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Single stage algorithms for pole placement using static output feedback

by

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Abstract: Paper presents effective formulae, enabling transfer matrix pole assignment. This is accomplished by the use of generalized matrix inverses and the Kronecker product. All the desired pole distributions are covered – namely simple and multiple, real and complex. On this basis one-stage algorithms were developed, avoiding commonly used reduced orthogonality condition. Computational example of the presented algorithms is given.

Keywords: time-invariant linear systems, static output feedback, pole placement, Kronecker product, generalized inverse.

1. Introduction

The problem of pole placement for a multivariable linear time-invariant system using static output feedback (as well as dynamic feedback) is a long-standing important question in modern control theory (Rosenthal and Willems, 1998). During the last thirty years, hundreds of papers on this subject have appeared, some of which are reviewed in Syrmos et al. (1997) and listed in bibliography (in particular, Fletcher and Magni, 1987; Fletcher, 1987; Magni, 1987). The problem in question is connected to the more general eigenstructure assignment problem (Fahmy and O'Reilly, 1982, 1983, 1988; Roppenecker and O'Reilly, 1989; Fletcher et al., 1985; Askarpour and Owens, 1997, 1998, 1999; Clarke et al., 2003; Clarke and Griffin, 2004). Most of the algorithms developed till now employ the so-called reduced orthogonality condition (Kimura, 1977). This requires taking into consideration right and left eigenvectors of the closed-loop system or corresponding right and left parameter vectors (Roppenecker and O'Reilly, 1989; Askarpour and Owens, 1997; Clarke et al., 2003). Such an approach enables solution of the eigenstructure assignment problem using the above mentioned orthogonality condition and theorems proved in the cited papers. This leads, however, to a two-stage computational algorithm. Furthermore, specifying simultaneously right and left eigenvectors (parameters) can be considered an excessive requirement in the case of the pole placement problem. In connection with the above in the following sections of this article two one-stage algorithms for solving pole placement problem were developed. Their specific feature is abandonment of the orthogonality condition. This is achieved by specifying for the n-th order system: in first algorithm n right parameter vectors ($\mathbf{f}_i$), and in second algorithm n left parameter vectors ($\mathbf{h}_i$). This implies the consequential use of generalized matrix inverses. At the same time in order to facilitate the choice of parameter vectors from allowed subspaces using solvability conditions of corresponding matrix equations, formulae involving Kronecker product are used (Van Loan, 2000).

2. Statement of the problem

Consider the linear time-invariant control system, described by equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$.

System (1)-(2) is assumed to be fully controllable and observable. Moreover, it is assumed that the following condition holds:

$$m + p > n \tag{3}$$

and matrices $\mathbf{B}$ and $\mathbf{C}$ have full rank.

The system described by (1) and (2) has the following transfer matrix:

$$\mathbf{G}_o(s) = \mathbf{C}(s\mathbf{1}_n - \mathbf{A})^{-1}\mathbf{B} \tag{4}$$

and characteristic polynomial:

$$M_o(s) = \det(s\mathbf{1}_n - \mathbf{A}). \tag{5}$$

To assign a desired set of poles $\Lambda = \{s_1, s_2, \ldots, s_n\}$, static output feedback is introduced, resulting in the control law

$$\mathbf{u} = \mathbf{K}\mathbf{y}, \qquad \text{where } \mathbf{K} \in \mathbb{R}^{m\times p}. \tag{6}$$

Consequently, the closed-loop system is described by the equation:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C})\,\mathbf{x}(t) \tag{7}$$

and the closed-loop characteristic polynomial is:

$$M(s) = \det(s\mathbf{1}_n - \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}). \tag{8}$$

To calculate matrix $\mathbf{K}$, ensuring that the characteristic equation holds:

$$M(s) = 0 \tag{9}$$

for $s = s_i$, $i = 1, 2, \ldots, n$, a properly generalized method from Fahmy and O'Reilly (1982, 1983, 1988) is used.

In the works mentioned above, and also in Roppenecker and O'Reilly (1989) and Askarpour and Owens (1997) it is proved that equation (9) holds for $s = s_i$, $i = 1, 2, \ldots, n$, if only for those values of s there holds:

$$\det[\mathbf{1}_m - \mathbf{K}\mathbf{G}_o(s)] = 0 \tag{10}$$

or

$$\det[\mathbf{1}_p - \mathbf{G}_o(s)\mathbf{K}] = 0 \tag{11}$$

assuming that the desired set Λ does not contain any of roots of equation

$$M_o(s) = 0. \tag{12}$$

3. Main results

The following considerations are split into two cases, namely

(i) when using (10), and

(ii) when using (11).

First, it is assumed that the set Λ contains only simple real or complex poles s_i; the case of multiple poles is treated separately. In the case of complex roots it is assumed that their set is self-conjugate i.e. $s_{i+1} = \bar{s}_i$.

3.1. Case (i)

Considering equation (10), one can claim that it holds if and only if columns of the matrix $[\mathbf{1}_m - \mathbf{K}\mathbf{G}_o(s_i)]$ are linearly dependent. This implies that for every s_i one can find non-null m-dimensional vector $\mathbf{f}_i$, that the following equations hold:

$$[\mathbf{1}_m - \mathbf{K}\mathbf{G}_o(s_i)]\mathbf{f}_i = \mathbf{0}_{m\times 1}, \quad i = 1, 2, \ldots, n \tag{13}$$

that is

$$\mathbf{K}\mathbf{G}_o(s_i)\mathbf{f}_i = \mathbf{f}_i \tag{14}$$

which can be jointly written as:

$$\mathbf{K}\mathbf{W} = \mathbf{F} \tag{15}$$

where

$$\mathbf{W} = [\mathbf{G}_o(s_1)\mathbf{f}_1, \mathbf{G}_o(s_2)\mathbf{f}_2, \ldots, \mathbf{G}_o(s_n)\mathbf{f}_n] \tag{16}$$

$$\mathbf{F} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n] \tag{17}$$

are matrices of dimension $p \times n$ and $m \times n$ respectively, and vectors $\mathbf{f}_i$ must be chosen in such a way as to assure the full rank of matrix $\mathbf{W}$ i.e.

$$\operatorname{rank} \mathbf{W} = p. \tag{18}$$

According to the Kronecker-Capelli theorem, equation (15) has a solution if and only if

$$\operatorname{rank} \begin{bmatrix} \mathbf{W} \\ \mathbf{F} \end{bmatrix} = \operatorname{rank} \mathbf{W}. \tag{19}$$

When conditions (18) and (19) hold, the solution of equation (15) is

$$\mathbf{K} = \mathbf{F}\mathbf{W}^{\dagger} \tag{20}$$

where $\mathbf{W}^{\dagger}$ is the right generalized inverse – for example Moore-Penrose pseudo-inverse – of the matrix $\mathbf{W}$ (Zielke, 1984):

$$\mathbf{W}^{\dagger} = \mathbf{W}^{\mathrm{T}} \left(\mathbf{W}\mathbf{W}^{\mathrm{T}}\right)^{-1}. \tag{21}$$

In view of the above, the main problem is to find vectors $\mathbf{f}_i, \quad i = 1, 2, \ldots, n$ comprising matrix $\mathbf{F}$ (equation (17)), and matrix $\mathbf{W}$ (equation (16)), such that the condition (19) is satisfied. One can achieve that by arbitrarily choosing (e.g. randomly) full-rank matrix $\mathbf{Q}$ of dimensions $n \times (n-p)$, for which the following condition must be satisfied

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{F} \end{bmatrix} \mathbf{Q} = \mathbf{0}_{(p+m)\times(n-p)}. \tag{22}$$

Thereafter, the vector notation is introduced by using operator vec $(\cdot)$, which maps a matrix to the vector consisting of its columns stacked one above the other e.g.

$$\operatorname{vec}(\mathbf{F}) = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix}. \tag{23}$$

Using the Kronecker product, and taking into account that

$$\mathbf{N}\mathbf{X}\mathbf{M} = \mathbf{Y} \;\Leftrightarrow\; \operatorname{vec}(\mathbf{Y}) = (\mathbf{M}^{\mathrm{T}} \otimes \mathbf{N})\operatorname{vec}(\mathbf{X}) \tag{24}$$

one can express (22) in the form of two equivalent equations (Van Loan, 2000):

$$\left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_p\right) \mathrm{vec}\,(\mathbf{W}) = \mathbf{0}_{p(n-p)\times 1} \tag{25}$$

$$\left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_m\right) \mathrm{vec}\,(\mathbf{F}) = \mathbf{0}_{m(n-p)\times 1} \tag{26}$$

At the same time, from (16) one can obtain:

$$\begin{aligned}\mathrm{vec}\,(\mathbf{W}) &= \begin{bmatrix} \mathbf{G}_o(s_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_o(s_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{G}_o(s_n) \end{bmatrix} \mathrm{vec}\,(\mathbf{F}) = \\ &= \left(\mathbf{G}_o(s_1) \oplus \mathbf{G}_o(s_2) \oplus \ldots \oplus \mathbf{G}_o(s_n)\right) \mathrm{vec}\,(\mathbf{F}).\end{aligned} \tag{27}$$

Consequently, equations (25), (26) and (27) take the form:

$$\mathbf{L}\ \mathrm{vec}\,(\mathbf{F}) = \mathbf{0}_{(p+m)(n-p)\times 1} \tag{28}$$

where $\mathbf{L}$ is a matrix of dimension $(p+m)(n-p) \times mn$, defined by equation:

$$\mathbf{L} = \begin{bmatrix} \left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_p\right) \left(\mathbf{G}_o(s_1) \oplus \mathbf{G}_o(s_2) \oplus \ldots \oplus \mathbf{G}_o(s_n)\right) \\ \mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_m \end{bmatrix}. \tag{29}$$

Equation (28) enables the choice of vec ($\mathbf{F}$), and, consequently, matrix $\mathbf{F}$, using standard procedures for finding the null-space of the matrix (linear operator).

