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Conditioning and error estimates in LQG design*

by

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Abstract: Efficient conditioning and error estimates are presented for the numerical solution of matrix Riccati equations in the continuous-time and discrete-time LQG design. The estimates implemented involve the solution of triangular Lyapunov equations along with usage of the LAPACK norm estimator.

Keywords: LQG control, Riccati equations, perturbation analysis, condition numbers, forward error estimation.

1. Introduction

The Linear-Quadratic Gaussian (LQG) design is the most efficient and widely used design approach in the field of linear stochastic control systems. It consists of computing a LQ state regulator and a Kalman filter for the controlled system, by solving a pair of dual matrix Riccati equations, Kwakernaak, Sivan (1972). The numerical solution of these equations, however, is still an open problem, which sometimes leads to serious difficulties in LQG design applications. First of all, the considered Riccati equations may be ill conditioned, i.e. small perturbations in their coefficients may lead to large variations in the solutions, Petkov, Christov, Konstantinov (1991). As it is well known, the conditioning of an equation depends neither on the method used for solving it, nor on the properties of the computer architecture. Therefore, it is necessary to have a quantitative characterization of the equation conditioning in order to estimate the accuracy of solution.

The second difficulty in the numerical solution of Riccati equations is connected with the stability of the numerical method used and the reliability of its implementation. It is well known (see Petkov, Christov, Konstantinov, 1991) that the methods for solving Riccati equations are generally unstable. This requires to have an estimate of the forward error in the computed solution.

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In this paper we present efficient algorithms for computing condition and error estimates pertaining to the numerical solution of the matrix Riccati equations in continuous-time and discrete-time LQG design. The algorithms are based on matrix norm estimator implemented in LAPACK (Anderson et al., 1995) and allow to obtain low cost condition and forward error estimates which are usually sufficiently accurate.

The following notation is used in the paper: $\mathcal{R}$ – the field of real numbers; $\mathcal{R}^{m\times n}$ – the space of $m\times n$ matrices $A=[a_{ij}]$ over $\mathcal{R}$; A^T – the transpose of A; $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ – the maximum and minimum singular values of A; $\|A\|_1$ – the 1-norm of the matrix A; $\|A\|_2=\sigma_{\max}(A)$ – the spectral norm of A; $\|A\|_F=(\sum|a_{ij}|^2)^{1/2})$ – the Frobenius norm of A; I_n – the unit $n\times n$ matrix; $A\otimes B$ – the Kronecker product of matrices A and B; $\mathrm{vec}(A)$ – the vector, obtained by stacking the columns of A in one vector; ε – the roundoff unit of the machine arithmetic.

2. LQG control problems

Consider the continuous-time linear stochastic system

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+v(t),\ t\geq 0\\ y(t) &= Cx(t)+w(t) \end{aligned} \tag{1}$$

where $x(t)\in\mathcal{R}^n$, $u(t)\in\mathcal{R}^m$, $y(t)\in\mathcal{R}^r$ are the system state, input and output vectors, respectively, $A\in\mathcal{R}^{n\times n}$, $B\in\mathcal{R}^{n\times m}$, $C\in\mathcal{R}^{r\times n}$ are known constant matrices, and $v(t)$, $w(t)$ are independent zero-mean Gaussian white-noise processes with variance matrices $\mathcal{E}\{v(t)v^T(s)\}=V\delta(t-s)$, $V\geq 0$, and $\mathcal{E}\{w(t)w^T(s)\}=W\delta(t-s)$, $W>0$. It is supposed that the pairs (A,B), $(A,V^{1/2})$ are stabilizable and the pair (A,C) is detectable.

The (continuous-time) LQG control problem consists in finding a control u that minimizes the quadratic performance index

$$J(u)=\lim_{T\to\infty}\mathcal{E}\left\{\frac{1}{T}\int_0^T\big(x^T(t)Qx(t)+u^T(t)Ru(t)\big)\,dt\right\} \tag{2}$$

having partial knowledge of the system state $x(t)$ via the output vector $y(t)$. It is assumed that $Q\geq 0$, $R>0$ and the pair $(A,Q^{1/2})$ is detectable.

