that this last term can also be integrated by parts. He proved that, if $\Omega \geq 0$ on K, then the matrix $CF = -H_{ux}F$ is symmetric (the necessary Goh condition of equality type), hence this term can be integrated. Namely, for any symmetric absolutely continuous matrix $S(t)$, one can write $\frac{d}{dt}(S\bar{y}, \bar{y}) = (\dot{S}\bar{y}, \bar{y}) + 2(S\bar{y}, \bar{u})$, hence

$$2\int_{t_0}^{t_1} (S\bar{y}, \bar{u})\, dt = \left. (S\bar{y}, \bar{y})\right|_{t_0}^{t_1} - \int_{t_0}^{t_1} (\dot{S}\bar{y}, \bar{y})\, dt,$$

so Ω (with Lipschitz continuous $S = CF$) is reduced to the form

$$\Omega = \tilde{g}(\bar{\xi}(t_0), \bar{\xi}(t_1), \bar{y}(t_1)) + \int_{t_0}^{t_1} (Q\bar{\xi}, \bar{\xi}) + 2(P\bar{\xi}, \bar{y}) + (R\bar{y}, \bar{y})\, dt, \tag{18}$$

with a new measurable bounded matrix $R(t)$ and a new terminal quadratic form $\tilde{g}$ in the space $\mathbb{R}^{2n+r}$. We do not write here the expressions for all coefficients in (18) through initial ones, which can be easily done if necessary. The practice shows that, in solving concrete problems, one need not use those expressions; it is easier to perform the Goh transformation in each situation, rather than to calculate the new matrices defining the quadratic form in terms of the old ones by means of general formulae.

The cone K in the new variables is given by the differential equations (10), (17), and the terminal constraints

$$a_i\, \bar{\xi}(t_0) + b_i\, (\bar{\xi}(t_1) + F\, \bar{y}(t_1)) = 0, \qquad i = 1, \ldots, \mu, \tag{19}$$

$$a'_j\, \bar{\xi}(t_0) + b'_j\, (\bar{\xi}(t_1) + F\, \bar{y}(t_1)) \le 0, \qquad j = 1, \ldots, \nu. \tag{20}$$

Thus, we arrive at a situation, where both the investigated quadratic form and the order of minimum depend only on $\bar{\xi}$ and $\bar{y}$, connected by equation (17) and terminal relations (19), (20), but do not explicitly depend on $\bar{u}$. The control variation $\bar{u}$ comes only in the state equation (10) for $\bar{y}$, which in fact just means that $\bar{y}$ is an arbitrary Lipschitz continuous function with initial value $\bar{y}(t_0) = 0$.

1.4. Extension of the space of critical variations

Now, we note that the space of Lipschitz functions, even with zero initial value, is dense in the space $L_2[t_0, t_1]$, and the integral part of Ω is continuous w.r.t. $||\bar{y}||_2$, hence we can consider $\bar{y} \in L_2^r[t_0, t_1]$, while the terminal value $\bar{y}(t_1)$ in the endpoint quadratic form $\tilde{g}$ should be replaced by a parameter $\bar{h} \in \mathbb{R}^r$, since the variety of pairs $(\bar{y}(\cdot),\, \bar{y}(t_1))$, where $\bar{y}(\cdot)$ is a Lipschitz function with zero initial value, is dense in the space $L_2^r[t_0, t_1] \times \mathbb{R}^r$.

Thus, we come to a functional

$$\widetilde{\Omega}(\bar{\xi}(t_0), \bar{y}, \bar{h}) = \tilde{g}(\bar{\xi}(t_0), \bar{\xi}(t_1), \bar{h})) + \int_{t_0}^{t_1} (Q\bar{\xi}, \bar{\xi}) + 2(P\bar{\xi}, \bar{y}) + (R\bar{y}, \bar{y})\, dt, \tag{21}$$

where $\bar{y} \in L_2^r[t_0, t_1]$, $\bar{h} \in \mathbb{R}^r$, and $\bar{\xi}$ is expressed through $\bar{y}$ by equation (17) with an arbitrary initial condition $\bar{\xi}(t_0)$. This new quadratic form is defined on the extended space $\widetilde{W} = \mathbb{R}^n \times L_2^r \times \mathbb{R}^r$ with elements $(\bar{\xi}(t_0), \bar{y}(\cdot), \bar{h})$ and is obviously continuous w.r.t. the norm $|\bar{\xi}(t_0)| + ||\bar{y}||_2 + |h|$. The extended cone

$\widetilde{K}$ in this space is given by equation (17) and the terminal constraints

$$a_i\,\bar\xi(t_0) + b_i\,(\bar\xi(t_1) + F\,\bar h) = 0, \qquad i = 1,\dots,\mu, \tag{22}$$

$$a'_j\,\bar\xi(t_0) + b'_j\,(\bar\xi(t_1) + F\,\bar h) \le 0, \qquad j = 1,\dots,\nu. \tag{23}$$

What remains to be checked is that this new cone $\widetilde{K}$ contains the old cone K, given by (17), (19), (20), as a dense subset. This fact is a particular case of the following general assertion, which is of independent interest.

Lemma 1 (on denseness) *Let in a locally convex linear topological space X be given a finite-faced cone C, and a linear variety (algebraic subspace) L dense in X. Then the cone $C \cap L$ is dense in C.*

Proof. Consider first the case when the cone C is a subspace given by one equation $(p,x) = 0$, $p \in X^*$, $p \ne 0$. Take any point $x_0 \in C$ and its convex neighborhood $\mathcal{O}(x_0)$. We have to show that $\exists\, x \in C \cap L \cap \mathcal{O}(x_0)$. Since the set $(p,x) < 0$ is open, its intersection with $\mathcal{O}(x_0)$ is open, too, and obviously nonempty, hence it contains a point x_1 from the set L, because the last one is dense in X. Similarly, the intersection of the set $(p,x) > 0$ with $\mathcal{O}(x_0)$ contains a point $x_2 \in L$. Since $\mathcal{O}(x_0)$ is convex, it contains the whole segment $[x_1,\, x_2]$, that also lies in L, since the last one is a linear variety. But this segment obviously contains a point x such that $(p,x) = 0$, which then belongs to C and to $L \cap \mathcal{O}(x_0)$, *q.e.d.*