The problem gets more complicated when some roots $s_i \in \Lambda$ are required to be complex. This implies that the corresponding matrices $\mathbf{G}_o(s_i)$ become complex. As mentioned above, it is assumed that the desired complex roots occur in conjugate pairs, i.e. $s_{i+1} = \bar{s}_i$. The roots can be ordered in such way that of the n desired roots s_i, first r are real, and remaining $n-r$ complex. Furthermore it is assumed that roots forming conjugate pairs are adjacent. To avoid complex calculus let us introduce the matrix:

$$\mathbf{T} = \frac{1}{2} \begin{bmatrix} 1+j & 1-j \\ -1+j & -1-j \end{bmatrix} \tag{30}$$

which transforms complex matrix

$$\begin{bmatrix} a-jb & 0 \\ 0 & a+jb \end{bmatrix}$$

to a similar real matrix:

$$\mathbf{T} \begin{bmatrix} a-jb & 0 \\ 0 & a+jb \end{bmatrix} \mathbf{T}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{31}$$

and, additionally, the following holds:

$$\mathbf{T}\begin{bmatrix} a-jb \\ a+jb \end{bmatrix} = \begin{bmatrix} a+b \\ b-a \end{bmatrix}. \tag{32}$$

The similarity transformation can be generalized by introducing the matrix $\mathbf{U}$ of dimension $n \times n$ defined by:

$$\mathbf{U} = \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\frac{(n-r)}{2}} \otimes \mathbf{T} \end{bmatrix}. \tag{33}$$

As can be easily checked, $\mathbf{T}$ and $\mathbf{U}$ are unitary (i.e. $\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T} = \mathbf{1}_2$ and $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{1}_n$). By introducing the abbreviated notation:

$$\begin{gathered} \mathbf{G}_o(s_i) = \mathbf{G}_i \quad \text{for} \quad 1 \leqslant i \leqslant r \\ \mathbf{G}_o(s_i) = \mathrm{Re}\mathbf{G}_o(s_i) + j\mathrm{Im}\mathbf{G}_o(s_i) = \Re\mathbf{G}_i + j\Im\mathbf{G}_i \quad \text{dla} \quad r < i \leqslant n \end{gathered} \tag{34}$$

from (31), taking into account that $s_{i+1} = \bar{s}_i$, one can write:

$$(\mathbf{T} \otimes \mathbf{1}_p)\begin{bmatrix} \mathbf{G}_o(s_i) & \mathbf{0}_{p\times m} \\ \mathbf{0}_{p\times m} & \mathbf{G}_o(s_{i+1}) \end{bmatrix}(\mathbf{T}^* \otimes \mathbf{1}_m) = \begin{bmatrix} \Re\mathbf{G}_i & \Im\mathbf{G}_i \\ \Im\mathbf{G}_{i+1} & \Re\mathbf{G}_{i+1} \end{bmatrix}. \tag{35}$$

Using matrix $\mathbf{U}$, in the aforementioned context, (28) can be reformulated as:

$$\begin{bmatrix} \left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_p\right)\left(\mathbf{U}^* \otimes \mathbf{1}_p\right)\left(\mathbf{G}_1 \oplus \ldots \oplus \mathbf{G}_r \oplus \begin{bmatrix} \Re\mathbf{G}_{r+1} & \Im\mathbf{G}_{r+1} \\ \Im\mathbf{G}_{r+2} & \Re\mathbf{G}_{r+2} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re\mathbf{G}_{n-1} & \Im\mathbf{G}_{n-1} \\ \Im\mathbf{G}_n & \Re\mathbf{G}_n \end{bmatrix}\right)(\mathbf{U} \otimes \mathbf{1}_m) \\ \left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_m\right)\left(\mathbf{U}^* \otimes \mathbf{1}_m\right)(\mathbf{U} \otimes \mathbf{1}_m) \end{bmatrix} \times$$
$$\times \operatorname{vec}(\mathbf{F}) = \mathbf{0}_{(p+m)(n-p)\times 1} \tag{36}$$

which, after simple calculations, becomes:

$$\begin{bmatrix} \left(\mathbf{Q}^{\mathrm{T}}\mathbf{U}^* \otimes \mathbf{1}_p\right)\left(\mathbf{G}_1 \oplus \ldots \oplus \mathbf{G}_r \oplus \begin{bmatrix} \Re\mathbf{G}_{r+1} & \Im\mathbf{G}_{r+1} \\ \Im\mathbf{G}_{r+2} & \Re\mathbf{G}_{r+2} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re\mathbf{G}_{n-1} & \Im\mathbf{G}_{n-1} \\ \Im\mathbf{G}_n & \Re\mathbf{G}_n \end{bmatrix}\right) \\ \left(\mathbf{Q}^{\mathrm{T}}\mathbf{U}^* \otimes \mathbf{1}_m\right) \end{bmatrix} \times$$
$$\times (\mathbf{U} \otimes \mathbf{1}_m) \operatorname{vec}(\mathbf{F}) = \mathbf{0}_{(p+m)(n-p)\times 1}. \tag{37}$$

Using the following definitions:

$$\mathbf{Q}^{\mathrm{T}}\mathbf{U}^* = \tilde{\mathbf{Q}}^{\mathrm{T}} \qquad \Rightarrow \qquad \tilde{\mathbf{Q}} = \bar{\mathbf{U}}\mathbf{Q} \tag{38}$$

$$(\mathbf{U} \otimes \mathbf{1}_m) \operatorname{vec}(\mathbf{F}) = \operatorname{vec}(\tilde{\mathbf{F}}) \quad \Leftrightarrow \quad \tilde{\mathbf{F}} = \mathbf{F}\mathbf{U}^{\mathrm{T}} \tag{39}$$

and

$$\tilde{\mathbf{L}} = \begin{bmatrix} \left(\tilde{\mathbf{Q}}^{\mathrm{T}} \otimes \mathbf{1}_p\right)\left(\mathbf{G}_1 \oplus \ldots \oplus \mathbf{G}_r \oplus \begin{bmatrix} \Re\mathbf{G}_{r+1} & \Im\mathbf{G}_{r+1} \\ \Im\mathbf{G}_{r+2} & \Re\mathbf{G}_{r+2} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re\mathbf{G}_{n-1} & \Im\mathbf{G}_{n-1} \\ \Im\mathbf{G}_n & \Re\mathbf{G}_n \end{bmatrix}\right) \\ \left(\tilde{\mathbf{Q}}^{\mathrm{T}} \otimes \mathbf{1}_m\right) \end{bmatrix} \tag{40}$$

we get

$$\tilde{\mathbf{L}} \operatorname{vec}(\tilde{\mathbf{F}}) = \mathbf{0}_{(p+m)(n-p)\times 1}. \tag{41}$$

Analogously to (27), it is assumed that

$$\text{vec}\,(\tilde{\mathbf{W}}) = \left(\mathbf{G}_1 \oplus \ldots \oplus \mathbf{G}_r \oplus \begin{bmatrix} \Re\mathbf{G}_{r+1} & \Im\mathbf{G}_{r+1} \\ \Im\mathbf{G}_{r+2} & \Re\mathbf{G}_{r+2} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re\mathbf{G}_{n-1} & \Im\mathbf{G}_{n-1} \\ \Im\mathbf{G}_{n} & \Re\mathbf{G}_{n} \end{bmatrix}\right) \text{vec}\,(\tilde{\mathbf{F}}) \tag{42}$$

and, as can be easily checked

$$\tilde{\mathbf{W}} = \mathbf{W}\mathbf{U}^{\mathrm{T}}. \tag{43}$$

Eventually, we obtain

$$\mathbf{K} = \tilde{\mathbf{F}}\tilde{\mathbf{W}}^{\dagger} \tag{44}$$

where

$$\tilde{\mathbf{W}}^{\dagger} = \tilde{\mathbf{W}}^{*}\left(\tilde{\mathbf{W}}\tilde{\mathbf{W}}^{*}\right)^{-1} = \left(\mathbf{W}\mathbf{U}^{\mathrm{T}}\right)^{*}\left(\left(\mathbf{W}\mathbf{U}^{\mathrm{T}}\right)\left(\mathbf{W}\mathbf{U}^{\mathrm{T}}\right)^{*}\right)^{-1} = \\ = \mathbf{U}^{\mathrm{T}}\mathbf{W}^{*}\left(\mathbf{W}\mathbf{W}^{*}\right)^{-1} = \mathbf{U}^{\mathrm{T}}\mathbf{W}^{\dagger}. \tag{45}$$

