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Turnpike properties of approximate solutions of autonomous variational problems*

by

Alexander J. Zaslavski

Department of Mathematics, The Technion-Israel Institute of Technology
32000 Haifa, Israel
e-mail: ajzasl@tx.technion.ac.il

Abstract: In this work we study the structure of approximate solutions of autonomous variational problems with vector-valued functions. We are interested in turnpike properties of these solutions, which are independent of the length of the interval, for all sufficiently large intervals. We show that the turnpike properties are stable under small perturbations of integrands.

Keywords: good function, infinite horizon problem, integrand, turnpike property.

1. Introduction

The study of variational and optimal control problems defined on infinite (large) intervals has recently been a rapidly growing area of research. See, for example, Belkina and Rotar (2006), Blot and Cartigny (2000), Blot and Cretezz (2004), Blot and Michel (2003), Glizer and Shinar (1993), Leizarowitz (1985), Mordukhovich (1990), Mordukhovich and Shvartsman (2004), Pickenhain and Lukina (2006), Zaslavski (1996, 1997, 1998, 1999, 2000) and the references therein. These problems arise in engineering (see Anderson and Moore, 1971; Leizarowitz, 1986), in models of economic growth (see Atsumi, 1965; Gale, 1967; Makarov and Rubinov, 1977; McKenzie, 2002; Samuelson, 1965; Weizsacker, 1965; Zaslavski, 2005), in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals (see Aubry and Le Daeron, 1983; Zaslavski, 1987) and in the theory of thermodynamical equilibrium for materials (see Coleman, Marcus and Mizel, 1992; Leizarowitz and Mizel, 1989; Marcus and Zaslavski, 1999, 2002).

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In this paper we analyze the structure of extremals of the variational problems

$$\int_0^T f(z(t), z'(t))dt \to \min,\ z(0) = x,\ z(T) = y, \tag{P}$$
$$z:\ [0, T] \to R^n \text{ is an absolutely continuous (a.c.) function,}$$

where $T > 0$, $x, y \in R^n$ and $f : R^n \times R^n \to R^1$ is an integrand. We are interested in turnpike properties of the extremals which are independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions.

Turnpike properties are well known in mathematical economics (see McKenzie, 2002; Zaslavski, 2005). The term was first coined by Samuelson in 1948 (see Samuelson, 1965) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics. See, for example, Makarov and Rubinov (1977), McKenzie (2002) and the references therein. Many turnpike results can be found in Zaslavski (2005).

Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$. Denote by $\mathcal{A}$ the set of all continuous functions $f :\ R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) for each $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1$ is convex;

A(ii) $f(x, u) \ge \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x, u) \in R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1, u_1) - f(x_2, u_2)| \le \epsilon \max\{f(x_1, u_1), f(x_2, u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1, 2,\ |u_i| \ge \Gamma,\ i = 1, 2, \quad |x_1 - x_2|,\ |u_1 - u_2| \le \delta.$$

It is easy to show that an integrand $f = f(x, u) \in C^1(R^{2n})$ belongs to $\mathcal{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 :\ [0, \infty) \to [0, \infty)$ such that

$$\max\{|\partial f/\partial x(x, u)|,\ |\partial f/\partial u(x, u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $x, u \in R^n$.

For the set $\mathcal{A}$ we consider the uniformity, which is determined by the following base:

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{A}\times\mathcal{A} :\ |f(x,u)-g(x,u)| \le \epsilon \text{ for all } u,x \in R^n \text{ satisfying } |x|,|u| \le N\}$$
$$\cap\{(f,g) \in \mathcal{A}\times\mathcal{A} :\ (|f(x,u)|+1)(|g(x,u)|+1)^{-1} \in [\lambda^{-1},\lambda] \text{ for all } x,u \in R^n \text{ satisfying } |x| \le N\},$$

where $N,\epsilon > 0$ and $\lambda > 1$. It was shown in Zaslavski (1996) that the uniform space $\mathcal{A}$ is metrizable and complete.

We consider functionals of the form

$$I^f(T_1,T_2,x) = \int_{T_1}^{T_2} f(x(t),x'(t))dt \tag{1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$ and $x :\ [T_1,T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y,z \in R^n$ and real numbers T_1,T_2 satisfying $T_1 < T_2$ we set

$$U^f(T_1,T_2,y,z) = \inf\{I^f(T_1,T_2,x) :\ x :\ [T_1,T_2] \to R^n \text{ is an a.c. function satisfying } x(T_1)=y,\ x(T_2)=z\}. \tag{2}$$

It is easy to see that $-\infty < U^f(T_1,T_2,y,z) < \infty$ for each $f \in \mathcal{A}$, each $y,z \in R^n$ and all numbers T_1,T_2 satisfying $-\infty < T_1 < T_2 < \infty$.

Let $f \in \mathcal{A}$. For any a.c. function $x : [0,\infty) \to R^n$ we set

$$J(x) = \liminf_{T\to\infty} T^{-1} I^f(0,T,x). \tag{3}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(x) :\ x : [0,\infty) \to R^n \text{ is an a.c. function}\}. \tag{4}$$

Clearly $-\infty < \mu(f) < \infty$. By a simple modification of the proof of Proposition 4.4 in Leizarowitz and Mizel (1989) (see Zaslavski, 1996, Theorems 8.1, 8.2) we obtain the representation formula

$$U^f(0,T,x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y),\quad x,y \in R^n,\ T \in (0,\infty), \tag{5}$$

where $\pi^f : R^n \to R^1$ is a continuous function and $(T,x,y) \to \theta^f_T(x,y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x,y \in R^n$,

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T] :\ v : [0,\infty) \to R^n \text{ is an a.c. function satisfying } v(0)=x\},\ x \in R^n \tag{6}$$

and for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta^f_T(x,y) = 0$.

An a.c. function $x : [0, \infty) \to R^n$ is called (f)-good if the function $T \to I^f(0, T, x) - \mu(f)T$, $T \in (0, \infty)$ is bounded. In Zaslavski (1996) we showed that for each $f \in \mathcal{A}$ and each $z \in R^n$ there exists an (f)-good function $v : [0, \infty) \to R^n$ satisfying $v(0) = z$.

Propositions 1.1 and 3.2 of Zaslavski (1996) imply the following result.