Now, let C be an arbitrary finite-faced cone, given by inequality constraints $(q_j, x) \le 0$, $j = 1,\dots,\nu$. Suppose first, that $\exists\, \hat x \in C$ such that all $(q_j, \hat x) < 0$, hence $\hat x \in int\,C$. Take any $x_0 \in C$ and any its neighborhood $\mathcal{O}(x_0)$. We have to find a point $x \in C \cap \mathcal{O}(x_0) \cap L$. We know that, for any $\varepsilon > 0$, the point $x_\varepsilon = x_0 + \varepsilon \hat x$ lies in $int\,C$ (a simple property of convex cones), and $\exists\, \varepsilon > 0$ such that this point lies also in $\mathcal{O}(x_0)$. Thus, the open set $int\,C \cap \mathcal{O}(x_0)$ is nonempty, and then it contains a point x from a dense set L, *q.e.d.*

Suppose now that the above point $\hat x \in int\,C$ does not exist, i.e., the strict inequality constraints do not intersect. If some $q_j = 0$, we can remove j-th inequality, so we assume that all $q_j \ne 0$. In this case, by the Dubovitskii–Milyutin theorem, there exist multipliers $\alpha_j \ge 0$, $j = 1,\dots,\nu$, not all zero, such that the following EL equation holds: $\alpha_1\, q_1 + \ldots + \alpha_\nu\, q_\nu = 0$. Take any j with $\alpha_j > 0$; let it be $j = \nu$. Then, for all $x \in C$ we actually have $(q_\nu, x) = 0$, not just ≤ 0. (Otherwise, if $x \in C$ and $(q_\nu, x) < 0$, one should multiply the EL equation by this x and take into account that all other terms $(q_j, x) \le 0$, while the sum equals 0, a contradiction.) This means that the cone C can be given by the constraints $(q_j, x) \le 0$, $j = 1,\dots,\nu-1$, $(q_\nu, x) = 0$. But, passing to the subspace $(q_\nu, x) = 0$ we obtain in it, as was shown above, a dense linear variety and a cone given by a smaller number $\nu - 1$ of inequality constraints. The equality $(q_\nu, x) = 0$ is thus removed. Applying the induction arguments, we arrive at the situation when either all the inequality constraints are transformed, one by one, into equalities and then removed, or the strict inequalities have a nonempty intersection. Since the last case is already considered, the proof is complete. ∎

In our situation we have $X = \mathbb{R}^n \times L_2^r \times \mathbb{R}^r$, $L = \{\, (\bar{\xi}_0, \bar{y}(\cdot), \bar{y}(t_1)) \mid \bar{\xi}_0 \in \mathbb{R}^n,\ \bar{y}$ is a Lipschitz function $\}$, and $C = \widetilde{K}$.

1.5. Passing to a new control

From Lemma 1 and the continuity of $\widetilde{\Omega}$ in the extended space it follows that the sign definiteness of Ω on K w.r.t. γ is equivalent to that of $\widetilde{\Omega}$ on $\widetilde{K}$ w.r.t. the same γ.

But now, looking at the new functional $\widetilde{\Omega}$ and the cone $\widetilde{K}$, we notice that the control $\bar{u}$ completely disappeared, and the role of control is now taken by the variable $\bar{y} \in L_2$, since it does not obey any differential equation, so the role of state variable is left only for $\bar{\xi}$. Accepting this, we then notice that $\widetilde{\Omega}$ now contains the Legendre term w.r.t. the new control, $(R\bar{y}, \bar{y})$, and therefore, the nonnegativity of $\widetilde{\Omega}$ on $\widetilde{K}$ immediately implies the Legendre condition: $R(t) \geq 0$ a.e. on $[t_0, t_1]$. This is the second necessary Goh condition, of inequality type, in the case of one-dimensional control obtained earlier by H.J. Kelley (see Kelley, Kopp, Moyer, 1967). (To be precise, Goh, 1966, proved his conditions for a particular case of problem A without inequality constraints (4), hence for K being a subspace. For the case of cone, the Goh conditions, both of equality and inequality type, were proved in Dmitruk, 1977, 1978.)

The order of minimum γ now contains the square of the control variation, exactly as in CCV (see Levitin, Milyutin, Osmolovskii, 1978, and Milyutin, Osmolovskii, 1998). If we hope to obtain the positive definiteness of $\widetilde{\Omega}$ w.r.t. γ, we must assume at least the strengthened Legendre condition w.r.t. the new control:

$$R(t) \geq const \cdot I \qquad \text{a.e. on } [t_0, t_1], \qquad const > 0, \tag{24}$$

where I is the identity matrix. In what follows, we do assume it holds. (In this case $\widetilde{\Omega}$ is called a Legendre quadratic functional, Hestenes, 1951).

Thus, we actually come to a situation, in which we should determine the sign definiteness of a quadratic functional $\widetilde{\Omega}$ of the form (21), satisfying the strengthened Legendre condition, on a cone $\widetilde{K}$ given by (17), (22), (23). The difference from CCV is in the following two features: a) $\widetilde{K}$ is a cone, not a subspace, and b) $\widetilde{\Omega}$ and relations (22), (23) include an additional parameter $\bar{h}$.

Our further aim is to develop Jacobi type conditions for this case. To simplify the exposition, let us assume here that the inequalities (23) are absent, i.e. $\widetilde{K}$ is a subspace. Recall that this indeed is the case if at least all but one multipliers $\alpha_i > 0$. (Jacobi type conditions for the case of a general quadratic form on a cone are obtained in Dmitruk, 1981, 1984.)