From the properties of matrix $\mathbf{U}$ it follows that in any choice of matrix $\mathbf{F}$ in which adjacent columns numbered $r+1$ to n are conjugate, the resulting matrix $\tilde{\mathbf{F}}$ is real. On account of the definition of matrix $\mathbf{W}$, when $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{K}$ are real, matrix $\tilde{\mathbf{W}}$ is also real. Conversely, if matrix $\mathbf{Q}$ is chosen in such a way that its *rows* numbered $r+1$ to n are complex conjugate (remaining real), or − which is equivalent − matrix $\tilde{\mathbf{Q}}$ is an arbitrarily chosen real matrix, then obtained from (41) matrix $\tilde{\mathbf{F}}$ and corresponding matrix $\tilde{\mathbf{W}}$ in view of equations (39) and (43) specify matrices $\mathbf{F}$ and $\mathbf{W}$ satisfying (13)-(17) with real matrix $\mathbf{K}$ specified by (44).

Remark 1

In the special case, when $p = n$, solving of equation (15) is much simpler, because matrix $\mathbf{W}$ becomes square. Consequently, in order to find matrix $\mathbf{K}$ using this equation, one should choose vectors $\mathbf{f}_i$, $\quad i = 1, 2, \ldots, n$, to satisfy the condition

$$\det \mathbf{W} \neq 0. \tag{46}$$

In this situation

$$\mathbf{K} = \mathbf{F}\mathbf{W}^{-1}. \tag{47}$$

In the case of $m = 1$ and valid assumption (3), one can solve the stated problem for $p = n$ if the following conditions hold. In that case equation (15) can be rewritten as

$$\mathbf{K}[\mathbf{G}_o(s_1), \mathbf{G}_o(s_2), \ldots, \mathbf{G}_o(s_n)] = [1, 1, \ldots, 1]. \tag{48}$$

Then, one can compute $\mathbf{K}$ if and only if

$$\det[\mathbf{G}_o(s_1), \mathbf{G}_o(s_2), \ldots, \mathbf{G}_o(s_n)] \neq 0. \tag{49}$$

This confirms the well-known fact, proved in Fahmy and O'Reilly (1982), that $\mathbf{K}$ is independent of the choice of $\mathbf{f}_i$ and is uniquely determined if (49) holds.

When the desired characteristic polynomial roots are not simple, it is assumed that q roots s_i, comprising self-conjugate set (i.e. $s_{i+1} = \bar{s}_i$), have algebraic multiplicities ν_i, that is

$$\sum_{i=1}^{q} \nu_i = n \tag{50}$$

and the characteristic polynomial is:

$$M(s) = \prod_{i=1}^{q} (s - s_i)^{\nu_i}. \tag{51}$$

In the case under consideration, the numbers s_i, $i = 1, 2, \ldots, q$ must be the roots of equation (10) of multiplicities $\nu_i, i = 1, 2, \ldots, q$, which requires:

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k} \det\left(\mathbf{1}_m - \mathbf{K}\mathbf{G}_o(s)\right)\Bigg|_{s=s_i} = 0, \quad i = 1, 2, \ldots, q \quad k = 0, 1, 2, \ldots, \nu_i - 1. \tag{52}$$

As proven in Fahmy and O'Reilly, 1983 (equations (10), (11)) this implies the existence of generally non-null vectors $\mathbf{f}_i^{(0)}, \mathbf{f}_i^{(1)}, \ldots, \mathbf{f}_i^{(\nu_i - 1)}$ $i = 1, 2, \ldots, q$ of dimensions $m \times 1$, such that:

$$\begin{gathered}
\mathbf{K}\mathbf{G}(s_i)\mathbf{f}_i^{(0)} = \mathbf{f}_i^{(0)} \\
-\frac{1}{k!}\mathbf{K}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathbf{G}(s_i)\mathbf{f}_i^{(0)} - \frac{1}{(k-1)!}\mathbf{K}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}\mathbf{G}(s_i)\mathbf{f}_i^{(1)} - \ldots - \mathbf{K}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}(s_i)\mathbf{f}_i^{(\nu_i - 2)} + \\
+ \left(\mathbf{1}_m - \mathbf{K}\mathbf{G}(s_i)\right)\mathbf{f}_i^{(\nu_i - 1)} = \mathbf{0}_{m \times 1}, \\
i = 1, 2, \ldots, q \quad k = 1, 2, \ldots, \nu_i - 1
\end{gathered} \tag{53}$$

where, for simplicity, the following notation is used:

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathbf{G}(s_i) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathbf{G}_o(s)\Bigg|_{s=s_i}. \tag{54}$$

From (4), one can obtain:

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathbf{G}(s_i) = (-1)^k (k!) \mathbf{C}(s_i \mathbf{1}_n - \mathbf{A})^{-(k+1)}\mathbf{B}, \quad k = 0, 1, 2, \ldots, \nu_i - 1 \tag{55}$$

and then, using notation

$$\mathbf{G}_i^{(k)} = (-1)^k \mathbf{C}(s_i \mathbf{1}_n - \mathbf{A})^{-(k+1)}\mathbf{B},$$
$$i = 1, 2, \dots, q, \quad k = 0, 1, 2, \dots, \nu_i - 1 \tag{56}$$

and substituting (55) in (53) and taking into account (56), it follows that:

$$\begin{gathered} \mathbf{K}\mathbf{G}_i^{(0)}\mathbf{f}_i^{(0)} = \mathbf{f}_i^{(0)} \\ \mathbf{K}\mathbf{G}_i^{(k)}\mathbf{f}_i^{(0)} + \mathbf{K}\mathbf{G}_i^{(k-1)}\mathbf{f}_i^{(1)} + \ldots + \mathbf{K}\mathbf{G}_i^{(0)}\mathbf{f}_i^{(k)} = \mathbf{f}_i^{(k)}, \\ i = 1, 2, \dots, q \quad k = 1, 2, \dots, \nu_i - 1. \end{gathered} \tag{57}$$

Thus, in the case under consideration, the matrices $\mathbf{F}$ and $\mathbf{W}$ in equation (15), are built-up from blocks corresponding to respective roots, namely:

$$\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_q] \tag{58}$$
$$\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_q]. \tag{59}$$

Respective blocks $\mathbf{F}_i, \mathbf{W}_i, i = 1, 2, \dots, q$ have dimensions $m \times \nu_i$ i $p \times \nu_i$, and are built of vectors $\mathbf{f}_i^{(k)}$ and $\mathbf{w}_i^{(k)}$, $k = 0, 1, \dots, \nu_i - 1$:

$$\mathbf{F}_i = [\mathbf{f}_i^{(0)}, \mathbf{f}_i^{(1)}, \dots, \mathbf{f}_i^{(\nu_i-1)}], \qquad i = 1, 2, \dots, q \tag{60}$$
$$\mathbf{W}_i = [\mathbf{w}_i^{(0)}, \mathbf{w}_i^{(1)}, \dots, \mathbf{w}_i^{(\nu_i-1)}], \qquad i = 1, 2, \dots, q \tag{61}$$

where, as follows from (57):

$$\mathbf{w}_i^{(k)} = \mathbf{G}_i^{(k)}\mathbf{f}_i^{(0)} + \mathbf{G}_i^{(k-1)}\mathbf{f}_i^{(1)} + \ldots + \mathbf{G}_i^{(1)}\mathbf{f}_i^{(k-1)} + \mathbf{G}_i^{(0)}\mathbf{f}_i^{(k)}, \quad i = 1, 2, \dots, q. \tag{62}$$

Matrix $\mathbf{K}$ can thus be computed from (20) and (21) using the formulae derived above. But to find $\mathbf{f}_i^{(k)} \quad i = 1, 2, \dots, q, \quad k = 0, 1, \dots, \nu_i - 1$, one must translate to a suitable form the equations (27), (28) and (29). For each root the following equation must hold:

$$\operatorname{vec}(\mathbf{W}_i) = \begin{bmatrix} \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} \\ \mathbf{G}_i^{(1)} & \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0}_{p\times m} \\ \mathbf{G}_i^{(\nu_i-1)} & \mathbf{G}_i^{(\nu_i-2)} & \cdots & \mathbf{G}_i^{(0)} \end{bmatrix} \operatorname{vec}(\mathbf{F}_i), \quad i = 1, 2, \dots, q. \tag{63}$$