PROPOSITION 1 *For any a.c. function $x : [0, \infty) \to R^n$ either $I^f(0, T, x) - T\mu(f) \to \infty$ as $T \to \infty$ or*

$$\sup\{|I^f(0, T, x) - T\mu(f)| : \ T \in (0, \infty)\} < \infty.$$

Moreover any (f)-good function $x : [0, \infty) \to R^n$ is bounded.

We denote by $d(x, B) = \inf\{|x - y| : \ y \in B\}$ for $x \in R^n$ and $B \subset R^n$ and by $\mathrm{dist}(A, B)$ the distance in the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$. For every bounded a.c. function $x : [0, \infty) \to R^n$ define

$$\begin{aligned} \Omega(x) = \{y \in R^n : \ & \text{there exists a sequence } \{t_i\}_{i=1}^{\infty} \subset (0, \infty) \\ & \text{for which } t_i \to \infty, \ x(t_i) \to y \text{ as } i \to \infty\}. \end{aligned} \tag{7}$$

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly (ATP), if $\Omega(v_2) = \Omega(v_1)$ for all (f)-good functions $v_i : [0, \infty) \to R^n$, $i = 1, 2$ (see Marcus and Zaslavski, 1999; Zaslavski 1996).

In Zaslavski (1996, Theorem 2.1) we established the following result.

THEOREM 1 *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ possesses (ATP).*

By Proposition 1 for each integrand $f \in \mathcal{A}$ which possesses (ATP) there exists a compact set $H(f) \subset R^n$ such that $\Omega(v) = H(f)$ for each (f)-good function $v : [0, \infty) \to R^n$.

Let $f \in \mathcal{A}$. We say that f has a weak turnpike property, or briefly (WTP), with a turnpike $D \subset R^n$, where D is a nonempty compact subset of R^n, if for each $M, \epsilon > 0$ there exist $\delta > 0$, $L > 0$, $l > 0$ such that the following assertion holds:

For each $T \geq L + l$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)| \leq M, \ |v(T)| \leq M, \ I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + \delta$$

there is a Lebesgue measurable set $\Omega \subset [0, T]$ such that $\mathrm{mes}(\Omega) \leq l$ and for each $\tau \in [0, T - L] \setminus \Omega$ the following inequality holds:

$$\mathrm{dist}(D, \{v(t) : \ t \in [\tau, \tau + L]\}) \leq \epsilon.$$

We showed in Zaslavski (1996, Theorem 2.4) that if $f \in \mathcal{A}$ possesses (ATP), then f has (WTP) with the turnpike $H(f)$. More precisely, we establish the following result (see Zaslavski, 1996, Theorem 2.4).

THEOREM 2 *Assume that $f \in \mathcal{A}$ has (ATP). Let $M_0, M_1, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, numbers $l, S > 0$ and integers $L,_* Q \geq 1$ such that for each $g \in \mathcal{U}$, each pair of numbers $T_1 \in [0, \infty)$, $T_2 \in [T_1 + L + lQ_*, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies*

$$|v(T_i)| \leq M_1, \ i = 1, 2, \quad I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + M_0$$

the following properties hold:

$$|v(t)| \leq S \text{ for all } t \in [T_1, T_2];$$

there exist sequences of numbers $\{b_i\}_{i=1}^{Q}$, $\{c_i\}_{i=1}^{Q} \subset [T_1, T_2]$ such that

$$Q \leq Q_*, \ 0 \leq c_i - b_i \leq l, \ i = 1, \dots, Q,$$
$$dist(H(f), \{v(t) : \ t \in [T, T + L]\}) \leq \epsilon$$
$$\text{for each } T \in [T_1, T_2 - L] \setminus \cup_{i=1}^{Q} [b_i, c_i].$$

COROLLARY 1 *Assume that $f \in \mathcal{A}$ has (ATP). Then f possesses (WTP) with the turnpike $H(f)$.*

The following theorem is our first main result.

THEOREM 3 *Suppose that $f \in \mathcal{A}$ has (WTP) with the turnpike $D \subset R^n$. Then f possesses (ATP) and $H(f) = D$.*

Corollary 1 and Theorem 3 mean that the properties (WTP) are (ATP) are equivalent. In view of Theorems 1 and 2, most integrands of the space $\mathcal{A}$ possess (WTP). It should be mentioned (see Zaslavski, 1999, 2005) that there are integrands in the space $\mathcal{A}$ which possess a turnpike property such that the set Ω is a union of two intervals containing the end points 0 and T, respectively.

More precisely, let $f \in \mathcal{A}$. We say that f has the turnpike property, or briefly (TP), with a turnpike $D \subset R^n$, where D is a nonempty compact subset of R^n, if for each $K, \epsilon > 0$ there exist $l_0 > l > 0$ and $\delta > 0$ such that the following assertion holds:

For each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, \ |v(T)| \leq K, \ I^f(0, T, v) = U^f(0, T, v(0), v(T))$$

the inequality

$$\text{dist}(D, \{v(t) : \ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{8}$$

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), D) \leq \delta$, then (8) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), D) \leq \delta$, then (8) holds for each $\tau \in [l_0, T - l]$.

Let $f \in \mathcal{A}$. We say that the integrand f has the strong turnpike property, or briefly (STP), with a turnpike $D \subset R^n$, where D is a nonempty compact subset of R^n, if for each $\epsilon, K > 0$ there exist real numbers $\delta > 0$ and $l_0 > l > 0$ such that the following assertion holds:

For each $T \geq 2l_0$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq K, \; I^f(0,T,v) \leq U^f(0,T,v(0),v(T)) + \delta$$

the inequality

$$\mathrm{dist}(D, \{v(t) : \; t \in [\tau, \tau + l]\}) \leq \epsilon \tag{9}$$

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), D) \leq \delta$, then (9) holds for all $\tau \in [0, T - l_0]$ and if $d(v(T), D) \leq \delta$, then (9) holds for each $\tau \in [l_0, T - l]$.

Note that the property (TP) deals only with exact minimizers of the variational problems while the property (STP) also describes the structure of approximate solutions.

Assume that $f \in \mathcal{A}$ possesses (STP) with the turnpike $D \subset R^n$. Then f possesses (TP) with the turnpike D, possesses (WTP) with the turnpike D and by Theorem 3, f possesses (ATP) with $H(f) = D$.