As it is well known this problem splits into two parts: an optimal estimation of the system state and a linear quadratic control problem (see Kwakernaak, Sivan, 1972, and Fleming, Rishel, 1975). The optimal control u^* is given by

$$u^*(t)=-R^{-1}B^TX\hat{x}(t) \tag{3}$$

where $\hat{x}(t)$ is the optimal estimate of $x(t)$ obtained using the Kalman-Bucy filter

$$\dot{\hat{x}}(t)=A\hat{x}(t)+YC^TW^{-1}[y(t)-C\hat{x}(t)],\ t\geq 0. \tag{4}$$

Here X and Y are the unique non-negative definite solutions of the dual matrix Riccati equations

$$A^T X + XA + Q - XBR^{-1}B^T X = 0 \tag{5}$$

and

$$AY + YA^T + V - YC^T W^{-1} CY = 0. \tag{6}$$

In the discrete-time analog of the LQG control problem (1), (2), one studies the linear stochastic system

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i + v_i,\ i \geq 0 \\ y_i &= Cx_i + w_i \end{aligned} \tag{7}$$

with the performance index

$$J(u) = \lim_{N\to\infty} \mathcal{E}\left\{ \frac{1}{N}\sum_{i=0}^{N}\left(x_i^T Q x_i + u_i^T R u_i\right)\right\} \tag{8}$$

where $\{v_i\}$ and $\{w_i\}$ are independent zero-mean Gaussian white-noise sequences with variance matrices $V \geq 0$ and $W > 0$, respectively. In this case the optimal control is

$$u_i^* = -(R + B^T XB)^{-1} B^T XA\hat{x}_i \tag{9}$$

where the optimal estimate $\hat{x}_i$ is given by the Kalman filter

$$\hat{x}_{i+1} = A\hat{x}_i + Bu_i + AYC^T(W + CYC^T)^{-1}(y_i - C\hat{x}_i) \tag{10}$$

and X and Y are the unique non-negative definite solutions of the discrete dual matrix Riccati equations

$$A^T XA - X + Q - A^T XB(R + B^T XB)^{-1} B^T XA = 0 \tag{11}$$

$$AYA^T - Y + V - AYC^T(W + CYC^T)^{-1} CYA^T = 0. \tag{12}$$

In what follows we shall consider the conditioning and error estimation for the matrix Riccati equation (5) and the discrete matrix Riccati equation (11). The corresponding results for the Riccati equations (6) and (12) can be obtained using the duality of (5) and (6), and of (11) and (12), respectively. In the sequel we shall write equations (5) and (11) as

$$A^T X + XA + Q - XSX = 0 \tag{13}$$

and

$$X = Q + A^T X(I_n + SX)^{-1} A, \tag{14}$$

where $S = BR^{-1}B^T$.

3. Conditioning of Riccati equations

Suppose that the matrices A, Q, S in (13), (14) are subject to perturbations ΔA, ΔQ, ΔS, respectively, so that instead of the initial data we have the matrices

$$\tilde{A} = A + \Delta A, \;\; \tilde{Q} = Q + \Delta Q, \;\; \tilde{S} = S + \Delta S.$$

The perturbation analysis of (13), (14) is aimed at studying the variation ΔX in the solution $\tilde{X} = X + \Delta X$ due to the perturbations ΔA, ΔQ, ΔS. If small perturbations in the data lead to small variations in the solution, the corresponding equation is said to be well-conditioned, and if these perturbations lead to large variations in the solution, this equation is ill-conditioned. In the perturbation analysis of the Riccati equations it is supposed that the perturbations preserve the symmetric structure of the equation, i.e. the perturbations ΔQ and ΔS are symmetric. If $\|\Delta A\|$, $\|\Delta Q\|$ and $\|\Delta S\|$ are sufficiently small, then the perturbed solution $\tilde{X} = X + \Delta X$ is well defined.

Consider first the Riccati equation (13). The condition number of this equation is defined as (Byers, 1985)

$$K_R = \lim_{\delta \to 0} \sup \left\{ \frac{\|\Delta X\|}{\delta \|X\|} : \|\Delta A\| \le \delta \|A\|, \; \|\Delta Q\| \le \delta \|Q\|, \; \|\Delta S\| \le \delta \|S\| \right\}.$$

For sufficiently small δ we have (within first order terms)

$$\frac{\|\Delta X\|}{\|X\|} \le K_R \delta.$$

Denote by $\bar{X}$ the solution of the Riccati equation computed by a numerical method in finite arithmetic with relative precision ε. If the method is backward stable, the relative error in the solution can be estimated by

$$\frac{\|\bar{X} - X\|}{\|X\|} \le p(n) K_R \varepsilon$$

where $p(n)$ is some low-order polynomial of n. This shows the importance of the condition numbers in the numerical solution of Riccati equations.