By denoting $\mathbf{P}_i$ the following matrix of dimensions $p\nu_i \times m\nu_i$:

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} \\ \mathbf{G}_i^{(1)} & \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0}_{p\times m} \\ \mathbf{G}_i^{(\nu_i-1)} & \mathbf{G}_i^{(\nu_i-2)} & \cdots & \mathbf{G}_i^{(0)} \end{bmatrix}, \quad i = 1, 2, \dots, q \tag{64}$$

one obtains

$$\text{vec}\,(\mathbf{W}_i) = \mathbf{P}_i \text{vec}\,(\mathbf{F}_i). \tag{65}$$

Thus, because

$$\text{vec}\,(\mathbf{F}) = \begin{bmatrix} \text{vec}\,(\mathbf{F}_1) \\ \text{vec}\,(\mathbf{F}_2) \\ \vdots \\ \text{vec}\,(\mathbf{F}_q) \end{bmatrix} \qquad \text{vec}\,(\mathbf{W}) = \begin{bmatrix} \text{vec}\,(\mathbf{W}_1) \\ \text{vec}\,(\mathbf{W}_2) \\ \vdots \\ \text{vec}\,(\mathbf{W}_q) \end{bmatrix} \tag{66}$$

there follows

$$\text{vec}\,(\mathbf{W}) = \begin{bmatrix} \mathbf{P}_1 & & & & \\ & \mathbf{P}_2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & \mathbf{P}_q \end{bmatrix} \text{vec}\,(\mathbf{F}) \tag{67}$$

that is,

$$\text{vec}\,(\mathbf{W}) = (\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \ldots \oplus \mathbf{P}_q)\ \text{vec}\,(\mathbf{F}). \tag{68}$$

In view of the above, $\mathbf{f}_i^{(k)}$ $i = 1, 2, \ldots, q,\ k = 0, 1, \ldots, \nu_i - 1$, can be obtained from (28), where

$$\mathbf{L} = \begin{bmatrix} \left(\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_p\right) (\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \ldots \oplus \mathbf{P}_q) \\ \mathbf{Q}^{\mathrm{T}} \otimes \mathbf{1}_m \end{bmatrix}. \tag{69}$$

In the case of assignment of multiple complex roots, it is convenient to order vectors $\mathbf{f}_i^{(k)}$ and $\mathbf{w}_i^{(k)}$ according to the following scheme. For roots s_i and $s_{i+1} = \bar{s}_i$ one defines matrices

$$\mathbf{F}_{i,i+1} = [\mathbf{f}_i^{(0)}, \mathbf{f}_{i+1}^{(0)}, \mathbf{f}_i^{(1)}, \mathbf{f}_{i+1}^{(1)}, \ldots, \mathbf{f}_i^{(\nu_i-1)}, \mathbf{f}_{i+1}^{(\nu_i-1)}] \tag{70}$$

and

$$\mathbf{W}_{i,i+1} = [\mathbf{w}_i^{(0)}, \mathbf{w}_{i+1}^{(0)}, \mathbf{w}_i^{(1)}, \mathbf{w}_{i+1}^{(1)}, \ldots, \mathbf{w}_i^{(\nu_i-1)}, \mathbf{w}_{i+1}^{(\nu_i-1)}]. \tag{71}$$

This enables writing

$$\text{vec}\,(\mathbf{W}_{i,i+1}) = \begin{bmatrix} \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} \\ \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(0)} & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} \\ \mathbf{G}_i^{(1)} & \mathbf{0}_{p\times m} & \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & & \vdots & \vdots \\ \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(1)} & \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(0)} & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ & \vdots & & & & & \\ \mathbf{G}_i^{(\nu_i-1)} & \mathbf{0}_{p\times m} & \mathbf{G}_i^{(\nu_i-2)} & \mathbf{0}_{p\times m} & \cdots & \mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} \\ \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(\nu_i-1)} & \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(\nu_i-2)} & \cdots & \mathbf{0}_{p\times m} & \mathbf{G}_{i+1}^{(0)} \end{bmatrix} \text{vec}\,(\mathbf{F}_{i,i+1}). \tag{72}$$

Using the approach presented in the section devoted to the assignment of simple complex roots, one can avoid the complex calculus, required in the case of complex $\mathbf{G}_i^{(k)}$ and $\mathbf{G}_{i+1}^{(k)}$. The transformation matrix $\mathbf{U}$, analogous to that defined by (33), should be introduced. Assuming that first r roots are real, and the remaining complex roots are ordered in such a way that conjugate pairs are adjacent, and denoting:

$$\mu = \sum_{i=1}^{r} \nu_i \tag{73}$$

it should be assumed that

$$\mathbf{U} = \begin{bmatrix} \mathbf{1}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\frac{(n-\mu)}{2}} \otimes \mathbf{T} \end{bmatrix}. \tag{74}$$

From equations (39), (43) and (68) it follows that

$$\mathrm{vec}\,(\tilde{\mathbf{W}}) = (\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \ldots \oplus \mathbf{P}_r \oplus \mathbf{P}_{r+1,r+2} \oplus \ldots \oplus \mathbf{P}_{q-1,q})\mathrm{vec}\,(\tilde{\mathbf{F}}) \tag{75}$$

where, according to (72)

$$\mathbf{P}_{i,i+1} = \begin{bmatrix} \Re\mathbf{G}_i^{(0)} & \Im\mathbf{G}_i^{(0)} & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} \\ \Im\mathbf{G}_{i+1}^{(0)} & \Re\mathbf{G}_{i+1}^{(0)} & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} & \cdots & \mathbf{0}_{p\times m} & \mathbf{0}_{p\times m} \\ \Re\mathbf{G}_i^{(1)} & \Im\mathbf{G}_i^{(1)} & \Re\mathbf{G}_i^{(0)} & \Im\mathbf{G}_i^{(0)} & & \vdots & \vdots \\ \Im\mathbf{G}_{i+1}^{(1)} & \Re\mathbf{G}_{i+1}^{(1)} & \Im\mathbf{G}_{i+1}^{(0)} & \Re\mathbf{G}_{i+1}^{(0)} & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ \Re\mathbf{G}_i^{(\nu_i-1)} & \Im\mathbf{G}_i^{(\nu_i-1)} & \Re\mathbf{G}_i^{(\nu_i-2)} & \Im\mathbf{G}_i^{(\nu_i-2)} & \cdots & \Re\mathbf{G}_i^{(0)} & \Im\mathbf{G}_i^{(0)} \\ \Im\mathbf{G}_{i+1}^{(\nu_i-1)} & \Re\mathbf{G}_{i+1}^{(\nu_i-1)} & \Im\mathbf{G}_{i+1}^{(\nu_i-2)} & \Re\mathbf{G}_{i+1}^{(\nu_i-2)} & \cdots & \Im\mathbf{G}_{i+1}^{(0)} & \Re\mathbf{G}_{i+1}^{(0)} \end{bmatrix},$$

$$i = r+1, r+3, \ldots, q-1. \tag{76}$$

Taking into account (41), one obtains:

$$\tilde{\mathbf{L}} = \begin{bmatrix} \left(\tilde{\mathbf{Q}}^{\mathrm{T}} \otimes \mathbf{1}_p\right)(\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \ldots \oplus \mathbf{P}_r \oplus \mathbf{P}_{r+1,r+2} \oplus \ldots \oplus \mathbf{P}_{q-1,q}) \\ \tilde{\mathbf{Q}}^{\mathrm{T}} \otimes \mathbf{1}_m \end{bmatrix} \tag{77}$$

where $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{F}}$ are given by (38) and (39) with $\mathbf{U}$ as in equation (74).