We can show that if $f \in \mathcal{A}$ has (ATP) and $H(f)$ is a singleton, then f possesses (STP) with the turnpike $H(f)$. These properties hold, for example, if $f \in \mathcal{A}$ is strictly convex (see Zaslavski, 2007). In Zaslavski (1999) we considered an important subset $\mathcal{M} \subset \mathcal{A}$ and showed that if $f \in \mathcal{M}$ possesses (ATP), then f possesses (STP) with the turnpike $H(f)$.

In this paper we will establish the following two results. The first of them shows that the property (STP) is stable under small perturbations of integrands. The second result implies that the properties (TP) and (STP) are equivalent.

THEOREM 4 *Suppose that $f \in \mathcal{A}$ has (STP) with a turnpike $H(f) \subset R^n$. Let $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, $l_1 > l > 0$ and $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_1$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq K, \; I^g(0,T,v) \leq U^g(0,T,v(0),v(T)) + \delta$$

the inequality

$$dist(H(f), \{v(t) : \; t \in [\tau, \tau + l]\}) \leq \epsilon \tag{10}$$

holds for each $\tau \in [l_1, T - l_1]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (10) holds for each $\tau \in [0, T - l_1]$ and if $d(v(T), H(f)) \leq \delta$, then (10) holds for each $\tau \in [l_1, T - l]$.

THEOREM 5 *Let $f \in \mathcal{A}$ have the property (TP) with the turnpike $D \subset R^n$. Then f possesses (ATP), $H(f) = D$ and f has (STP) with the turnpike $H(f)$.*

Combining Theorems 4 and 5 we obtain the following result.

THEOREM 6 *Suppose that $f \in \mathcal{A}$ has the property (TP) with a turnpike $D \subset R^n$. Then f possesses (ATP), $H(f) = D$ and for each $\epsilon, K > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$, $l_1 > l > 0$ and $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_1$ and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq K, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta$$

the inequality (10) holds for each $\tau \in [l_1, T - l_1]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (10) holds for each $\tau \in [0, T - l_1]$ and if $d(v(T), H(f)) \leq \delta$, then (10) holds for each $\tau \in [l_1, T - l]$.

Note that all our results can be applied for an integrand $f \in \mathcal{A}$ which is strictly convex. See for details Zaslavski (2007). They can also be applied for most elements (in the sense of Baire category) of certain spaces of integrands studied in Zaslavski (1999).

2. Auxiliary results

In this paper we need the following results obtained in Zaslavski (1996) and in Zaslavski (1998).

PROPOSITION 2 *(Zaslavski, 1996, Proposition 5.1). Let $g \in \mathcal{A}$, $y : [0, \infty) \to R^n$ be a (g)-good function and let $\epsilon > 0$. Then there exists $T_0 > 0$ such that for each $T \geq T_0$ and each $\bar{T} > T$*

$$I^g(T, \bar{T}, y) \leq U^g(T, \bar{T}, y(T), y(\bar{T})) + \epsilon.$$

PROPOSITION 3 *(Zaslavski, 1996, Theorem 6.1). Let $f \in \mathcal{A}$. Then the function*

$$(T_1, T_2, x, y) \to U^f(T_1, T_2, x, y)$$

is continuous for $T_1 \in [0, \infty)$, $T_2 \in (T_1, \infty)$, $x, y \in R^n$.

PROPOSITION 4 *(Zaslavski, 1998, Proposition 2.4). Let $M_1, \epsilon > 0$ and let $0 < \tau_0 < \tau_1$. Then there exists a positive number δ such that for each $f \in \mathcal{A}$ and each pair of numbers $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [\tau_0, \tau_1]$ the following property holds:*

If an a.c. function $x : [T_1, T_2] \to R^n$ satisfies $I^f(T_1, T_2, x) \leq M_1$ and if $t_1, t_2 \in [T_1, T_2]$ satisfies $|t_1 - t_2| \leq \delta$, then $|x(t_1) - x(t_2)| \leq \epsilon$.

PROPOSITION 5 *(Zaslavski, 1998, Theorem 1.3). Let $f \in \mathcal{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, each $T_2 \in [T_1 + c, \infty)$ and each a.c. function $v : [T_1, T_2] \to R^n$ satisfying*

$$|v(T_i)| \leq M_1, \ i = 1, 2, \ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + M_2$$

the following inequality holds:

$$|v(t)| \leq S, \ t \in [T_1, T_2].$$

PROPOSITION 6 *(Zaslavski, 1996, Lemma 10.2). Let $f \in \mathcal{A}$ possess (ATP), $\epsilon_0 \in (0,1)$, $K_0 > 0$, $M_0 > 0$ and let l be a positive integer such that for each (f)-good function $x : [0,\infty) \to R^n$ the inequality*

$$dist(H(f), \{x(t) : \ t \in [T, T+l]\}) \le 8^{-1}\epsilon_0$$

holds for all large T (the existence of l follows from Theorem 5.1 of Zaslavski, 1996). Then there exists an integer $N \ge 10$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$, each $S \in [0,\infty)$ and each a.c. function $x : [S, S+Nl] \to R^n$ satisfying

$$|x(S)|, |x(S+Nl)| \le K_0, \ I^g(S, S+Nl, x)$$
$$\le U^g(S, S+Nl, x(S), x(S+Nl)) + M_0$$

there exists an integer $i_0 \in [0, N-8]$ such that

$$dist(H(f), \{x(t) : \ t \in [T, T+l]\}) \le \epsilon_0$$

for all $T \in [S + i_0 l, S + (i_0+7)l]$.

PROPOSITION 7 *(Zaslavski, 1998, Proposition 2.8). Let $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$ and let $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a.c. function $x : [T_1, T_2] \to R^n$ satisfying*

$$\min\{I^g(T_1, T_2, x), \ I^f(T_1, T_2, x)\} \le D$$

the inequality $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \le \epsilon$ holds.

PROPOSITION 8 *(Zaslavski, 1998, Proposition 2.9). Let $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$ and let $c_3, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $z, y \in R^n$ satisfying $|y|, |z| \le c_3$ the inequality*

$$|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \le \epsilon$$

holds.