The computation of the exact value of K_R requires the construction and manipulation of $n^2 \times n^2$ matrices which is not practical for large n. That is why it is useful to derive approximations of K_R that can be obtained inexpensively.

In first order approximation ΔX can be represented as

$$\Delta X = -\Omega^{-1}(\Delta Q) - \Theta(\Delta A) + \Pi(\Delta S) \tag{15}$$

where

$$\Omega(Z) = A_c^T Z + Z A_c, \;\; \Theta(Z) = \Omega^{-1}(Z^T X + XZ), \;\; \Pi(Z) = \Omega^{-1}(XZX)$$

are linear operators in the space of $n \times n$ matrices, which determine the sensitivity of X with respect to the perturbations in Q, A, S, respectively, and $A_c = A - SX$. Based on (15) it is possible to use the approximate condition number

$$K_B := \frac{\|\Omega^{-1}\|\|Q\| + \|\Theta\|\|A\| + \|\Pi\|\|S\|}{\|X\|} \tag{16}$$

where $\|\Omega^{-1}\|, \|\Theta\|, \|\Pi\|$ are the corresponding induced operator norms. Note that

$$\|\Omega^{-1}\|_F = \frac{1}{\text{sep}_F(A_c^T, -A_c)}$$

where

$$\text{sep}_F(A_c^T, -A_c) := \min_{Z \neq 0} \frac{\|A_c^T Z + Z A_c\|_F}{\|Z\|_F}.$$

The condition number of the discrete Riccati equation (14) is defined in an analogous way. In this case the operators Ω, Θ and Π are determined from

$$\begin{aligned} &\Omega(Z) = A_c^T Z A_c - Z, \;\; \Theta(Z) = \Omega^{-1}(Z^T X A_c + A_c^T X Z) \\ &\Pi(Z) = \Omega^{-1}(A_c^T X Z X A_c), \end{aligned} \tag{17}$$

where $A_c = (I_n + SX)^{-1} A$.

4. Conditioning estimation

The quantities $\|\Omega^{-1}\|_1$, $\|\Theta\|_1$, $\|\Pi\|_1$ arising in the sensitivity analysis of Riccati equations can be efficiently estimated by using the norm estimator, proposed in Higham (1988), which estimates the norm $\|T\|_1$ of a linear operator T, given the ability to compute Tv and $T^T w$ quickly for arbitrary v and w. This estimator is implemented in the LAPACK subroutine `xLACON` (Anderson et al., 1995), which is called via a reverse communication interface, providing the products Tv and $T^T w$.

Consider first the Riccati equation (13). With respect to the computation of

$$\|\Omega^{-1}\|_F = \|P^{-1}\|_2 = \frac{1}{\text{sep}_F(A_c^T, -A_c)}$$

the use of `xLACON` means solving the linear equations

$$Py = v, \;\; P^T z = v$$

where

$$P = I_n \otimes A_c^T + A_c^T \otimes I_n, \;\; P^T = I_n \otimes A_c + A_c \otimes I_n,$$

v being determined by `xLACON`. This is equivalent to the solution of the Lyapunov equations

$$\begin{aligned} A_c^T Y + Y A_c &= V \\ A_c Z + Z A_c^T &= V, \end{aligned} \tag{18}$$

where $\mathrm{vec}(V) = v,\ \mathrm{vec}(Y) = y,\ \mathrm{vec}(Z) = z$.

The solution of these Lyapunov equations can be obtained in a numerically stable way using the Bartels-Stewart algorithm (Bartels, Stewart, 1972; Petkov, Christov, Konstantinov, 1991). Note that in (18) the matrix V is symmetric, which allows for a reduction in complexity by operating on vectors v of length $n(n+1)/2$ instead of n^2.