3.2. Case (ii)

In this case equation (11) should be employed, which holds if and only if rows of the matrix $[\mathbf{1}_p - \mathbf{G}_o(s_i)\mathbf{K}]$ are linearly dependent. Thus, to each s_i corresponds a non-null p-dimensional vector $\mathbf{h}_i$, such that:

$$\mathbf{h}_i{}^{\mathrm{T}}\left[\mathbf{1}_p - \mathbf{G}_o(s_i)\mathbf{K}\right] = \mathbf{0}_{1\times p} \quad \text{dla} \quad i = 1, 2, \ldots, n \tag{78}$$

that is

$$\mathbf{h}_i{}^{\mathrm{T}}\mathbf{G}_o(s_i)\mathbf{K} = \mathbf{h}_i{}^{\mathrm{T}} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{79}$$

This can be rewritten in the matrix form as

$$\mathbf{VK} = \mathbf{H} \tag{80}$$

where

$$\mathbf{V} = \begin{bmatrix} \mathbf{h}_1{}^{\mathrm{T}}\mathbf{G}_o(s_1) \\ \mathbf{h}_2{}^{\mathrm{T}}\mathbf{G}_o(s_2) \\ \cdots \\ \mathbf{h}_n{}^{\mathrm{T}}\mathbf{G}_o(s_n) \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} \mathbf{h}_1{}^{\mathrm{T}} \\ \mathbf{h}_2{}^{\mathrm{T}} \\ \vdots \\ \mathbf{h}_n{}^{\mathrm{T}} \end{bmatrix} \tag{81}$$

are matrices of dimension $n \times m$ and $n \times p$ correspondingly. Vectors $\mathbf{h}_i$ must be chosen in such a way that the following rank condition holds

$$\operatorname{rank} \mathbf{V} = m. \tag{82}$$

According to the Kronecker-Capelli theorem, equation (80) has a solution if and only if

$$\operatorname{rank} [\mathbf{V}\ \mathbf{H}] = \operatorname{rank} \mathbf{V}. \tag{83}$$

Assuming that conditions (82) and (83) are valid, the solution of the equation (80) can be expressed as:

$$\mathbf{K} = \mathbf{V}^{\dagger}\mathbf{H} \tag{84}$$

where $\mathbf{V}^{\dagger}$ is any left generalized inverse – for example Moore-Penrose pseudo-inverse – of matrix $\mathbf{V}$:

$$\mathbf{V}^{\dagger} = \left(\mathbf{V}^{\mathrm{T}}\mathbf{V}\right)^{-1}\mathbf{V}^{\mathrm{T}}. \tag{85}$$

In view of the above, analogously to Section 3.1, the main problem consists in finding vectors $\mathbf{h}_i$, $i = 1, 2, \ldots, n$, composing matrix $\mathbf{H}$ and matrix $\mathbf{V}$ (equation (81)), such that equation (83) holds. One can achieve that by choosing arbitrarily full-rank matrix $\mathbf{R}$ of dimensions $(n - m) \times n$, for which the following equality is valid:

$$\mathbf{R}[\mathbf{V}\ \mathbf{H}] = \mathbf{0}_{(n-m)\times(m+p)} \tag{86}$$

which can be also written in the form:

$$\begin{bmatrix} \mathbf{V}^{\mathrm{T}} \\ \mathbf{H}^{\mathrm{T}} \end{bmatrix} \mathbf{R}^{\mathrm{T}} = \mathbf{0}_{(m+p)\times(n-m)}. \tag{87}$$

Equation (87) is equivalent to the pair of equations:

$$(\mathbf{R} \otimes \mathbf{1}_m)\,\mathrm{vec}\,(\mathbf{V}^{\mathrm{T}}) = \mathbf{0}_{(n-m)m\times 1} \quad (88)$$

$$(\mathbf{R} \otimes \mathbf{1}_p)\,\mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}) = \mathbf{0}_{(n-m)p\times 1}. \quad (89)$$

Taking into account that:

$$\mathrm{vec}\,(\mathbf{V}^{\mathrm{T}}) = \begin{bmatrix} \mathbf{G}_o(s_1)^{\mathrm{T}}\mathbf{h}_1 \\ \mathbf{G}_o(s_2)^{\mathrm{T}}\mathbf{h}_2 \\ \dots \\ \mathbf{G}_o(s_n)^{\mathrm{T}}\mathbf{h}_n \end{bmatrix} = \\ = \begin{bmatrix} \mathbf{G}_o^{\mathrm{T}}(s_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_o^{\mathrm{T}}(s_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{G}_o^{\mathrm{T}}(s_n) \end{bmatrix} \mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}) = \\ = \left(\mathbf{G}_o^{\mathrm{T}}(s_1) \oplus \mathbf{G}_o^{\mathrm{T}}(s_2) \oplus \dots \oplus \mathbf{G}_o^{\mathrm{T}}(s_n)\right) \mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}), \quad (90)$$

equations (89) and (90) can be rewritten as

$$\mathbf{D}\mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}) = \mathbf{0}_{(n-m)(m+p)\times 1} \quad (91)$$

where

$$\mathbf{D} = \begin{bmatrix} (\mathbf{R} \otimes \mathbf{1}_m)\left(\mathbf{G}_o^{\mathrm{T}}(s_1) \oplus \mathbf{G}_o^{\mathrm{T}}(s_2) \oplus \dots \oplus \mathbf{G}_o^{\mathrm{T}}(s_n)\right) \\ \mathbf{R} \otimes \mathbf{1}_p \end{bmatrix}. \quad (92)$$

Analogously as in Section 3.1, equation (91) enables the choice of vec $(\mathbf{H}^{\mathrm{T}})$, and consequently of matrix $\mathbf{H}$, using standard procedures for null-space computation.

In cases, where some of the demanded characteristic polynomial roots are complex, the same transformation matrix $\mathbf{U}$, as the one introduced in Section 3.1 (equation (33)) can be employed. In the case under consideration, the equivalent of equation (37), ipso facto equation (91), takes the form:

$$\begin{bmatrix} (\mathbf{RU} \otimes \mathbf{1}_m)\left(\mathbf{G}_1{}^{\mathrm{T}} \oplus \dots \oplus \mathbf{G}_r{}^{\mathrm{T}} \oplus \begin{bmatrix} \Re\mathbf{G}_{r+1}{}^{\mathrm{T}} & \Im\mathbf{G}_{r+2}{}^{\mathrm{T}} \\ \Im\mathbf{G}_{r+1}{}^{\mathrm{T}} & \Re\mathbf{G}_{r+2}{}^{\mathrm{T}} \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} \Re\mathbf{G}_{n-1}{}^{\mathrm{T}} & \Im\mathbf{G}_{n}{}^{\mathrm{T}} \\ \Im\mathbf{G}_{n-1}{}^{\mathrm{T}} & \Re\mathbf{G}_{n}{}^{\mathrm{T}} \end{bmatrix}\right) \\ \mathbf{RU} \otimes \mathbf{1}_p \end{bmatrix} \times \\ \times (\mathbf{U} \otimes \mathbf{1}_p)\,\mathrm{vec}\,\left(\mathbf{H}^{\mathrm{T}}\right) = \mathbf{0}_{(m+p)(n-m)\times 1} \quad (93)$$

where the earlier introduced, abbreviated notation for complex matrices $\mathbf{G}_o(s_i)$ (equation (34)) is used. Denoting in the following:

$$\tilde{\mathbf{R}} = \mathbf{RU} \quad (94)$$

$$(\mathbf{U} \otimes \mathbf{1}_m)\,\mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}) = \mathrm{vec}\,(\tilde{\mathbf{H}}^{\mathrm{T}}) \Leftrightarrow \tilde{\mathbf{H}} = \mathbf{UH} \quad (95)$$

and

$$\tilde{\mathbf{D}} = \begin{bmatrix} \left(\tilde{\mathbf{R}} \otimes \mathbf{1}_m\right) \left({\mathbf{G}_1}^{\mathrm{T}} \oplus \ldots \oplus {\mathbf{G}_r}^{\mathrm{T}} \oplus \begin{bmatrix} \Re{\mathbf{G}_{r+1}}^{\mathrm{T}} & \Im{\mathbf{G}_{r+2}}^{\mathrm{T}} \\ \Im{\mathbf{G}_{r+1}}^{\mathrm{T}} & \Re{\mathbf{G}_{r+2}}^{\mathrm{T}} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re{\mathbf{G}_{n-1}}^{\mathrm{T}} & \Im{\mathbf{G}_{n}}^{\mathrm{T}} \\ \Im{\mathbf{G}_{n-1}}^{\mathrm{T}} & \Re{\mathbf{G}_{n}}^{\mathrm{T}} \end{bmatrix} \right) \\ \tilde{\mathbf{R}} \otimes \mathbf{1}_p \end{bmatrix} \tag{96}$$

one obtains

$$\tilde{\mathbf{D}} \operatorname{vec} \left(\tilde{\mathbf{H}}^{\mathrm{T}}\right) = \mathbf{0}_{(m+p)(n-m)\times 1}. \tag{97}$$

At the same time

$$\operatorname{vec} (\tilde{\mathbf{V}}^{\mathrm{T}}) = \tag{98}$$
$$= \left({\mathbf{G}_1}^{\mathrm{T}} \oplus \ldots \oplus {\mathbf{G}_k}^{\mathrm{T}} \oplus \begin{bmatrix} \Re{\mathbf{G}_{k+1}}^{\mathrm{T}} & \Im{\mathbf{G}_{k+2}}^{\mathrm{T}} \\ \Im{\mathbf{G}_{k+1}}^{\mathrm{T}} & \Re{\mathbf{G}_{k+2}}^{\mathrm{T}} \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} \Re{\mathbf{G}_{n-1}}^{\mathrm{T}} & \Im{\mathbf{G}_{n}}^{\mathrm{T}} \\ \Im{\mathbf{G}_{n-1}}^{\mathrm{T}} & \Re{\mathbf{G}_{n}}^{\mathrm{T}} \end{bmatrix} \right) \operatorname{vec} (\tilde{\mathbf{H}}^{\mathrm{T}})$$

and

$$\tilde{\mathbf{V}} = \mathbf{U}\mathbf{V}. \tag{99}$$

Thus, on the basis of equations (84) and (85) it holds that

$$\mathbf{K} = \tilde{\mathbf{V}}^{\dagger}\tilde{\mathbf{H}} \tag{100}$$

where

$$\tilde{\mathbf{V}}^{\dagger} = \left(\tilde{\mathbf{V}}^{\mathrm{T}}\tilde{\mathbf{V}}\right)^{-1} \tilde{\mathbf{V}}^{\mathrm{T}}. \tag{101}$$

Remark 2.