PROPOSITION 9 *(Zaslavski, 1998, Theorem 1.2). For each $f \in \mathcal{A}$ there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and a number $M > 0$ such that for each $g \in \mathcal{U}$ and each (g)-good function $x : [0,\infty) \to R^n$ the relation $\limsup_{t\to\infty} |x(t)| < M$ holds.*

PROPOSITION 10 *(Zaslavski, 1998, Proposition 2.5). Assume that $f \in \mathcal{A}$, $M_1 > 0$, $0 \le T_1 < T_2$ and that $x_i : [T_1, T_2] \to R^n$, $i = 1, 2, \ldots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \le M_1$, $i = 1, 2 \ldots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $x : [T_1, T_2] \to R^n$ such that*

$$I^f(T_1, T_2, x) \le M_1,$$

$x_{i_k}(t) \to x(t)$ as $k \to \infty$ uniformly on $[T_1, T_2]$ and $x'_{i_k} \to x'$ as $k \to \infty$ weakly in $L^1(R^n; (T_1, T_2))$.

Let $f \in \mathcal{A}$. For each pair of real numbers $T_2 > T_1$ and each a.c. function $x : [T_1, T_2] \to R^n$ set

$$\sigma^f(T_1, T_2, x) = I^f(T_1, T_2, x) - (T_2 - T_1)\mu(f) - \pi^f(x(T_1)) + \pi^f(x(T_2)). \quad (11)$$

By (11), (2) and (5),

$$\sigma^f(T_1, T_2, v) \geq 0$$
$$\text{for each } T_1 \in R^1, \text{ each } T_2 > T_1 \text{ and each a.c. function } v : [T_1, T_2] \to R^n. \quad (12)$$

3. Proof of Theorem 3

Let $v : [0, \infty) \to R^n$ be an (f)-good function. We show that $\Omega(v) = D$. By Proposition 9 the function v is bounded. Thus, there is

$$M > \sup\{|v(t)| : t \in [0, \infty)\}. \quad (13)$$

First we show that $D \subset \Omega(v)$. Let

$$\epsilon > 0, \ z \in D \quad (14)$$

and let $L, l, \delta > 0$ be as guaranteed by the property (WTP). By Proposition 2 there is $\tau_0 > 0$ such that

$$I^f(S_1, S_2, v) \leq U^f(S_1, S_2, v(S_1), v(S_2)) + \delta$$

for each pair of numbers S_1, S_2 satisfying $S_2 > S_1 \geq \tau_0$. Let $T \geq \tau_0$. Then by the choice of τ_0

$$I^f(T, T + 2(L + l), v) \leq U^f(T, T + 2(L + l), v(T), v(T + 2(L + l))) + \delta.$$

It follows from this inequality, (13), the choice of L, l, δ and the property (WTP) that there is $t \in [T, T + 2(l + L)]$ such that $d(v(t), z) \leq \epsilon$. This implies that $d(z, \Omega(v)) \leq \epsilon$. Since ϵ is any positive number and z is an arbitrary element of D we conclude that $D \subset \Omega(v)$.

Now we show that $\Omega(v) \subset D$. Let us assume the contrary. Then there is

$$z \in \Omega(v) \setminus D. \quad (15)$$

There is $\epsilon > 0$ such that

$$d(z, D) \geq 4\epsilon \quad (16)$$

and there is a sequence $\{t_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that

$$t_{i+1} - t_i \geq 16, \ i = 1, 2, \ldots, \ \lim_{i \to \infty} v(t_i) = z. \quad (17)$$

We may assume without loss of generality that

$$d(v(t_i), D) \geq 3\epsilon, \ i = 1, 2, \dots \tag{18}$$

In view of Proposition 2 there is $\tau_0 > 0$ such that

$$I^f(S_1, S_2, v) \leq U^f(S_1, S_2, v(S_1), v(S_2)) + 1 \tag{19}$$

for each pair of numbers S_1, S_2 satisfying $S_2 > S_1 \geq \tau_0$. We may assume without loss of generality that $t_1 > \tau_0$. Relations (13), (19) and Proposition 3 imply that there is a number $M_1 > M$ such that

$$I^f(s, s+2, v) \leq M_1 \text{ for each } s \geq \tau_0. \tag{20}$$

It follows from (20) and Proposition 4 that there is $\gamma \in (0, 1/2)$ such that for each $s \geq \tau_0$, each $\bar{t}_1, \bar{t}_2 \in [s, s+1]$ which satisfy $|\bar{t}_1 - \bar{t}_2| \leq \gamma$ the inequality $|v(\bar{t}_1) - v(\bar{t}_2)| \leq \epsilon$ holds. By the choice of γ and (18), for each integer $i \geq 1$ and each $t \in [t_i, t_i + \gamma]$,

$$d(v(t), D) \geq 2\epsilon. \tag{21}$$

Let $L, l, \delta > 0$ be as guaranteed by the property (WTP). In view of Proposition 2 there is $\tau_1 > \tau_0$ such that

$$I^f(s_1, s_2, v) \leq U^f(s_1, s_2, v(s_1), v(s_2)) + \delta$$

for each pair of numbers s_1, s_2 satisfying $s_2 > s_1 \geq \tau_1$.

Fix an integer $i_0 \geq 1$ such that $t_{i_0} > \tau_1$. Let $q \geq 2$ be a natural number. By the choice of τ_1,

$$I^f(t_{i_0}, t_{i_0+q} + l + L, v) \leq U^f(t_{i_0}, t_{i_0+q} + l + L, v(t_{i_0}), v(t_{i_0+q} + l + L)) + \delta.$$

It follows from this inequality, (13), the choice of L, δ, l and the property (WTP) that there is a Lebesgue measurable set $\Omega \subset [t_{i_0}, t_{i_0+q} + l + L]$ such that $\mathrm{mes}(\Omega) \leq l$ and for each $\tau \in [0, T-L] \setminus \Omega$ we have

$$\mathrm{dist}(D, \{v(t) : \ t \in [\tau, \tau + L]\}) \leq \epsilon.$$

Combined with (21) this implies that

$$\cup_{i=i_0}^{i_0+q-1} [t_i, t_i + \gamma] \subset \Omega$$

and

$$l \geq \mathrm{mes}(\Omega) \geq \mathrm{mes}(\cup_{i=i_0}^{i_0+q-1} [t_i, t_{i+1} + \gamma]) = \gamma q \text{ and } q \leq l\gamma^{-1}.$$

Since q is any natural number satisfying $q \geq 2$ we have reached a contradiction. Therefore $\Omega(v) \subset D$. Theorem 3 is proved.