An estimate of $\|\Theta\|_1$ can be obtained in a similar way by solving the Lyapunov equations

$$\begin{aligned} A_c^T Y + Y A_c &= V^T X + XV \\ A_c Z + Z A_c^T &= V^T X + XV. \end{aligned} \tag{19}$$

To estimate $\|\Pi\|_1$ via `xLACON`, it is necessary to solve the equations

$$\begin{aligned} A_c^T Y + Y A_c &= XVX \\ A_c Z + Z A_c^T &= XVX, \end{aligned} \tag{20}$$

where the matrix V is again symmetric and we can again work with shorter vectors.

To avoid overflows, instead of estimating the condition number (16) an estimate of the reciprocal condition number

$$\frac{1}{\tilde{K}_B} = \frac{\widetilde{\mathrm{sep}}_1(\bar{A}_c^T, -\bar{A}_c)\|\bar{X}\|_1}{\|Q\|_1 + \widetilde{\mathrm{sep}}_1(\bar{A}_c^T, -\bar{A}_c)(\|\tilde{\Theta}\|_1\|A\|_1 + \|\tilde{\Pi}\|_1\|S\|_1)}$$

is determined. Here $\bar{A}_c$ and $\bar{X}$ are the computed matrices A_c and X, and the estimated quantities are denoted by tilde.

The estimation of $\|\Omega\|_1, \|\Theta\|_1, \|\Pi\|_1$ for the discrete Riccati equation (14) is done in a similar way. In this case instead of (18) - (20) it is necessary to solve the corresponding discrete Lyapunov equations

$$\begin{aligned} A_c^T Y A_c - Y &= V \\ A_c Z A_c^T - Z &= V \end{aligned} \tag{21}$$

$$\begin{aligned} A_c^T Y A_c - Y &= V^T X A_c + A_c^T X V \\ A_c Z A_c^T - Z &= V^T X A_c + A_c^T X V \end{aligned} \tag{22}$$

$$\begin{aligned} A_c^T Y A_c - Y &= A_c^T X V X A_c \\ A_c Z A_c^T - Z &= A_c^T X V X A_c. \end{aligned} \tag{23}$$

The solution of these equations can be obtained in a numerically reliable way using the discrete counterpart of the Bartels-Stewart algorithm, proposed by Barraud (Barraud, 1977; Petkov, Christov, Konstantinov, 1991).

The accuracy of the estimates that we obtain via this approach depends on the ability of `xLACON` to find a right-hand side vector v which maximizes the ratios

$$\frac{\|y\|}{\|v\|},\ \frac{\|z\|}{\|v\|}$$

when solving the equations $Py = v,\ P^T z = v$. As in the case of other condition estimators it is always possible to find special examples when the value produced by `xLACON` underestimates the true value of the corresponding norm by an arbitrary factor. Note, however, that this may happens in rare circumstances.

5. Error estimation

A posteriori error bounds for the computed solution of the matrix equations (13), (14) can be obtained in several ways. One of the most efficient and reliable ways to get an estimate of the solution error is to use practical error bounds, similar to the case of solving linear systems of equations (Arioli, Demmel, Duff, 1989; Anderson et al., 1995) and matrix Sylvester equations (Higham, 1993).

Consider again the Riccati equation (13). Let

$$R = A^T\bar{X} + \bar{X}A + Q - \bar{X}S\bar{X}$$

be the exact residual matrix associated with the computed solution $\bar{X}$. Setting $\bar{X} := X + \Delta X$, where X is the exact solution and ΔX is the absolute error in the solution, one obtains

$$R = (A - S\bar{X})^T\Delta X + \Delta X(A - S\bar{X}) + \Delta X S \Delta X.$$

If we neglect the second order term in ΔX, we obtain the linear system of equations

$$\bar{P}\mathrm{vec}(\Delta X) = \mathrm{vec}(R)$$

where $\bar{P} = I_n \otimes \bar{A}_c^T + \bar{A}_c^T \otimes I_n$, $\bar{A}_c = A - S\bar{X}$. In this way we have

$$\|\mathrm{vec}(X - \bar{X})\|_\infty = \|\bar{P}^{-1}\mathrm{vec}(R)\|_\infty \le \|\ |\bar{P}^{-1}|\ |\mathrm{vec}(R)|\ \|_\infty.$$

As it is known (see Arioli, Demmel, Duff, 1989) this bound is optimal if we ignore the signs in the elements of $\bar{P}^{-1}$ and $\mathrm{vec}(R)$.