In the special case, where $m = n$, matrix $\mathbf{V}$ becomes square. Consequently, the resolution of equation (80) is much simpler. Then, one must choose vectors $\mathbf{h}_i$, $i = 1, 2, \ldots, n$, for which the following condition is valid:

$$\det \mathbf{V} \neq 0, \tag{102}$$

which enables the computation of matrix $\mathbf{K}$ from equation

$$\mathbf{K} = \mathbf{V}^{-1}\mathbf{H}. \tag{103}$$

In turn, when $p = 1$, which, due to assumption (3) implies $m = n$, vectors $\mathbf{h}_i,\ i = 1, 2, \ldots, n$, become scalars, and equation (80) takes the form

$$\begin{bmatrix} \mathbf{G}_o(s_1) \\ \mathbf{G}_o(s_2) \\ \vdots \\ \mathbf{G}_o(s_n) \end{bmatrix} \mathbf{K} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{104}$$

In this case $\mathbf{K}$ can be computed, when:

$$\det\left[\mathbf{G}_o{}^{\mathrm{T}}(s_1)\ \mathbf{G}_o{}^{\mathrm{T}}(s_2)\ \dots\ \mathbf{G}_o{}^{\mathrm{T}}(s_n)\right] \neq 0. \tag{105}$$

In cases, where the required roots s_i are not simple, namely have multiplicities ν_i, $\quad i = 1, 2, \dots, q$, the following conditions must hold:

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[\mathbf{1}_p - \mathbf{G}_o(s)\mathbf{K}\right]\Bigg|_{s=s_i} = 0 \quad \text{dla} \quad i = 1, 2, \dots, q, \quad k = 0, 1, \dots, \nu_i - 1. \tag{106}$$

Using the formulae from Fahmy and O'Reilly (1983) and the notation introduced in Section 3.1, one obtains from equation (106):

$$\begin{gathered}(\mathbf{h}_i^{(0)})^{\mathrm{T}}\mathbf{G}_i^{(0)}\mathbf{K} = (\mathbf{h}_i^{(0)})^{\mathrm{T}} \\ \dots \\ \left[(\mathbf{h}_i^{(0)})^{\mathrm{T}}\mathbf{G}_i^{(k)} + (\mathbf{h}_i^{(1)})^{\mathrm{T}}\mathbf{G}_i^{(k-1)} + \dots + (\mathbf{h}_i^{(k-1)})^{\mathrm{T}}\mathbf{G}_o^{(1)} + (\mathbf{h}_i^{(k)})^{\mathrm{T}}\mathbf{G}_o^{(0)}\right]\mathbf{K} = (\mathbf{h}_i^{(k)})^{\mathrm{T}} \\ i = 1, 2, \dots, q \quad k = 0, 1, \dots, \nu_i - 1.\end{gathered} \tag{107}$$

Thus, matrices $\mathbf{H}$ and $\mathbf{V}$ can be expressed as

$$\mathbf{H} = \begin{bmatrix}\mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_q\end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix}\mathbf{V}_1 \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_q\end{bmatrix} \tag{108}$$

where subsequent blocks $\mathbf{H}_i$ and $\mathbf{V}_i$ of dimension $\nu_i \times p$ i $\nu_i \times p$ are built of vectors $\mathbf{h}_i^{(k)}$ and $\mathbf{v}_i^{(k)}$, $i = 1, 2, \dots, q, \quad k = 0, 1, \dots, \nu_i - 1$, i.e.

$$\mathbf{H}_i = \begin{bmatrix}(\mathbf{h}_i^{(0)})^{\mathrm{T}} \\ (\mathbf{h}_i^{(1)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{h}_i^{(\nu_i-1)})^{\mathrm{T}}\end{bmatrix} \qquad \mathbf{V}_i = \begin{bmatrix}(\mathbf{v}_i^{(0)})^{\mathrm{T}} \\ (\mathbf{v}_i^{(1)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{v}_i^{(\nu_i-1)})^{\mathrm{T}}\end{bmatrix} \tag{109}$$

and, as follows from equation (107)

$$\begin{gathered}\mathbf{v}_i^{(k)} = (\mathbf{h}_i^{(0)})^{\mathrm{T}}\mathbf{G}_i^{(k)} + (\mathbf{h}_i^{(1)})^{\mathrm{T}}\mathbf{G}_i^{(k-1)} + \dots + (\mathbf{h}_i^{(k-1)})^{\mathrm{T}}\mathbf{G}_i^{(1)} + (\mathbf{h}_i^{(k)})^{\mathrm{T}}\mathbf{G}_i^{(0)} \\ \text{for} \quad i = 1, 2, \dots, q, \quad k = 0, 1, \dots, \nu_i - 1.\end{gathered} \tag{110}$$

Using equations (108)–(110) one can compute matrix $\mathbf{K}$ from formulae (84) and (85). This requires the prior choice of $\mathbf{h}_i^{(k)}$ for $i = 1, 2, \dots, q$; $k = 0, 1, \dots, \nu_i - 1$.

For this purpose equation (91) must be transformed, according to (88) and (89), holding in this case. From equation (108) it follows that

$$\mathrm{vec}\,(\mathbf{H}^{\mathrm{T}}) = \begin{bmatrix} \mathrm{vec}\,({\mathbf{H}_1}^{\mathrm{T}}) \\ \mathrm{vec}\,({\mathbf{H}_2}^{\mathrm{T}}) \\ \vdots \\ \mathrm{vec}\,({\mathbf{H}_q}^{\mathrm{T}}) \end{bmatrix} \qquad \mathrm{vec}\,(\mathbf{V}^{\mathrm{T}}) = \begin{bmatrix} \mathrm{vec}\,({\mathbf{V}_1}^{\mathrm{T}}) \\ \mathrm{vec}\,({\mathbf{V}_2}^{\mathrm{T}}) \\ \vdots \\ \mathrm{vec}\,({\mathbf{V}_q}^{\mathrm{T}}) \end{bmatrix} \tag{111}$$

and, from equation (109), that

$$\mathrm{vec}\,({\mathbf{H}_i}^{\mathrm{T}}) = \begin{bmatrix} \mathbf{h}_i^{(0)} \\ \mathbf{h}_i^{(1)} \\ \vdots \\ \mathbf{h}_i^{(\nu_i-1)} \end{bmatrix}. \tag{112}$$

At the same time, the following holds:

$$\mathrm{vec}\,({\mathbf{V}_i}^{\mathrm{T}}) = \begin{bmatrix} \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0} & \cdots & \mathbf{0} \\ \left(\mathbf{G}_i^{(2)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \left(\mathbf{G}_i^{(\nu_i-1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(\nu_i-2)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(\nu_i-3)}\right)^{\mathrm{T}} & \cdots & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} \end{bmatrix} \mathrm{vec}\,({\mathbf{H}_i}^{\mathrm{T}}) \tag{113}$$

which, using notation:

$$\mathbf{E}_i = \begin{bmatrix} \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0} & \ldots & \mathbf{0} \\ \left(\mathbf{G}_i^{(2)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \ldots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \left(\mathbf{G}_i^{(\nu_i-1)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(\nu_i-2)}\right)^{\mathrm{T}} & \left(\mathbf{G}_i^{(\nu_i-3)}\right)^{\mathrm{T}} & \ldots & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} \end{bmatrix} \tag{114}$$

can be written as

$$\mathrm{vec}\,({\mathbf{V}_i}^{\mathrm{T}}) = \mathbf{E}_i \mathrm{vec}\,({\mathbf{H}_i}^{\mathrm{T}}) \tag{115}$$

and thereby

$$\operatorname{vec}(\mathbf{V}^{\mathrm{T}}) = \begin{bmatrix} \mathbf{E}_1 & & & \\ & \mathbf{E}_2 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \ddots & \\ & & & & \mathbf{E}_q \end{bmatrix} \operatorname{vec}(\mathbf{H}^{\mathrm{T}}) =$$
$$= (\mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \ldots \oplus \mathbf{E}_q) \operatorname{vec}(\mathbf{H}^{\mathrm{T}}). \quad (116)$$

Thus, in the case under consideration, with valid (91), it follows that:

$$\mathbf{D} = \begin{bmatrix} (\mathbf{R} \otimes \mathbf{1}_m)\,(\mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \ldots \oplus \mathbf{E}_q) \\ (\mathbf{R} \otimes \mathbf{1}_p) \end{bmatrix} \quad (117)$$

which enables the computation of **H** and **V**.