2. Jacobi type conditions for the sign definiteness of a quadratic functional

2.1. Linear-quadratic situation with a parameter

Stating the situation (17), (21), (22), let us pass once again to a simpler and more convenient notation, by changing $(\bar{\xi}, \bar{y}, \bar{h}) \longmapsto (x, u, h)$. Then we have a quadratic form (denoted again by Ω) of the type

$$\Omega = g(x(t_0), x(t_1), h)) + \int_{t_0}^{t_1} (Qx, x) + 2(Px, u) + (Ru, u)\, dt, \tag{25}$$

satisfying (24), that should be considered on a subspace K in the space of variables $(x(t_0), u, h) \in \mathbb{R}^n \times L_2^r \times \mathbb{R}^r$ given by a linear differential equation

$$\dot{x} = Ax + Bu \tag{26}$$

and terminal equality constraints of the form

$$a_i x(t_0) + b_i x(t_1) + c_i h = 0, \qquad i = 1, \ldots, m. \tag{27}$$

Our task is to verify the sign definiteness of Ω on K w.r.t. the quadratic order

$$\gamma(x, u, h) = |x(t_0)|^2 + |h|^2 + \int_{t_0}^{t_1} |u(t)|^2 dt, \tag{28}$$

which is the square of norm in the space $\mathbb{R}^n \times L_2^r \times \mathbb{R}^r$. The procedure of this verification is based on the abstract approach to Jacobi theory proposed in Hestenes (1951) and depends on the specificity of terminal constraints. If the terminal constraints are in the general form (27), this procedure is very cumbersome; it will be treated by the author elsewhere. Here we consider two most important cases, where equations (27) result from the problem A with at least one endpoint fixed. The specificity of such terminal constraints allows one to make this procedure simpler.

Note that the absence of formal symmetry between these cases is caused by the fact that the order γ and the Goh transformation are not symmetrical w.r.t. the left and the right endpoints.

2.2. The left endpoint fixed

This means that (15) reads $\bar{x}(t_0) = 0, \quad b_i \bar{x}(t_1) = 0$, which in (19) yields $\bar{\xi}(t_0) = 0$, $b_i(\bar{\xi}(t_1) + F\bar{y}(t_1)) = 0$, and so, (27) can be represented in the form

$$x(t_0) = 0, \tag{29}$$

$$\Lambda x(t_1) + Nh = 0, \tag{30}$$

with some matrices Λ, N of dimensions $m \times n$, $m \times r$. (Note that this last x has nothing in common with the initial x!) We can even allow now arbitrary dimensions for u and h : $\dim u = r$, $\dim h = q$, not necessarily equal (so, $\dim N = m \times q$). Note that here all components of $x(t_0)$, both essential and inessential, automatically disappear from γ, since $x(t_0) = 0$.

The terminal quadratic form g can be represented as

$$g(x(t_1), h) = (S_{xx}x(t_1), x(t_1)) + 2(S_{hx}x(t_1), h) + (S_{hh}h, h), \tag{31}$$

where S_{xx}, S_{hx}, S_{hh} are matrices of corresponding dimensions.

2.3. The Hestenes scheme

We have to study Ω of the form (25), (31) with respect to the quadratic order (28) under the relations (26), (29), (30).

Let us fix the terminal time t_1 and vary initial time t_0. The smaller is t_0, the larger is the interval $[t_0, t_1]$, and the broader is the set of triples $x(t), u(t), h$ satisfying (26), (29), (30), because any such triple can be naturally extended to the larger interval by zero value of $x(t)$ and $u(t)$. Moreover, since h does not come into the integrand in (25), Ω considered at any triple for an initial interval $[t_0, t_1]$, takes the same value at this triple extended to the larger interval, thus Ω has more chances to be negative on the larger interval. So, the sign definiteness of Ω *monotonically depends on* t_0, and this is a key point for application of the Hestenes approach, that will allow us to define a point t_0^* conjugate to t_1.

Note that, since $x(t_0) = 0$, the state variable $x(t)$ on the interval $[t_0, t_1]$ is uniquely determined by $u(t)$ from equation (26), and so, Ω uniquely depends on (u, h). Set $T = t_1 - t_0$. For any $s \in [0, T]$ define $t_0' = t_1 - s$ and consider the Hilbert space H_s consisting of elements $(u, h) \in L_2^r[t_0', t_1] \times \mathbb{R}^q$ such that the corresponding solution to (26) with $x(t_0') = 0$ has $x(t_1)$ satisfying (30). (The change of parameters $t_0' \mapsto s$ is taken for the sake of unification of the Hestenes scheme.)

The space H_s obviously expands as s grows: if $s' < s''$, then $H_{s'} \subset H_{s''}$ with the natural embedding. Moreover, it continuously depends on s in the following sense:

$$H_s = \bigcap_{s' > s} H_{s'} \qquad \text{and} \qquad H_s = \overline{\bigcup_{s' < s} H_{s'}}\,.$$

For $s = 0$ we have a finite-dimensional space $H_0 = \{0\} \times \mathbb{R}^q$.

Recall that a quadratic form is said to be positive definite on a subspace if it is estimated from below by the square of norm on this subspace.

Suppose that Ω is positive definite on H_0. Then, according to Hestenes (1951) (see also Dmitruk, 1976, 1982, 1984; Zeidan, 1994; Stefani, Zezza, 1997; Stefani, 2003, 2004; Rosenblueth, 2003), we should find a minimal $s > 0$, for which the functional Ω has a nonzero stationary point on H_s. Denoting this value by s^*, we can say that Ω is positive definite on H_s for any $s < s^*$, and $\Omega \geq 0$ on H_{s^*}; moreover, there is a nonzero pair $(u,h) \in H_{s^*}$ such that $\Omega(u,h) = 0$. (Obviously, this pair is a stationary point of Ω on H_{s^*}.)

(For the detailed proof see the above papers. It is based on the following key facts valid for the Legendre quadratic forms: a) if $\Omega(u,h) > 0$ for all nonzero $(u,h) \in H_s$, then Ω is positive definite on H_s, and b) if Ω is positive definite on H_s, then it remains positive definite on $H_{s'}$ for some $s' > s$. Also, we use the following simple fact: if (u,h) is a stationary point of Ω on a subspace, then $\Omega(u,h) = 0$.)