4. Proof of Theorem 4

It was mentioned in Introduction that f possesses (ATP).

By Proposition 5 there exist a neighborhood $\mathcal{U}_1$ of f in $\mathcal{A}$ and a number $M > K$ such that for each $g \in \mathcal{U}_1$, each $T_1 \geq 0$, each $T_2 \geq T_1 + 1$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_i)| \leq 2K + 4,\ i = 1, 2,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + 4 \tag{22}$$

the following inequality holds:

$$|v(t)| \leq M,\ t \in [T_1, T_2]. \tag{23}$$

Since f has (STP) there exist $\delta \in (0, 1)$, $l_0 > 1$, $l > 0$ such that $l_0 > l$ and the following property holds:

(P1) For each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq M,\ I^f(0, t, v) \leq U^f(0, T, v(0), v(T)) + 4\delta \tag{24}$$

the inequality

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{25}$$

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (25) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta$, then (25) holds for each $\tau \in [l_0, T - l]$.

By Proposition 6 there exist a positive number N and a neighborhood $\mathcal{U}_2$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}_2$, each $s \in [0, \infty)$ and each a.c. function $v : [s, s + N] \to R^n$ satisfying

$$\begin{aligned} &|v(s)|,\ |v(s + N)| \leq 2M + 2, \\ &I^g(s, s + N, v) \leq U^g(s, s + N, v(s), v(s + N)) + 8 \end{aligned} \tag{26}$$

there is $\tau \in [s, s + N]$ such that

$$d(v(\tau), H(f)) \leq \delta. \tag{27}$$

Set

$$l_1 = 16(l_0 + 2 + N). \tag{28}$$

By Proposition 3 there is $M_0 > M$ such that for each $\tau \in [1, 4l_1]$, each $x, y \in R^n$ satisfying $|x|, |y| \leq 2M + 4$

$$|U^f(0, \tau, x, y)| \leq M_0. \tag{29}$$

By Proposition 8 there exists a neighborhood $\mathcal{U}_3$ of f in $\mathcal{A}$ such that for each $\tau \in [1, 4l_1]$, each $g \in \mathcal{U}_3$ and each $x, y \in R^n$ satisfying $|x|, |y| \le 2M + 4$

$$|U^f(0, \tau, x, y) - U^g(0, \tau, x, y)| \le \delta/2. \tag{30}$$

By Proposition 7 there exists a neighborhood $\mathcal{U}_4$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}_4$, each $\tau \in [1, 4l_1]$ and each a.c. function $v : [0, \tau] \to R^n$ which satisfies

$$\min\{I^f(0, \tau, v),\ I^g(0, \tau, v)\} \le 2M_0 + 4 \tag{31}$$

the inequality

$$|I^f(0, \tau, v) - I^g(0, \tau, v)| \le \delta/4 \tag{32}$$

holds. Set

$$\mathcal{U} = \cap_{i=1}^4 \mathcal{U}_i. \tag{33}$$

Assume that $g \in \mathcal{U}$, $T \ge 2l_1$ and a.c. function $v : [0, T] \to R^n$ satisfies

$$|v(0)|, |v(T)| \le K,\ I^g(0, T, v) \le U^g(0, T, v(0), v(T)) + \delta. \tag{34}$$

In view of (33), (34) and the definition of $\mathcal{U}_1$ (see (22), (23))

$$|v(t)| \le M,\ t \in [0, T]. \tag{35}$$

Assume that $S_1, S_2 \in [0, T]$ satisfy

$$2l_0 \le S_2 - S_1 \le 4l_1,\ d(v(S_i), H(f)) \le \delta,\ i = 1, 2. \tag{36}$$

By (35), (36) and the choice of M_0 (see (29)),

$$|U^f(S_1, S_2, v(S_1), v(S_2))| \le M_0. \tag{37}$$

It follows from (33), the choice of $\mathcal{U}_3$ (see (30)), (35) and (36) that

$$|U^f(S_1, S_2, v(S_1), v(S_2)) - U^g(S_1, S_2, v(S_1), v(S_2))| \le \delta/2. \tag{38}$$

Combined with (34) and (37) this inequality implies that

$$\begin{aligned} &I^g(S_1, S_2, v) \le U^g(S_1, S_2, v(S_1), v(S_2)) + \delta \\ &\le U^f(S_1, S_2, v(S_1), v(S_2)) + (3/2)\delta \le M_0 + (3/2)\delta \le M_0 + 2. \end{aligned} \tag{39}$$

Together with (36), (33) and the choice of $\mathcal{U}_4$ this inequality implies that

$$|I^g(S_1, S_2, v) - I^f(S_1, S_2, v)| \le \delta/4. \tag{40}$$

Relations (34), (38) and (40) imply that

$$I^f(S_1, S_2, v) \le I^g(S_1, S_2, v) + \delta/4 \le U^g(S_1, S_2, v(S_1), v(S_2)) + (5/4)\delta$$
$$\le U^f(S_1, S_2, v(S_1), v(S_2)) + 2\delta. \tag{41}$$

It follows from (35), (36), (41) and the choice of δ, l_0, l (see (24), (25)) that for each $\tau \in [S_1, S_2 - l]$ the inequality (25) holds.

We showed that the following property holds:

(P2) For each $S_1, S_2 \in [0, T]$ satisfying (36) and each $\tau \in [S_1, S_2 - l]$ the inequality (25) holds.