In order to take into account the rounding errors in forming the residual matrix, instead of R we use

$$\bar{R} = fl(Q + A^T\bar{X} + \bar{X}A - \bar{X}S\bar{X}) = R + \Delta R$$

where

$$|\Delta R| \leq \varepsilon(4|Q| + (n+4)(|A^T|\,|\bar{X}| + |\bar{X}|\,|A|) + 2(n+1)|\bar{X}|\,|S|\,|\bar{X}|) =: R_\varepsilon$$

and fl denotes the result of a floating point computation. Here we made use of the well known error bounds for matrix addition and matrix multiplication.

In this way we have obtained the overall bound

$$\frac{\|X - \bar{X}\|_M}{\|\bar{X}\|_M} \leq \frac{\|\,|P^{-1}|\,(|\mathrm{vec}(\bar{R})| + \mathrm{vec}(R_\varepsilon))\|_\infty}{\|\bar{X}\|_M} \tag{24}$$

where $\|X\|_M = \max_{i,j} |x_{ij}|$.

The numerator in the right hand side of (24) is of the form $\|\,|P^{-1}|r\|_\infty$, and as in Arioli, Demmel, Duff (1989), and Higham (1993) we have

$$\begin{aligned} \|\,|\bar{P}^{-1}|\,r\,\|_\infty &= \|\,|\bar{P}^{-1}|\,D_R e\|_\infty = \|\,|\bar{P}^{-1}D_R|\,e\|_\infty \\ &= \|\,|\bar{P}^{-1}D_R|\,\|_\infty = \|\bar{P}^{-1}D_R\|_\infty \end{aligned}$$

where $D_R = \mathrm{diag}(r)$ and $e = [1, 1, \ldots, 1]^T$. This shows that $\|\,|P^{-1}|\,r\|_\infty$ can be efficiently estimated using the norm estimator `xLACON` in LAPACK, which estimates $\|Z\|_1$ at the cost of computing a few matrix-vector products involving Z and Z^T. This means that for $Z = \bar{P}^{-1}D_R$ we have to solve a few linear systems involving $\bar{P} = I_n \otimes \bar{A}_c^T + \bar{A}_c^T \otimes I_n$ and $\bar{P}^T = I_n \otimes \bar{A}_c + \bar{A}_c \otimes I_n$ or, in other words, we have to solve several Lyapunov equations

$$\begin{aligned} &\bar{A}_c^T X + X\bar{A}_c = V \\ &\bar{A}_c X + X\bar{A}_c^T = W. \end{aligned} \tag{25}$$

Note that the Schur form of $\bar{A}_c$ is already available from the condition estimation of the Riccati equation, so that the solution of the Lyapunov equations can be obtained efficiently via the Bartels-Stewart algorithm. Also, due to the symmetry of the matrices $\bar{R}$ and R_ε, we only need the upper (or lower) part of the solution of this Lyapunov equations, which allows to reduce the complexity by manipulating only vectors of length $n(n+1)/2$ instead of n^2.

The error estimation in the solution of the discrete Riccati equation (14) is done in an analogous way. In this case instead of (25) we have to solve a few discrete Lyapunov equations

$$\begin{aligned} &\bar{A}_c^T X \bar{A}_c - X = V \\ &\bar{A}_c X \bar{A}_c^T - X = W. \end{aligned}$$

The software implementation of the condition and error estimates is based entirely on LAPACK (Anderson et al., 1995) and BLAS (Lawson et al., 1979; Dongarra et al., 1990) subroutines.

6. Numerical examples

In this section we present two examples which demonstrate the performance of the estimates implemented in the solution of families of Riccati equations whose conditioning vary very much. All computations were carried out on a Sun workstation with relative machine precision $\varepsilon = 2.22 \times 10^{-16}$.