The point $t_0^* = t_1 - s^*$ is said to be conjugate to t_1.

2.4. The Euler–Jacobi equation

To write out the stationarity equation, let us impose the following assumption on the control system (26), (29), (30):

A1) For any $t_0' < t_1$ the system (26), (29), (30) is *controllable on* $[t_0', t_1]$, i.e., the mapping $\Gamma : (u,h) \longmapsto \Lambda x(t_1) + Nh$, where $x(t)$ satisfies (26) with $x(t_0') = 0$, is a surjection: $Im\,\Gamma = \mathbb{R}^m$.

Note that this system is assuredly controllable on the initial interval $[t_0, t_1]$, since we removed from (8) at least the condition $\varphi_0' \bar{p} = 0$. If the remaining system is uncontrollable, there would exist Lagrange multipliers that provide the EL equation without φ_0, which is impossible, because we assume the Lagrange multipliers in Problem A at w^0 to be unique and have $\alpha_0 > 0$. Assumption A1 requires that the above system is controllable for any $t_0' < t_1$, not only for the initially given t_0.

This requirement can also be formulated in a dual form: if a Lipschitz $n-$ vector function $\psi(t)$ and a vector $\beta \in \mathbb{R}^m$ satisfy the relations

$$-\dot{\psi} = A^*\psi, \qquad B^*\psi = 0 \quad \text{on } [t_0', t_1], \tag{32}$$

$$-\psi(t_1) = \Lambda^*\beta, \qquad N^*\beta = 0, \tag{33}$$

then $\psi(t) \equiv 0$ and $\beta = 0$.

Note also that Assumption A1 can be weakened: if it is somehow known apriori that $\Omega > 0$ on H_σ for some $\sigma > 0$, then it suffices to require A1 only for all $t_0' < t_1 - \sigma$.

Now, considering the so-called auxiliary problem

$$\Omega(u,h) \to \min, \qquad (u,h) \in H_s,$$

we obtain the following condition: if (u,h) is a stationary point in this problem, then there exist a Lipschitz $n-$ vector function $\psi(t)$ and a vector $\beta \in \mathbb{R}^m$ satisfying the following Euler–Lagrange relations (for the linear-quadratic case they are also called Euler–Jacobi (EJ) equation) on $[t_0', t_1]$, where $t_0' = t_1 - s$:

$$-\dot{\psi} = A^*\psi - Qx - P^*u, \tag{34}$$
$$B^*\psi - Px - Ru = 0, \tag{35}$$
$$-\psi(t_1) = S_{xx}x(t_1) + S_{xh}h + \Lambda^*\beta, \tag{36}$$
$$S_{hx}x(t_1) + S_{hh}h + N^*\beta = 0. \tag{37}$$

(They can be conveniently obtained by the formalism of Maximum Principle: composing the Pontryagin function $\Pi = \psi\,(Ax + Bu) - \frac{1}{2}\big((Qx, x) + 2(Px, u) + (Ru, u)\big)$ and the terminal Lagrange function $\lambda = \beta(\Lambda x(t_1) + Nh) + \frac{1}{2}\,g(x(t_1), h)$, we should write $-\dot{\psi} = \Pi_x$, $\Pi_u = 0$, $-\psi(t_1) = \lambda_{x(t_1)}$, $\lambda_h = 0$. Assumption A1 allows us to take the multiplier $1/2$ at the cost Ω.)

A quadruple $(u(t), h, \psi(t), \beta)$ satisfying (26), (29), (30), and (34)–(37) on $[t_0', t_1]$ will be called a solution to EJ equation on $[t_0', t_1]$.

We have to find a maximal $t_0' < t_1$ such that the EJ equation on $[t_0', t_1]$ has a solution (u, h, ψ, β) with a nonzero pair (u, h). Let us first show that the nontriviality of the pair (u, h) is equivalent to the nontriviality of the pair (x, ψ).

Lemma 2 *Let u, h, x, ψ, β satisfy relations (26), (29), (30), (34)–(37). Then* $(u, h) = (0, 0)$ *iff* $(x, \psi) = (0, 0)$.

Proof. a) If $u(t) = 0$, then (26), (29) yield $x(t) = 0$, and then relations (34)–(37) for $h = 0$ are exactly relations (32), (33), which imply $\psi(t) = 0$, $\beta = 0$.

b) If $(x, \psi) = (0, 0)$, then (35) yields $u(t) = 0$, since R is a nondegenerate (positive definite) matrix, and (37) is reduced to $S_{hh}h + N^*\beta = 0$, which means that h is a stationary point of $\Omega = g(0, h)$ on H_0. Since we assume that $\Omega > 0$ on H_0, the only stationary point is $h = 0$. ∎

An important consequence of the strengthened Legendre condition (24) is that equation (35) allows us to represent $u = R^{-1}(B^*\psi - Px)$. Substituting this expression into (26) and (34), we obtain a linear system w.r.t. (x, ψ) of the general form

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x + \mathcal{B}(t)\psi, \\ \dot{\psi} &= \mathcal{C}(t)x + \mathcal{D}(t)\psi, \end{aligned} \tag{38}$$

with some $n \times n-$ matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. Our task is to find a pair $(x, \psi) \neq (0, 0)$ and vectors β, h satisfying this system and terminal relations (29), (30), (36), (37) on $[t_0', t_1]$.

Since the system (38) is linear and homogeneous, the pair (x, ψ) is nontrivial iff $(x(t), \psi(t)) \neq (0, 0)$ $\forall\, t$. Note also that, if $\psi(t) \equiv 0$, then the first equation in (38) with account of (29), i.e. $x(t_0') = 0$, implies $x(t) \equiv 0$.

2.5. Equation for the conjugate point

Note that assumption A1 implies, in particular, that the $m \times (n+q)-$ matrix $||\Lambda|N||$ has the full rank m, and so the subspace of pairs $(x(t_1), h) \in \mathbf{R}^n \times \mathbf{R}^q$ satisfying (30) has dimension $n+q-m$.