Assume that

$$\bar{S}_1, \bar{S}_2 \in [0, T],$$
$$\bar{S}_2 \ge \bar{S}_1 + 2l_0,\ d(v(\bar{S}_i), H(f)) \le \delta,\ i = 1, 2. \tag{42}$$

We show that for each $\tau \in [\bar{S}_1, \bar{S}_2 - l]$ the inequality (25) holds. If $S_2 - S_1 \le 4l_1$, then our property follows from (P2). Therefore we may assume that

$$\bar{S}_2 - \bar{S}_1 > 4l_1. \tag{43}$$

Let

$$\tau \in [\bar{S}_1, \bar{S}_2 - l]. \tag{44}$$

Define $S_1, S_2 \in [0, T]$ as follows. If $\tau - \bar{S}_1 \le l_1$, then set

$$S_1 = \bar{S}_1. \tag{45}$$

If $\tau - \bar{S}_1 > l_1$, then it follows from (33), the choice of $\mathcal{U}_2$ (see (26)-(28)), (34) and (35) that there is a number S_1 such that

$$S_1 \in [\tau - l_1, N + \tau - l_1],\ d(v(S_1), H(f)) \le \delta. \tag{46}$$

If $\bar{S}_2 - \tau \le l_1$, then set

$$S_2 = \bar{S}_2. \tag{47}$$

If $\bar{S}_2 - \tau > l_1$, then it follows from (33), the choice of $\mathcal{U}_2$ (see (26), (29)), (35) and (22) that there is a number S_2 such that

$$S_2 \in [\tau + l_1 - N, \tau + l_1],\ d(v(S_2), H(f)) \le \delta. \tag{48}$$

In view of (42) and the choice of S_1, S_2 (see (45)-(48)),

$$d(v(S_i), H(f)) \le \delta,\ i = 1, 2, \tag{49}$$
$$S_2 - S_1 \in [2l_0, 4l_1],\ \tau \in [S_1, S_2 - l]. \tag{50}$$

Combined with the property (P2) these relations imply that the inequality (25) holds.

Thus we showed that the following property holds:

(P3) For each $\bar{S}_1, \bar{S}_2 \in [0,T]$ satisfying (23) and each $\tau \in [\bar{S}_1, \bar{S}_2 - l]$ the inequality (25) holds.

Now define $\bar{S}_1, \bar{S}_2 \in [0,T]$ as follows. If $d(v(S_1), H(f)) \leq \delta$, then $\bar{S}_1 = 0$. Otherwise by the choice of $\mathcal{U}_2$ (see (26), (27)), (33), (34) and (35) there is $\bar{S}_1 \in [0,N]$ such that $d(v(\bar{S}_1), H(f)) \leq \delta$. If $d(v(T_2), H(f)) \leq \delta$, then $\bar{S}_2 = T$. Otherwise by the choice of $\mathcal{U}_2$ (see (26), (27)), (35) and (34) there is $\bar{S}_2 \in [T-N, T]$ such that

$$d(v(\bar{S}_2), H(f)) \leq \delta.$$

By the property (P3) for each $\tau \in [\bar{S}_1, \bar{S}_2 - l]$ the inequality (25) holds. Theorem 4 is proved.

5. An auxiliary result for Theorem 5

PROPOSITION 11 *Let $f \in \mathcal{A}$ have (TP) with the turnpike $D \subset R^n$. Then f has (ATP) and $H(f) = D$.*

Proof. Let $v : [0,\infty) \to R^n$ be (f)-good function. We show that $\Omega(v) = D$. Let $z \in \Omega(v)$. There exists a sequence $\{t_i\}_{i=1}^{\infty} \subset [0,\infty)$ such that

$$t_{i+1} \geq t_i + 10, \ i = 1, 2, \ldots, \ \lim_{i\to\infty} v(t_i) = z. \tag{51}$$

For each integer $i \geq 1$ we define $u_i : [-t_i, \infty) \to R^n$ by

$$u_i(t) = v(t + t_i), \ t \in [-t_i, \infty). \tag{52}$$

By Proposition 9 there is $M > 0$ such that

$$|v(t)| \leq M \text{ for all } t \in [0,\infty). \tag{53}$$

By Proposition 2 there is $\tau_0 > 0$ such that for each pair of real numbers S_1, S_2 satisfying $S_2 > S_1 \geq \tau_0$

$$I^f(S_1, S_2, v) \leq U^f(S_1, S_2, v(S_1), v(S_2)) + 1.$$

Combined with (51)-(53) and Proposition 3 this implies that for each natural number q the sequence $\{I^f(-q, q, u_i)\}$, where $i \geq 1$ is an integer such that $t_i \geq q$, is bounded. Together with Proposition 10 this implies that there exist a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $u : R^1 \to R^n$ such that for each natural number q

$$u_{i_k}(t) \to u(t) \text{ as } k \to \infty \text{ uniformly in } [-q, q], \tag{54}$$

$$I^f(-q, q, u) \leq \liminf_{k\to\infty} I^f(-q, q, u_{i_k}). \tag{55}$$

In view of (54), (53) and (52)

$$|u(t)| \leq M \text{ for all } t \in R^1. \tag{56}$$

Relations (51), (52) and (54) imply that

$$u(0) = z.$$

It follows from (55), (52) and Proposition 3 that for each integer $q \geq 1$,

$$I^f(-q, q, u) \leq \liminf_{k\to\infty} I^f(-q, q, u_{i_k}) = \liminf_{k\to\infty} I^f(-q + t_{i_k}, q + t_{i_k}, v)$$
$$= \liminf_{k\to\infty} U^f(0, 2q, v(-q + t_{i_k}), v(q + t_{i_k}))$$
$$= \liminf_{k\to\infty} U^f(0, 2q, u_{i_k}(-q), u_{i_k}(q)) = U^f(0, 2q, u(-q), u(q)).$$

This implies that

$$I^f(-q, q, u) = U^f(2q, u(-q), u(q))$$

for each integer $q \geq 1$. Together with (56) and (TP) this implies that $z \in \{u(t) : t \in R^1\} \subset D$. Since z is any element of $\Omega(v)$ we conclude that $\Omega(v) \subset D$.

We show that $D \subset \Omega(v)$. Let us assume the contrary. Then there is

$$z \in D \setminus \Omega(v). \tag{57}$$

Hence, there exist $\epsilon, t_0 > 0$ such that

$$|v(t) - z| \geq 3\epsilon \text{ for all } t \geq t_0. \tag{58}$$

Choose a natural number $i_0 > t_0$. For each integer $i \geq i_0$ define $u_i : [-i, \infty) \to R^n$ by

$$u_i(t) = v(i + t),\ t \in [-i, \infty). \tag{59}$$

By Proposition 9 there is $M > 0$ such that

$$|v(t)| \leq M \text{ for all } t \in [0, \infty). \tag{60}$$

In view of Proposition 2 there is $\tau_0 > 0$ such that for each pair of numbers S_1, S_2 satisfying $S_2 > S_1 \geq \tau_0$ we have

$$I^f(S_1, S_2, v) \leq U^f(S_1, S_2, v(S_1), v(S_2)) + 1.$$

Combined with (59), (60) and Proposition 3 this implies that for each natural number q the sequence $\{I^f(-q, q, u_i)\}$, where i is a natural number such that $i \geq q$, i_0, is bounded. Together with Proposition 10 this implies that there exist a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ and an a.c. function $u : R^1 \to R^n$ such that for each natural number q the relations (54) and (55) are true.