In turn, when some of the required roots are multiple and complex, similarly to Section 3.1 it is convenient to order vectors $(\mathbf{h}_i^{(k)})^{\mathrm{T}}$ and $\mathbf{v}_i^{(k)}$ according to the following scheme. For roots s_i and $s_{i+1} = \bar{s}_i$ matrices:

$$\mathbf{H}_{i,i+1} = \begin{bmatrix} (\mathbf{h}_i^{(0)})^{\mathrm{T}} \\ (\mathbf{h}_{i+1}^{(0)})^{\mathrm{T}} \\ (\mathbf{h}_i^{(1)})^{\mathrm{T}} \\ (\mathbf{h}_{i+1}^{(1)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{h}_i^{(\nu_i-1)})^{\mathrm{T}} \\ (\mathbf{h}_{i+1}^{(\nu_i-1)})^{\mathrm{T}} \end{bmatrix} \qquad \mathbf{V}_{i,i+1} = \begin{bmatrix} (\mathbf{v}_i^{(0)})^{\mathrm{T}} \\ (\mathbf{v}_{i+1}^{(0)})^{\mathrm{T}} \\ (\mathbf{v}_i^{(1)})^{\mathrm{T}} \\ (\mathbf{v}_{i+1}^{(1)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{v}_i^{(\nu_i-1)})^{\mathrm{T}} \\ (\mathbf{v}_{i+1}^{(\nu_i-1)})^{\mathrm{T}} \end{bmatrix} \quad (118)$$

are introduced, which implies

$$\operatorname{vec}(\mathbf{H}_i{}^{\mathrm{T}}) = \begin{bmatrix} \mathbf{h}_i^{(0)} \\ \mathbf{h}_{i+1}^{(0)} \\ \mathbf{h}_i^{(1)} \\ \mathbf{h}_{i+1}^{(1)} \\ \vdots \\ \mathbf{h}_i^{(\nu_i-1)} \\ \mathbf{h}_{i+1}^{(\nu_i-1)} \end{bmatrix}. \quad (119)$$

Jointly this enables us to write

$$\operatorname{vec}\left(\mathbf{V}_{i,i+1}{}^{\mathrm{T}}\right) = \\ = \begin{bmatrix} \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(1)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(\mathbf{G}_i^{(\nu_i-1)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \left(\mathbf{G}_i^{(\nu_i-2)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \cdots & \left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} \\ \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(\nu_i-1)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(\nu_i-2)}\right)^{\mathrm{T}} & \cdots & \mathbf{0}_{m\times p} & \left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} \end{bmatrix} \times \\ \times \operatorname{vec}\left(\mathbf{H}_{i,i+1}{}^{\mathrm{T}}\right). \tag{120}$$

Assuming (73) and using transformation (74) and equations (91), (95), (113) one obtains

$$\operatorname{vec}\left(\tilde{\mathbf{V}}^{\mathrm{T}}\right) = \left(\mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \ldots \oplus \mathbf{E}_r \oplus \mathbf{E}_{r+1,r+2} \oplus \ldots \oplus \mathbf{E}_{q-1,q}\right) \operatorname{vec}\left(\tilde{\mathbf{H}}^{\mathrm{T}}\right) \tag{121}$$

where, according to equation (120):

$$\mathbf{E}_{i,i+1} = \\ = \begin{bmatrix} \Re\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \Im\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \Re\left(\mathbf{G}_i^{(1)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(1)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \Im\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(1)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} & \cdots & \mathbf{0}_{m\times p} & \mathbf{0}_{m\times p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Re\left(\mathbf{G}_i^{(\nu_i-1)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(\nu_i-1)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_i^{(\nu_i-2)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(\nu_i-2)}\right)^{\mathrm{T}} & \cdots & \Re\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} \\ \Im\left(\mathbf{G}_i^{(\nu_i-1)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(\nu_i-1)}\right)^{\mathrm{T}} & \Im\left(\mathbf{G}_i^{(\nu_i-2)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(\nu_i-2)}\right)^{\mathrm{T}} & \cdots & \Im\left(\mathbf{G}_i^{(0)}\right)^{\mathrm{T}} & \Re\left(\mathbf{G}_{i+1}^{(0)}\right)^{\mathrm{T}} \end{bmatrix}. \tag{122}$$

Thus in the case under consideration, conditions enabling the proper choice of matrix $\tilde{\mathbf{H}}$ reduce to equation (97), where

$$\tilde{\mathbf{D}} = \begin{bmatrix} \left(\tilde{\mathbf{R}} \otimes \mathbf{1}_m\right) \left(\mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \ldots \oplus \mathbf{E}_r \oplus \mathbf{E}_{r+1,r+2} \oplus \ldots \oplus \mathbf{E}_{q-1,q}\right) \\ \left(\tilde{\mathbf{R}} \otimes \mathbf{1}_p\right) \end{bmatrix} \tag{123}$$

with $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{H}}$ given by (94) and (95).

4. Algorithm of computation

On the basis of the above considerations and the derived formulae, one can propose the following algorithm in order to computationally solve the stated problem:

[1] Input data: n, m, p, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, q, r

[2] Input required set of characteristic equation roots, ordered as in Section 3:

(i) $\Lambda = \{s_1, s_2, \dots, s_n\}$

(ii) $\Lambda = \{s_1, s_2, \dots, s_q\}$ and ν_i for $i = 1, 2, \dots, q$

and specify the number of real roots r; set:

- $Z = 1$ for simple real roots;
- $Z = 2$ for simple real and complex roots;
- $Z = 3$ for multiple real roots;
- $Z = 4$ for multiple real and complex roots;

[3] Check:

(i) condition (3)

(ii) whether $\mathbf{B}$ and $\mathbf{C}$ have full rank

(iii) whether the system is fully controllable and observable

If any of the above conditions fail, then quit, else go to the next step

[4] Calculate matrices $\mathbf{G}_o(s_i)$ for $i = 1, 2, \dots, n$ using equation (4)

[5] If $Z = 1$, then go to next step, else jump to step [20]

[6] If $m = 1$, then go to next step, else jump to step [9]

[7] Check condition (49); if *false* then quit

[8] Compute $\mathbf{K}$ from (48) and quit

[9] If $p = n$, then go to next step, else jump to step [13]

[10] Choose random m–dimensional vectors $\mathbf{f}_i$ for $i = 1, 2, \dots, n$

[11] Compute matrix $\mathbf{W}$ from equation (16)

[12] Check condition (46); If *true*, then compute matrix $\mathbf{K}$ from equation (47) and quit, else jump to step [10]

[13] Choose randomly matrix $\mathbf{Q}$ of dimensions $n \times (n-p)$ and compute matrix $\mathbf{L}$ from equation (29)

[14] Find null-space of matrix $\mathbf{L}$ (in Matlab function null($\cdot$))

[15] Compute vec ($\mathbf{F}$), using any vector from basis computed in [14]

[16] Transform vec ($\mathbf{F}$) into $\mathbf{F}$ (in Matlab function reshape($\cdot$)) and corresponding vectors $\mathbf{f}_i$

[17] Compute matrix $\mathbf{W}$ from equation (16)

[18] Check condition (19); if *true*, then go to next step, else jump (return) to step [13]

[19] Compute matrix $\mathbf{W}^{\dagger}$ from (21), and then matrix $\mathbf{K}$ from (20); quit

[20] If $Z = 2$, then go to next step, else jump to [29]

[21] For $i = r+1, r+2, \ldots, n$ compute $\Re\mathbf{G}_i$ and $\Im\mathbf{G}_i$ from (34)

[22] Choose randomly $\tilde{\mathbf{Q}}$ of dimensions $n \times (n-p)$

[23] Compute matrix $\tilde{\mathbf{L}}$ from equation (40)

[24] Compute null-space of matrix $\tilde{\mathbf{L}}$ (null$(\cdot)$)

[25] Compute vec $(\tilde{\mathbf{F}})$, choosing any of the vectors from basis computed in step [24], and then transform into $\tilde{\mathbf{F}}$

[26] Compute vec $(\tilde{\mathbf{W}})$ from equation (42) and then transform into $\tilde{\mathbf{W}}$

[27] Check condition (19), substituting $\mathbf{F} \rightarrow \tilde{\mathbf{F}}$ and $\mathbf{W} \rightarrow \tilde{\mathbf{W}}$. If *true*, go to next step, else jump to step [22]