The conjugate point t_0^* can be found by the following procedure. Choose a basis

$$(x^i(t_1), h^i), \qquad i = 1, \ldots, n+q-m, \tag{39}$$

in that subspace, and also choose an arbitrary basis $\beta^1, \ldots, \beta^m$ in the space of $\beta \in \mathbf{R}^m$. Thus, we have $n+q$ basis vectors in the subspace of triples $(x(t_1), h, \beta) \in \mathbf{R}^{n+q+m}$ satisfying (30). These basis vectors generate $n+q$ vectors

$$(x^i(t_1), h^i, \psi^i(t_1), \theta^i), \qquad i = 1, \ldots, n+q, \tag{40}$$

where vectors $\psi^i(t_1)$ are obtained by (36), and $q-$ dimensional vectors θ^i by the relation $\theta = S_{hx}x(t_1) + S_{hh}h + N^*\beta$ in view of the left-hand side of (37).

For each vector (40), we should solve the system (38) backwards in time, starting with the terminal values $(x^i(t_1), \psi^i(t_1))$, thus obtaining vector-functions $(x^i(t), \psi^i(t))$. Adding here the constant vectors h^i and θ^i, we obtain vector-functions

$$(x^i(t), h^i, \psi^i(t), \theta^i), \qquad i = 1, \ldots, n+q, \tag{41}$$

that form a fundamental family of solutions to system (38) with terminal relations (30), (36). Now, we only have to satisfy relations $x(t_0') = 0$ and (37).

The vector-functions $(x^i(t), \theta^i)$, $i = 1, \ldots, n+q$, from (41) compose $(n+q) \times (n+q)-$ matrix $||\, X(t) \,|\, \Theta \,||$, and hence, the required relations are satisfied for some solution to system (38), (30), (36) if and only if

$$det\, ||\, X(t_0') \,|\, \Theta \,|| \;=\; 0.$$

The point t_0^* is then the maximal root $t_0' < t_1$ of this equation.

If the matrix $N : \mathbf{R}^q \to \mathbf{R}^m$ allows us to use some scalar relations of (37) to eliminate some components of β, then the number of basis vectors β^i can be reduced, and we need only pay attention to the remaining scalar relations of (37). Thus, the matrix $||\, X(t) \,|\, \Theta \,||$ would have a smaller dimension.

In the most favorable case, when $rank\, N = m$ (hence $m \le q$), the vector β can be totally expressed through $x(t_1)$ and h by some m equations from (37), in the form $\beta = Cx(t_1) + Dh$ with some matrices C, D. The left hand sides of remaining $q-m$ equations of (37) compose a reduced vector $\tilde\theta$ of dimension $q' = q-m$. Here we do not need to take an independent basis in the space of β; instead, we should choose the above basis $(x^i(t_1), h^i)$, $i = 1, \ldots, n+q-m$, then define the vectors $\beta^i = Cx^i(t_1) + Dh^i$, calculate the corresponding vectors

$\psi^i(t_1)$ by (36), and solve the system (38) backwards in time for each terminal value $(x^i(t_1),\, \psi^i(t_1))$, thus obtaining vector-functions $(x^i(t),\, \psi^i(t))$, $i = 1, \ldots, n+q-m$. Adding to them the corresponding vectors h^i and $\tilde{\theta}^i$, we obtain vector-functions

$$(x^i(t), h^i, \psi^i(t), \tilde{\theta}^i), \qquad i = 1, \ldots, n+q'.$$

This is a fundamental family of solutions to system (38) with terminal relations (30), (36), (37), and it only remains to satisfy the initial condition $x(t_0) = 0$ and $\tilde{\theta} = 0$. To do this, we should compose the $(n+q') \times (n+q')-$ matrix $||\, X(t) \,|\, \widetilde{\Theta} \,||$ and find the maximal $t < t_1$ satisfying the equation $det\, ||\, X(t) \,|\, \widetilde{\Theta} \,|| = 0$. This would be a point t_0^* conjugate to t_1.

In another favorable case, when $rank\, N = q \le m$, some q scalar relations of (30) can be used to express $h = Dx(t_1)$ with some matrix D, and $x(t_1)$ should only satisfy the remaining $m' = m - q$ equations of (30), which can be written in the form

$$G\, x(t_1) \,=\, 0 \tag{42}$$

with some $m' \times n-$ matrix G. The endpoint quadratic form g (and then Ω) does not contain h, and so, we have the standard classical case with zero left endpoint $x(t_0) = 0$ and equality constraints (42) for the right endpoint.

Assumption A1 takes here the following form: *the mapping* $\Gamma : u \longmapsto Gx(t_1)$, *where* $x(t)$ *is obtained from (26), (29), is a surjection:* $Im\, \Gamma = \mathbb{R}^{m'}$. The EJ equation reads here as follows: if u is a stationary point in the problem $\Omega(u) \to \min$, $u \in H_s$, then there exist a Lipschitz $n-$ vector function $\psi(t)$ and a vector $\beta \in \mathbb{R}^{m'}$ satisfying relations (34), (35), and the terminal transversality condition

$$-\psi(t_1) \,=\, S_{xx}\, x(t_1) \,+\, G^* \beta\,. \tag{43}$$

Here we may choose a basis $x^i(t_1)$, $i = 1, \ldots, n-m'$, in the subspace of vectors $x(t_1)$ satisfying (42), add to it a basis β^i, $i = 1, \ldots, m'$, in the space of $\beta \in \mathbb{R}^{m'}$, obtaining thus a basis $(x^i(t_1), \beta^i)$, $i = 1, \ldots, n$, in the subspace of pairs $(x(t_1), \beta)$ satisfying (42), and then calculate the corresponding vectors $\psi^i(t_1)$, $i = 1, \ldots, n$, by (43). Thus, we obtain n vectors $(x^i(t_1), \psi^i(t_1), \beta^i)$, $i = 1, \ldots, n$, satisfying (42) and (43). For each of these vectors, solving the system (38) backwards in time, we obtain vector-functions $(x^i(t), \psi^i(t))$, $i = 1, \ldots, n$. Next, compose $n \times n-$ matrix $X(t)$ of the vectors $x^i(t)$. Then, the conjugate point t_0^* is the maximal $t < t_1$ such that $det\, X(t) \,=\, 0$.