It follows from (54), (55), (59) and Propositions 2 and 3 that for each natural number q

$$\begin{aligned}
&I^f(-q,q,u) \le \liminf_{k\to\infty} I^f(-q,q,u_{i_k}) = \liminf_{k\to\infty} I^f(-q+i_k,q+i_k,v)\\
&= \liminf_{k\to\infty} U^f(0,2q,v(i_k-q),v(i_k+q))\\
&= \liminf_{k\to\infty} U^f(0,2q,u_{i_k}(-q),u_{i_k}(q)) = U^f(0,2q,u(-q),u(q)).
\end{aligned}$$

Thus

$$I^f(-q,q,u) = U^f(0,2q,u(-q),u(q)) \text{ for all natural numbers } q. \tag{61}$$

By (54), (59) and (60),

$$|u(t)| \le M \text{ for all } t \in R^1, \tag{62}$$
$$|u(t) - z| \ge 3\epsilon \text{ for all } t \in R^1. \tag{63}$$

Relations (61), (62), (57) and (TP) imply that there is $\tau \in R^1$ such that

$$|z - u(\tau)| \le \epsilon.$$

This contradicts (63). The contradiction we have reached shows that $D \subset \Omega(v)$. Proposition 11 is proved.

6. Proof of Theorem 5

LEMMA 1 *Let $\epsilon, M, S > 0$. Then there exists $\delta > 0$ such that for each a.c. function $v : [0,S] \to R^n$ satisfying*

$$|v(0)|,\ |v(S)| \le M,\ I^f(0,S,v) \le U^f(0,S,v(0),v(S)) + \delta$$

there is an a.c. function $u : [0,S] \to R^n$ such that

$$\begin{aligned}
&I^f(0,T,u) = U^f(0,S,v(0),v(S)),\\
&|v(t) - u(t)| \le \epsilon,\ t \in [0,S].
\end{aligned}$$

Proof. Let us assume the contrary. Then for each natural number m there exists an a.c. function $u_m : [0,S] \to R^n$ such that

$$|u_m(0)|,\ |u_m(S)| \le M, \tag{64}$$
$$I^f(0,S,u_m) \le U^f(0,S,u_m(0),u_m(S)) + 1/m \tag{65}$$

and

$$\sup\{|u_m(t) - u(t)| :\ t \in [0,S]\} > \epsilon \tag{66}$$

for each a.c. function $u : [0, S] \to R^n$ such that

$$I^f(0, S, u) = U^f(0, S, u(0), u(S)). \tag{67}$$

By (64), (65) and Proposition 3 the sequence $\{I^f(0, S, u_m)\}_{m=1}^{\infty}$ is bounded from above. It follows from Proposition 10 that there exists a subsequence $\{u_{m_i}\}_{i=1}^{\infty}$ and an a.c. function $u : [0, S] \to R^n$ such that

$$u_{m_i}(t) \to u(t) \text{ as } i \to \infty \text{ uniformly in } [0, S], \tag{68}$$

$$I^f(0, S, u) \le \liminf_{i \to \infty} I^f(0, S, u_{m_i}). \tag{69}$$

It follows from (68), (65), (69) and Proposition 3 that

$$I^f(0, S, u) \le \liminf_{i \to \infty} I^f(0, S, u_{m_i}) = \liminf_{i \to \infty} U^f(0, S, u_{m_i}(0), u_{m_i}(S))$$
$$= U^f(0, S, u(0), u(S)).$$

Thus

$$I^f(0, S, u) = U^f(0, S, u(0), u(S)).$$

By (68) there is a natural a number p such that

$$|u_p(t) - u(t)| \le \epsilon/2, \ t \in [0, S],$$

a contradiction (see (66)). The contradiction we have reached proves Lemma 1.

Proof of Theorem 5. By Proposition 11 f has (ATP) and $H(f) = D$. We may assume that

$$\epsilon < 1, \ K > \sup\{|z| : \ z \in H(f)\} + 2. \tag{70}$$

By Proposition 5 there exists $K_0 > K + 1$ such that for each $T \ge 1$ and each a.c. function $u : [0, T] \to R^n$ satisfying

$$|u(0)|, |u(T)| \le K + 1, \ I^f(0, T, u) \le U^f(0, T, u(0), u(T)) + 4 \tag{71}$$

the following inequality holds:

$$|v(t)| \le K_0, \ t \in [0, T]. \tag{72}$$

Since f has (TP) with the turnpike $D = H(f)$, there exist real numbers l_0, l, δ_0 such that

$$l_0 > l > 0, \ \delta_0 \in (0, 1), \ \delta_0 < \epsilon \tag{73}$$

and the following assertion holds:

(C1) For each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|,\ |v(T)| \leq K_0 + 2,\ I^f(0, T, v) = U^f(0, T, v(0), v(T)) \tag{74}$$

the inequality

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \leq \epsilon/8 \tag{75}$$

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta_0$, then (75) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta_0$, then (75) holds for each $\tau \in [l_0, T - l]$.