In order to have a closed form solution, the matrices in the Riccati equations are chosen as

$$A = TA_0T^{-1}, \quad Q = T^{-T}Q_0T^{-1}, \quad S = TS_0T^T$$

where A_0, Q_0, S_0 are diagonal matrices and T is a nonsingular transformation matrix. The solution is then given by $X = T^{-T}X_0T^{-1}$, where X_0 is a diagonal matrix whose elements are determined simply from the elements of A_0, Q_0, S_0. To avoid large rounding errors in constructing and inverting T, this matrix is chosen as $T = H_2\Sigma H_1$, where H_1 and H_2 are elementary reflectors and Σ is a diagonal matrix,

$$H_1 = I_n - 2ee^T/n, e = [1, 1, ..., 1]^T$$
$$H_2 = I_n - 2ff^T/n, f = [1, -1, 1, ..., (-1)^{n-1}]^T$$
$$\Sigma = \mathrm{diag}(1, s, s^2, ..., s^{n-1}), s > 1.$$

Using different values of the scalar s, it is possible to change the condition number of the matrix T with respect to inversion, $\mathrm{cond}_2(T) = s^{n-1}$.

Example 1 (Higham et al., 2004) Consider a family of Riccati equations of sixth order, constructed as described above with

$$A_0 = \mathrm{diag}(A_1, A_1), \quad Q_0 = \mathrm{diag}(Q_1, Q_1), \quad S_0 = \mathrm{diag}(S_1, S_1),$$

where

$$A_1 = \mathrm{diag}(-1 \times 10^{-k}, -2, -3 \times 10^k), \quad Q_1 = \mathrm{diag}(3 \times 10^{-k}, 5, 7 \times 10^k)$$
$$S_1 = \mathrm{diag}(10^{-k}, 1, 10^k).$$

The solution is given by $X_0 = \mathrm{diag}(X_1, X_1), \quad X_1 = \mathrm{diag}(1, 1, 1)$.

The conditioning of these equations deteriorates with the increase of k and s. The equations are solved by the Schur method (Laub, 1979; Petkov, Christov, Konstantinov, 1991).

Fig. 1 shows the ratio of the condition number estimate and the exact condition number and Fig. 2 – the ratio of the exact forward error in the solution and its estimate as functions of k and s. The results presented demonstrate the good performances of the condition number and error estimates for different k and s.

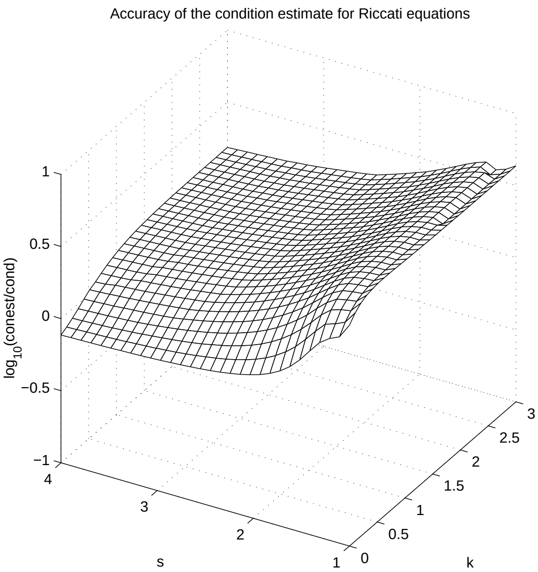

Figure 1. Accuracy of the condition number estimate for a family of Riccati equations

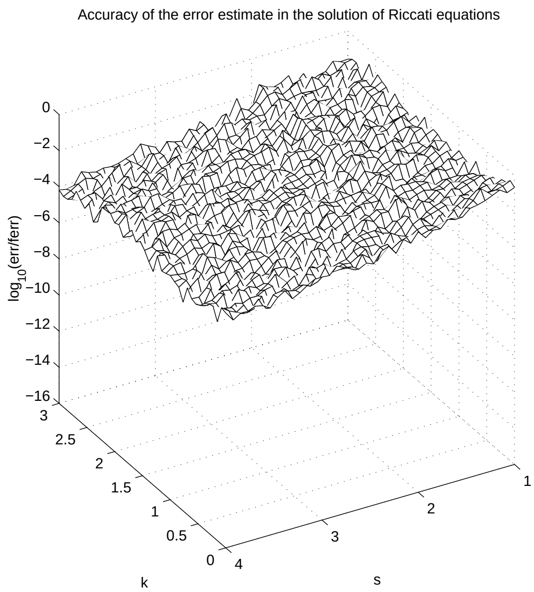

Figure 2. Accuracy of the forward error estimate for a family of Riccati equations

The next example illustrates the potential pessimism in the forward error estimate for the discrete Riccari equations.