2.6. Checking the nonnegativity of Ω

Now, suppose the conjugate point t_0^* is somehow found. In accordance with the abstract Jacobi theory, if $t_0^* < t_0$, then the quadratic functional (25) defined on the interval $[t_0, t_1]$ is positive definite w.r.t. γ on the subspace of (x, u, h) satisfying relations (26), (29), (30). However, if $t_0^* \in (t_0, t_1)$, we cannot, in general, say that Ω has negative values on this subspace. In other words, if $s^* = t_1 - t_0^*$ is the conjugate point for Ω on the one-parameter family of spaces H_s, then it can happen that Ω is still nonnegative on H_s for some $s > s^*$ (an easy example is given in Dmitruk, 1976). Let s^{**} be the maximum of such s. Obviously, $s^{**} \geq s^*$. The interval $[s^*, s^{**}]$ is called *the conjugate (or focal) interval.* In the case when it is nondegenerate $(s^{**} > s^*)$, the point s^* and the corresponding point t_0^* should better be called *the closest conjugate point*, while the point s^{**} and the corresponding point t_0^{**} should be called *the farthest conjugate point* (with respect to $s = 0$ and $t = t_1$ respectively). Fortunately, the farthest conjugate point s^{**} can be also determined by using the solutions to EJ equation, see Hestenes (1951), Dmitruk (1976) (and also Dmitruk, 1981 for the general case of finite-faced cone K). In the most favorable case these two points coincide: $s^{**} = s^*$, and there is no need to find the point s^{**} if the point s^* is already found.

To guarantee this coincidence, we impose one more assumption on our situation:

A2) If a vector-function $\psi(t)$ satisfies the equations $-\dot{\psi} = A^*\psi$, $B^*\psi = 0$ on an interval (t', t''), then $\psi(t) \equiv 0$ on this interval.

This is, of course, rather a strong assumption. It means that the system $\dot{x} = Ax + Bu$ is *completely controllable,* i.e., controllable on any nonzero interval $[t', t'']$ (which means that for any $a', a'' \in \mathbf{R}^n$ there exists a pair $(x(t), u(t))$ satisfying the above system with $x(t') = a'$ and $x(t'') = a''$). However, it often holds in problems of CCV.

One can propose a weaker assumption: any nontrivial solution (x, ψ, h, β) to EJ equation on an interval $[t', t_1]$ cannot remain a solution on a larger interval $[t'', t_1]$, $t'' < t'$, being extended on $[t'', t']$ by setting there $x(t) = 0$, $\psi(t) = 0$. However, the weakest assumption, in fact equivalent to the coincidence $s^{**} = s^*$, and hence to $t_0^{**} = t_0^*$, is as follows: *there exists* a nontrivial solution (x, ψ, h, β) to EJ equation on the interval $[t_0^*, t_1]$ that cannot be extended to a solution on a larger interval $[t'', t_1]$, $t'' < t_0^*$, by setting $x(t) = 0$, $\psi(t) = 0$ on $[t'', t_0^*]$. (This equivalence is a key assertion proved in Hestenes, 1951. It is worth noting that the similar assertion holds also true for the case involving terminal *inequality constraints* (23), see Dmitruk, 1981.)

Under assumptions A1 and A2, if we find the conjugate point t_0^*, we obtain, like in CCV, the complete information about the sign of quadratic functional Ω given by (25) and (31): a) if $[t_0, t_1]$ does not contain a point conjugate to t_1, then Ω is positive definite on the subspace of (x, u, h) satisfying relations (26), (29), (30); b) if $t_0^* \in (t_0, t_1)$, then Ω has negative values on this subspace, and c) if $t_0^* = t_0$, then $\Omega \geq 0$ on this subspace, and there exists a nonzero pair (u, h) such that $\Omega(u, h) = 0$.

Consider now another important case.

2.7. The right endpoint fixed

This case can be treated in two ways. In the first one, we can just reduce the situation to the preceding case by changing t_0 and t_1. The second way is as follows. Since $x(t_1)$ is fixed, for the critical directions we have $\bar{x}(t_1) = 0$ plus some equality constraints at the left endpoint, $G\,\bar{x}(t_0) = 0$. After the Goh transformation we get $G\bar{\xi}(t_0) = 0$, $\bar{\xi}(t_1) + F(t_1)\bar{h} = 0$, and so, equations (27) in the final notation have the form

$$G\,x(t_0) = 0, \qquad x(t_1) + F\,h = 0,$$

where G and F are some matrices of dimensions $m\times n$ and $n\times r$, respectively. (Here we can neglect the relation of F with the matrix $F(t,x(t))$ from the initial control system.)

Assume that $rank\,F = r$, i.e., the mapping $F : \mathbb{R}^r \to \mathbb{R}^n$ is injective (which is quite realistic). Then it has a right inverse $D : \mathbb{R}^n \to \mathbb{R}^r$, $DF = I$, and so, h can be expressed through $x(t_1)$ in the form $h = D\,x(t_1)$, while $x(t_1)$ should satisfy a constraint $\Lambda\,x(t_1) = 0$ with some $(n-r)\times n-$ matrix Λ. Thus, we have a quadratic form Ω of the type (25) with a finite-dimensional part g of the general type

$$g(x_0,x_1) = (S_{00}\,x_0,x_0) + 2(S_{01}\,x_0,x_1) + (S_{11}\,x_1,x_1),$$

where S_{00}, S_{01}, S_{11} are some $n\times n-$ matrices, and this form should be considered on the space of functions $(x(t),u(t))$ on $[t_0,t_1]$ satisfying equation (26) and terminal relations

$$G\,x(t_0) = 0, \qquad \Lambda\,x(t_1) = 0, \tag{44}$$

where G and Λ are some matrices of dimensions $m\times n$ and $p\times n$, respectively (with some integer p). So, we actually have the general linear-quadratic situation *with both endpoints independently variable.* If at least one endpoint is zero, say $x(t_0) = 0$ (i.e., $rank\,G = n$), we have completely the classical situation, so we should vary the point t_0 and find the conjugate point t_0^* by the standard classical procedure.