Since f has (ATP) Proposition 6 implies that there is a number $N > 0$ such that for each a.c. function $v : [0, N] \to R^n$ satisfying

$$|v(0)|, |v(N)| \leq K_0 + 1,\ I^f(0, N, v) \leq U^f(0, N, v(0), v(N)) + 8$$

we have

$$\inf\{d(v(t), H(f)) :\ t \in [0, N]\} \leq \delta_0/32. \tag{76}$$

Choose a number l_1 such that

$$l_1 \geq 8(l_0 + N + 2). \tag{77}$$

By Lemma 1 there is $\delta_1 > 0$ such that for each a.c. function $v : [0, l_1] \to R^n$ satisfying

$$|v(0)|, |v(l_1)| \leq K_0 + 1,\ I^f(0, l_1, v) \leq U^f(0, l_1, v(0), v(l_1)) + \delta_1 \tag{78}$$

there is an a.c. function $u : [0, l_1] \to R^n$ such that

$$I^f(0, l_1, u) = U^f(0, l_1, u(0), u(l_1)), \tag{79}$$

$$|u(t) - v(t)| \leq \delta_0/32,\ t \in [0, l_1] \tag{80}$$

Put

$$\delta = \min\{\delta_0, \delta_1\}/64. \tag{81}$$

Assume that $T \geq 2l_1$ and an a.c. function $v : [0, T] \to R^n$ satisfies

$$|v(0)|,\ |v(T)| \leq K,\ I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + \delta. \tag{82}$$

It follows from (82) and the choice of K_0 that (72) is true.

Assume now that $S_1, S_2 \in [0, T]$ satisfy

$$S_2 - S_1 \in [2l_0, l_1],\ d(v(S_i), H(f)) \leq \delta_0,\ i = 1, 2. \tag{83}$$

We show that for each $\tau \in [S_1, S_2 - l]$

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \le \epsilon. \tag{84}$$

Clearly, there is $\tilde{S}_1 \in [0, T]$ such that

$$[S_1, S_2] \subset [\tilde{S}_1, \tilde{S}_1 + l_1] \subset [0, T]. \tag{85}$$

By (82),

$$I^f(\tilde{S}_1, \tilde{S}_1 + l_1, v) \le U^f(\tilde{S}_1, \tilde{S}_1 + l_1, v(\tilde{S}_1), v(\tilde{S}_1 + l_2)) + \delta.$$

It follows from this inequality, (85), (72), (81) and the choice of δ_1 (see (78)-(80)) that there is an a.c. function $u : [\tilde{S}_1, \tilde{S}_1 + l_1] \to R^n$ such that

$$I^f(\tilde{S}_1, \tilde{S}_1 + l_1, u) = U^f(\tilde{S}_1, \tilde{S}_1 + l_1, u(\tilde{S}_1), u(\tilde{S}_1 + l_1)), \tag{86}$$

$$|u(t) - v(t)| \le \delta_0/32,\ t \in [\tilde{S}_1, \tilde{S}_1 + l_1]. \tag{87}$$

In view of (87), (73) and (72),

$$|u(t)| \le K_0 + 1,\ t \in [\tilde{S}_1, \tilde{S}_1 + l_1]. \tag{88}$$

Relations (85) and (86) imply that

$$I^f(S_1, S_2, u) = U^f(S_1, S_2, u(S_1), u(S_2)). \tag{89}$$

By (89), (88), the assertion (C1) and (83) for each $\tau \in [S_1, S_2 - l]$ the following inequality is true:

$$\mathrm{dist}(H(f), \{u(t) :\ t \in [\tau, \tau + l]\}) \le \epsilon/8. \tag{90}$$

Let $\tau \in [S_1, S_2 - l]$. Then (90) is true. Together with (87) and (73) the inequality (90) implies that

$$\mathrm{dist}(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \le \epsilon/8 + \delta_0/32 \le \epsilon/4. \tag{91}$$

Thus we have shown that the following assertion is true:

(C2) For each $S_1, S_2 \in [0, T]$ satisfying (83) and each $\tau \in [S_1, S_2 - l]$ the inequality (91) is true.

Assume that $S_1, S_2 \in [0, T]$ satisfy

$$S_2 - S_1 \ge 2l_0,\ d(v(S_i), H(f)) \le \delta_0,\ i = 1, 2. \tag{92}$$

We show that for each $\tau \in [S_1, S_2 - l]$ (91) is true. Note that if $S_2 - S_1 \le l_1$, then this is true in view of (C2). Therefore we may may assume that

$$S_2 - S_1 > l_1. \tag{93}$$

Let $\tau \in [S_1, S_2 - l]$. Define $\tilde{S}_1, \tilde{S}_2 \in [0, T]$ as follows:

If $\tau \leq 2l_0 + N + S_1$, then we set $\tilde{S}_1 = S_1$. Otherwise, by the choice of N (see (76)), (72), (73) and (82) there is $\tilde{S}_1 \in [\tau - 2l_0 - N, \tau - 2l_0]$ such that

$$d(v(\tilde{S}_1), H(f)) \leq \delta_0/32. \tag{94}$$

If $\tau \geq S_2 - 2l_0 - N$, then we set $\tilde{S}_2 = S_2$. Otherwise, by the choice of N (see (76)), (82) and (72) there is $\tilde{S}_2 \in [\tau + 2l_0, \tau + 2l_0 + N]$ such that

$$d(v(\tilde{S}_2), H(f)) \leq \delta_0/32. \tag{95}$$

It follows from the choice of $\tilde{S}_1, \tilde{S}_2$, (93), (77), (92), (94) and (95) that

$$l_1 \geq \tilde{S}_2 - \tilde{S}_1 \geq 2l_0,$$
$$d(v(\tilde{S}_i), H(f)) \leq \delta_0, \ i = 1, 2, \ \tau \in [\tilde{S}_1, \tilde{S}_2 - l].$$

It follows from these inequalities and (C2) (applied for $\tilde{S}_1, \ \tilde{S}_2$) that (91) is true. Thus the following property holds:

(C3) For each $S_1, S_2 \in [0, T]$ satisfying (92) and each $\tau \in [S_1, S_2 - l]$ the inequality (91) is true.

Define real numbers τ_1, τ_2 as follows. If $d(v(0), H(f)) \leq \delta$, then $\tau_1 = 0$. Otherwise by the choice of N (see (76)), (72) and (82) there is $\tau_1 \in [0, N]$ such that

$$d(v(\tau_1), H(f)) \leq \delta_0.$$

If $d(v(T), H(f)) \leq \delta$, then set $\tau_2 = T$. Otherwise by the choice of N (see (76)), (72) and (82) there is $\tau_2 \in [T - N, T]$ such that $d(v(\tau_2), H(f)) \leq \delta_0$. It follows from (C3) that for each $\tau \in [\tau_1, \tau_2 - l]$ the inequality (91) is true. This completes the proof of Theorem 5.

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