EXAMPLE 2 Consider a family of discrete Riccati equations whose matrices A_0, Q_0, S_0 are chosen as

$$A_0 = \mathrm{diag}(A_1, A_1), \quad Q_0 = \mathrm{diag}(Q_1, Q_1), \quad S_0 = \mathrm{diag}(S_1, S_1),$$

where

$$A_1 = \mathrm{diag}(0,\ 1,\ 2)$$
$$Q_1 = \mathrm{diag}(10^k,\ 1,\ 10^{-k})$$
$$S_1 = \mathrm{diag}(10^{-k},\ 10^{-2k},\ 10^{-k}).$$

The conditioning of these equations deteriorates with the increase of k and s.

The accuracy of the condition number estimate for the discrete Riccati equations is shown in Fig. 3. As in the previous example, the condition number estimate is close to the true condition number. The results related to the forward error estimate, presented in Fig. 4, show that for the given discrete Riccati equations the error estimate may be very pessimistic for large k and s. In any case, however, we are sure that the actual forward error in the solution is less than the estimate obtained.

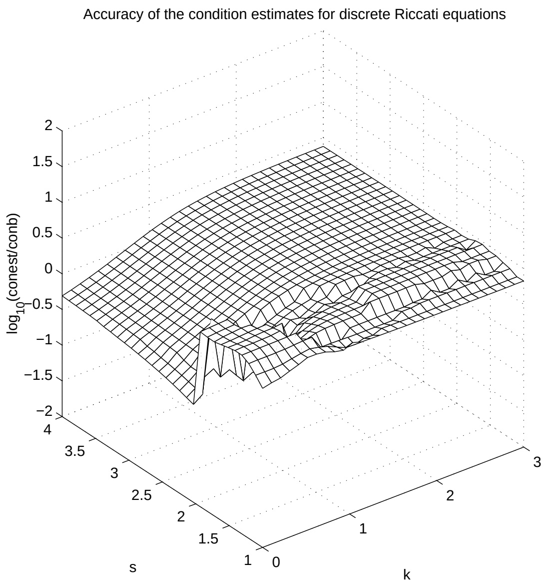

Figure 3. Accuracy of the condition number estimate for a family of discrete Riccati equations

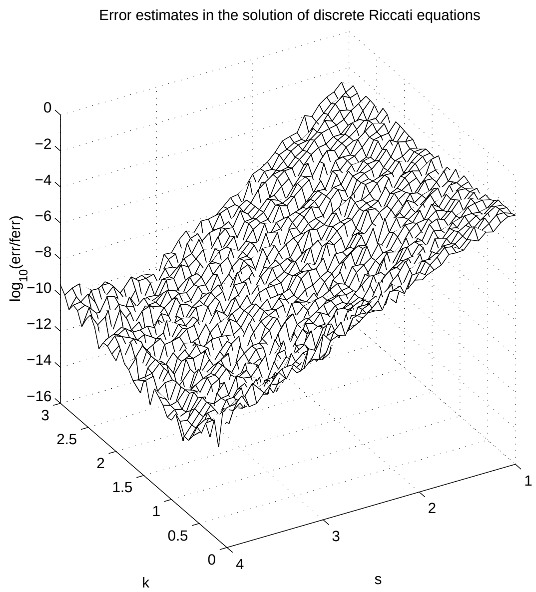

Figure 4. Accuracy of the forward error estimate for a family of discrete Riccati equations

7. Conclusions

The results presented in the paper show that it is possible to use successfully the LAPACK matrix norm estimator in the condition and forward estimation for the Riccati matrix equations arising in LQG design. The numerical experiments show that the condition estimates are always of the same order as the true condition numbers. However, the forward error estimates may be pessimistic just as in the solution of linear systems of equations. It should be pointed out that theoretically the forward error estimates may underestimate the true errors in the solution of the Riccati equations due to the neglecting of the higher order terms in the analysis. Such phenomenon was never observed in practice which shows that the forward error estimates are sufficiently reliable.

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