However, in the general case, when both endpoints are indeed variable, the sign definiteness of Ω on $[t_0,t_1]$ *does not depend monotonically on* t_0, and so, the implementation of the Hestenes approach in this case should be modified. A natural modification follows the idea (noted e.g. in Dmitruk, 1984) that one should vary *the support of* $u(t)$, but keep the integration over the whole fixed interval $[t_0,t_1]$.

To be more precise, consider the Hilbert space $\mathcal{H} = L_2^r[t_0,t_1]\times\mathbb{R}^n$ with elements $(u(t),b)$ such that the corresponding $x(t)$, determined by equation (26) with initial condition $x(t_1) = b$, satisfies (44). Set $T = t_1 - t_0$. For any $s\in[0,T]$ define a subspace $H_s\subset\mathcal{H}$ consisting of all $(u,b)\in\mathcal{H}$ such that $\operatorname{supp} u(t)\subset[t_1-s,t_1]$, i.e. $u(t) = 0$ a.e. on $[t_0,t_1-s]$. Obviously, $\{H_s\}$ is a continuously expanding family of subspaces, $H_0 = 0$, and $H_T = \mathcal{H}$. The quadratic form $\Omega : \mathcal{H} \to \mathbb{R}$ does not depend on s.

As before, one should first check that Ω is positive definite on H_0, and then find s^* as the minimal $s > 0$ for which the functional Ω has a nonzero stationary point on H_s. The difference from the case with $x(t_0) = 0$ is that, since the integration is taken over the whole initial interval $[t_0, t_1]$, the EJ equation for Ω on H_s should be also considered on this whole interval, not only on $[t_1 - s, t_1]$. Alternatively, one can reduce the integration to $[t_1 - s, t_1]$, but then one should take into account the contribution of the integrand term (Qx, x) on the "inactive" subinterval $[t_0, t_1 - s]$. (The other integrand terms, involving u, give no contribution.) Since $x(t)$ on this subinterval satisfies the equation $\dot{x} = Ax$, then $x(t) = \Phi(t_1 - s, t)\, x(t_1 - s)$, where Φ is the transition matrix of this equation ($\Phi_t(\tau, t) = A(t)\, \Phi(\tau, t)$, $\Phi(\tau, \tau) = I$), and so, the integral of (Qx, x) is a quadratic form of $x(t_1 - s)$. Moreover, since $x(t_0) = \Phi(t_1 - s, t_0)\, x(t_1 - s)$, the endpoint quadratic form $g(x(t_0), x(t_1))$ can be expressed through $x(t_1 - s)$ and $x(t_1)$. Thus, the integration in Ω can be reduced to $[t_1 - s, t_1]$ at the expense of changing some coefficients in the endpoint quadratic form: $g(x(t_0), x(t_1)) \longmapsto \tilde{g}(x(t_1 - s), x(t_1))$. The first relation in (44) can be also expressed in the form $G_s\, x(t_1 - s) = 0$ with some matrix G_s. Actually, here one must deal not with a single quadratic form Ω, but with a family of quadratic forms Ω_s having the same integral part while their endpoint part depends on the parameter s. The transversality conditions in the EJ equation would then also involve s. For a detailed exposition of this procedure see, e.g., Zeidan (1994), and Stefani, Zezza (1997).

EXAMPLE 1 Consider the following problem:

$$\dot{x}_1 = x_2 + f_1(x_1, x_2), \qquad x_1(t_0) = 0,$$
$$\dot{x}_2 = u + f_2(x_1, x_2), \qquad x_2(t_0) = 0,$$
$$J = 2s\, x_1(T)\, x_2(T) + \int_{t_0}^{T} (x_1^2 + x_2^2 + 2p\, x_2 u)\, dt \to \min,$$

where $p > 0$ and s are parameters, the functions f_1, f_2 are twice smooth, vanishing at zero together with their first and second derivatives, and analyze the process w^0 : $x_1^0 = x_2^0 = u^0 = 0$. To represent formally this problem as an example of problem A, introduce an additional state variable x_3 satisfying the equation

$$\dot{x}_3 = x_1^2 + x_2^2 + 2p\, x_2 u, \qquad x_3(t_0) = 0,$$

whence the cost takes the terminal form $J = 2s\, x_1(T)\, x_2(T) + x_3(T) \to \min$.

Obviously, the reference process w^0 satisfies the MP with a unique (up to normalization) collection of multipliers $\psi_1 = \psi_2 = 0$, $\psi_3 = 1$, $\alpha_0 = 1/2$,

$\beta_1 = \beta_2 = 0, \quad \beta_3 = -1,$ and with the corresponding functions

$$l = \alpha_0\big(2s\, x_1(T)\, x_2(T) + x_3(T)\big) + \beta_1 x_1(t_0) + \beta_2 x_2(t_0) + \beta_3 x_3(t_0),$$
$$H = \psi_1(x_2 + f_1) + \psi_2(u + f_2) + \psi_3(x_1^2 + x_2^2 + 2p\, x_2 u).$$

The Lagrange function here is $\mathcal{L} = l + \int \psi_3(\dot{x}_3 - (x_1^2 + x_2^2 + 2p\, x_2 u))\, dt,$ and its second variation at w^0 is

$$\Omega(\bar{x}, \bar{u}) = 2s\, \bar{x}_1(T)\, \bar{x}_2(T) + \int_{t_0}^{T} (\bar{x}_1^2 + \bar{x}_2^2 + 2p\, \bar{x}_2 \bar{u})\, dt. \tag{45}$$

The critical subspace K is defined by the relations

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2\,, & \bar{x}_1(t_0) &= 0, \\ \dot{\bar{x}}_2 &= \bar{u}\,, & \bar{x}_2(t_0) &= 0, \\ \dot{\bar{x}}_3 &= 0\,, & \bar{x}_3(t_0) &= 0, \end{aligned}$$

and $\bar{x}_3(T) \le 0$ (linearization of the cost). Since the multiplier at the cost $\alpha_0 > 0,$ we can put $\bar{x}_3(T) = 0,$ whence the artificial variable $\bar{x}_3(t) \equiv 0,$ so we remove it from all relations.

Introducing the variable $\bar{y}$ subject to $\dot{\bar{y}} = \bar{u}, \;\; \bar{y}(t_0) = 0,$ we should try to estimate Ω from below by $\gamma = \bar{y}^2(t_1) + \int_{t_0}^{T} \bar{y}^2\, dt$ for $(\bar{x}_1, \bar{x}_2, \bar{u}) \in K$. Theorem 1 says (in view of $dim\, u = 1$), that if Ω is positive definite in this sense, the reference process w^0 gives a Pontryagin minimum in the problem, and if Ω admits negative values, w^0 does not give even a weak minimum.

Perform the Goh transformation by setting $\bar{x}_1 = \bar{\xi}_1$ and $\bar{x}_2 = \bar{\xi}_2 + \bar{y}$. Then $\dot{\bar{\xi}}_2 = 0,\ \bar{\xi}_2(t_0) = 0,$ hence $\bar{\xi}_2(t) \equiv 0,$ and we obtain $\bar{x}_2 = \bar{y}, \quad \dot{\bar{\xi}}_1 = \bar{y}, \quad \bar{\xi}_1(t_0) = 0,$ while Ω takes the form

$$\begin{aligned} \Omega &= 2s\, \bar{\xi}_1(T)\, \bar{y}(T) + \int_{t_0}^{T} (\bar{\xi}_1^2 + \bar{y}^2 + 2p\, \bar{y}\bar{u})\, dt = \\ &= p\, \bar{y}^2(T) + 2s\, \bar{\xi}_1(T)\, \bar{y}(T) + \int_{t_0}^{T} (\bar{\xi}_1^2 + \bar{y}^2)\, dt \end{aligned}$$

(we took into account that $\int_{t_0}^{T} 2\bar{y}\bar{u}\, dt = \bar{y}^2(T)$). The strong Legendre condition is satisfied here: $R(t) \equiv 1.$

Changing the variables $(\bar{\xi}_1, \bar{y}, \bar{y}(T)) \longmapsto (\bar{\xi}_1, \bar{y}, \bar{h}) \longmapsto (x, u, h),$ we finally come to the quadratic form

$$\Omega(u, h) = \; ph^2 + \; 2s\, x(T)\, h + \int_{t_0}^{T} (x^2 + u^2)\, dt, \tag{46}$$

where $\qquad \dot{x} = u, \qquad x(t_0) = 0,$

which should be estimated by the order $\gamma(u,h) = h^2 + \int_{t_0}^{T} u^2\,dt$ on the space of all $(u,h) \in L_2[t_0,\,T] \times \mathbb{R}^n$. If $u(t) \equiv 0$ (i.e., (u,h) lies in the space H_0), then $\Omega = ph^2$ is really positively definite, due to the assumption $p>0$. Next, since the left endpoint is zero, we can vary t_0 and try to find the conjugate point t_0^*. Since the system $\dot{x} = u$ is obviously completely controllable, there are no conjugate intervals, so it suffices to find just the conjugate point t_0^* only.

In order to write out the EJ equation, define

$$\Pi = \psi\, u - \frac{1}{2}\,(x^2 + u^2), \qquad \lambda = \frac{1}{2}\,p\,h^2 + s\,x(T)\,h,$$

and so, the collection (ψ, x, u, h) should satisfy the relations

$$\dot{\psi} = -\Pi_x = x, \qquad \psi(T) = -sh,$$
$$\psi - u = 0, \qquad \lambda_h = ph + s\,x(T) = 0,$$

whence $x(t)$ should satisfy the equation

$$\ddot{x} = x, \qquad \text{with} \quad \dot{x}(T) = \frac{s^2}{p}\,x(T).$$

From here we get $x(t) = a\,\cosh(t-T) + b\,\sinh(t-T)$, $b = s^2 a/p$, and so, the nontrivial solution is, up to normalization:

$$x(t) = p\,\cosh(t-T) + s^2\,\sinh(t-T).$$

We have to find the maximal $t<T$ such that $x(t)=0$, i.e.,

$$\tanh(t-T) \;=\; -\frac{p}{s^2}\,. \tag{47}$$

This equation determines the conjugate point t_0^*. Obviously, it has a (unique) solution $t<T$ if and only if $s^2>p$. Moreover, one can easily see that, if $|s| \to \infty$, then $t \to T-0$, which means that, for any fixed interval $[t_0, t_1]$ the Ω has negative values if s is sufficiently large. This result is in a good accordance with the apriori considerations of (45): choosing any process with $\bar{x}_1(T)\,\bar{x}_2(T) \neq 0$ and taking a large enough s, one can easily make Ω to be negative.

Note that the conjugate point does not depend on the sign of s, which can be seen both from (47) and (46): if $s \mapsto -s$, we can also change $h \mapsto -h$, and Ω stays invariant.

In particular case, when $s^2 = p$, the conjugate point is absent, which means that for any $t_0 < T$ the functional

$$\Omega(u,h) = \; s^2\,h^2 + \; 2sh\,x(T) + \int_{t_0}^{T} (x^2 + u^2)\,dt \tag{48}$$

is positive definite. This fact can be also obtained from the inequality

$$\int_{t_0}^{T} (x^2 + u^2)\, dt \; > \; x^2(T)$$

for all nonzero $u(t)$ (then (48) majorates the square of $sh + x(T)$), i.e.,

$$-x^2(T) + \int_{t_0}^{T} (x^2 + u^2)\, dt \; > 0.$$

This inequality follows from the absence of conjugate point of the quadratic form in its left hand side, which is of classical type. (Here the conjugate point should satisfy the equation $\tanh (t - T) \; = \; -1,$ which has no solution.)

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