[28] Compute $\tilde{\mathbf{W}}^{\dagger}$ from equation (45) and then $\mathbf{K}$ from (44). Quit

[29] For $i = 1, 2, \ldots, q$, $k = 0, 1, \ldots, \nu_i - 1$ compute matrices $\mathbf{G}_i^{(k)}$ from equation (56)

[30] If $Z = 3$, go to next step, else jump to step [40]

[31] For $i = 1, 2, \ldots, q$ assembly matrices $\mathbf{P}_i$ according to (64)

[32] Choose randomly matrix $\mathbf{Q}$ of dimensions $n \times (n-p)$

[33] Compute matrix $\mathbf{L}$ from equation (69)

[34] Find null-space of matrix $\mathbf{L}$ (null$(\cdot)$)

[35] Compute vec $(\mathbf{F})$, choosing any of the vectors from basis computed in step [34], and then transform into matrix $\mathbf{F}$

[36] For $i = 1, 2, \ldots, q$; $k = 0, 1, \ldots, \nu_i - 1$ compute vectors $\mathbf{w}_i^{(k)}$ using equation (62)

[37] For $i = 1, 2, \ldots, q$ assembly matrices $\mathbf{W}_i$ using equation (61); assembly matrix $\mathbf{W}$ using (59)

[38] Check condition (19). If *true*, then go to next step, else jump to step [32]

[39] Compute matrix $\mathbf{W}^{\dagger}$ using equation (21); compute matrix $\mathbf{K}$ using equation (20). Quit

[40] For $i = r+1, r+2, \ldots, q$, $k = 0, 1, \ldots, \nu_i - 1$ compute $\Re\mathbf{G}_i^{(k)} = \mathrm{Re}\mathbf{G}_i^{(k)}$, $\Im\mathbf{G}_i^{(k)} = \mathrm{Im}\mathbf{G}_i^{(k)}$

[41] For $i = 1, 2, \ldots, r$ compute matrices $\mathbf{P}_i$ from equation (64) and for $i = r+1, r+3, \ldots, q-1$ compute matrices $\mathbf{P}_{i,i+1}$ from equation (76)

[42] Choose randomly matrix $\tilde{\mathbf{Q}}$ of dimensions $n \times (n-p)$

[43] Compute matrix $\tilde{\mathbf{L}}$ using equation (77)

[44] Compute null-space of matrix $\tilde{\mathbf{L}}$ (null$(\cdot)$)

[45] Determine vec $(\tilde{\mathbf{F}})$, choosing any of the vectors from computed null-space basis and then transform into matrix $\tilde{\mathbf{F}}$

[46] Compute vec $(\tilde{\mathbf{W}})$ using equation (75), and then transform into matrix $\tilde{\mathbf{W}}$

[47] Check condition (19), substituting $\mathbf{F} \to \tilde{\mathbf{F}}$ and $\mathbf{W} \to \tilde{\mathbf{W}}$. If *true*, then go to next step, else jump to step [42]

[48] Compute matrix $\tilde{\mathbf{W}}^{\dagger}$ using equation (45) and matrix $\mathbf{K}$ using equation (44). Quit

Another variant of the algorithm is as follows:

[1] Input data: n, m, p, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, q, r

[2] Input required set of characteristic equation roots, ordered as in Section 3:

(i) $\Lambda = \{s_1, s_2, \ldots, s_n\}$

(ii) $\Lambda = \{s_1, s_2, \ldots, s_q\}$ i ν_i for $i = 1, 2, \ldots, q$

and specify the number of real roots r; set:

- $Z = 1$ for simple real roots;
- $Z = 2$ for simple real and complex roots;
- $Z = 3$ for multiple real roots;
- $Z = 4$ for multiple real and complex roots;

[3] Check:

(i) condition (3)

(ii) whether $\mathbf{B}$ and $\mathbf{C}$ have full rank

(iii) whether system is fully controllable and observable

If any of the above conditions fail then quit, else go to next step

[4] Calculate matrices $\mathbf{G}_o(s_i)$ for $i = 1, 2, \ldots, n$ using equation (4)

[5] If $Z = 1$, then go to next step, else go to step [20]

[6] If $p = 1$, then go to next step, else jump to step [9]

[7] Check condition (105). If *true*, then go to next step, else quit

[8] Compute matrix $\mathbf{K}$ using equation (104) and quit

[9] If $m = n$, then go to next step, else jump to step [13]

[10] For $i = 1, 2, \ldots, n$ choose randomly a p-dimensional vector $\mathbf{h}_i$

[11] Assembly matrix $\mathbf{H}$ and compute matrix $\mathbf{V}$ using equation (81)

[12] Check condition (102). If *true*, then compute matrix $\mathbf{K}$ using equation (103) and quit, else jump to step [10]

[13] Choose randomly matrix $\mathbf{R}$ of dimensions $(n-m) \times n$

[14] Compute matrix $\mathbf{D}$ using equation (92)

[15] Compute null-space of the matrix $\mathbf{D}$ (null($\cdot$))

[16] Determine vec ($\mathbf{H}^{\mathrm{T}}$), choosing any of the vectors from basis computed in step [15].

[17] Calculate matrix $\mathbf{H}^{\mathrm{T}}$ (reshape($\cdot$)) and transform it into vectors ${\mathbf{h}_i}^{\mathrm{T}}$ $i = 1, 2, \ldots, n$; then calculate matrices $\mathbf{H}$ i $\mathbf{V}$ using (81)

[18] Check condition (83). If *true*, then go to next step, else go to step [13]

[19] Calculate matrix $\mathbf{V}^{\dagger}$ using (85), then matrix $\mathbf{K}$ using equation (84). Quit

[20] Check condition $Z = 2$. If *true*, then go to next step, else go to step [30]

[21] For $i = r+1, r+2, \ldots, n$ determine $\Re\mathbf{G}_i$ and $\Im\mathbf{G}_i$ using equation (34)

[22] Choose randomly matrix $\tilde{\mathbf{R}}$ of dimensions $(n-m) \times n$

[23] Calculate matrix $\tilde{\mathbf{D}}$, using equation (96)

[24] Determine basis of null-space of matrix $\tilde{\mathbf{D}}$ calculated in [23] (null($\cdot$))

[25] Determine vector vec ($\tilde{\mathbf{H}}^{\mathrm{T}}$), choosing any of vectors from basis, calculated in [24]; then transform it into matrix $\tilde{\mathbf{H}}^{\mathrm{T}}$ (reshape($\cdot$))

[26] Calculate vec ($\tilde{\mathbf{V}}^{\mathrm{T}}$), using equation (98) and transform into matrix $\tilde{\mathbf{V}}^{\mathrm{T}}$.

[27] Check condition (83), substituting $\mathbf{V} \to \tilde{\mathbf{V}}$ and $\mathbf{H} \to \tilde{\mathbf{H}}$. If true, then go to next step, else jump to step [22]

[28] Calculate matrix $\tilde{\mathbf{V}}^{\dagger}$ using (101), and then $\mathbf{K}$, using (100). Quit

[29] For $i = 1, 2, \ldots, q$, $k = 0, 1, \ldots, \nu_i - 1$ calculate matrices $\mathbf{G}_i^{(k)}$, using equation (56)

[30] Check condition $Z = 3$. If *true*, then go to next step, else jump to step [40]

[31] For $i = 1, 2, \ldots, q$ calculate $\mathbf{E}_i$, using equation (114)

[32] Choose randomly matrix $\mathbf{R}$ of dimensions $(n-m) \times n$

[33] Calculate matrix $\mathbf{D}$, using equation (117)

[34] Find basis of null-space of matrix $\mathbf{D}$ calculated in step [33] (null ($\cdot$))

[35] Determine vector vec ($\mathbf{H}^{\mathrm{T}}$), choosing any of vectors from basis calculated in step [34]

[36] Transform vec ($\mathbf{H}^{\mathrm{T}}$) into matrix $\mathbf{H}^{\mathrm{T}}$ (reshape($\cdot$)).

[37] Calculate vector vec $(\mathbf{V}^{\mathrm{T}})$, using equation (116) and then transform it into matrix $\mathbf{V}^{\mathrm{T}}$

[38] Check condition (83). If *true*, then go to next step, else jump to step [32]

[39] Calculate matrix $\mathbf{V}^{\dagger}$, using equation (85), and then matrix $\mathbf{K}$, using (84). Quit

[40] For $i = 1, 2, \ldots, r$ calculate matrices $\mathbf{E}_i$, using equation (114) and then for $i = r+1, r+3, \ldots, q-1$, matrices $\mathbf{E}_{i,i+1}$, using equation (122)

[41] Choose randomly matrix $\tilde{\mathbf{R}}$ of dimensions $(n-m) \times n$

[42] Calculate matrix $\tilde{\mathbf{D}}$, using equation (123)

[43] Compute basis of null-space of the matrix $\tilde{\mathbf{D}}$ calculated in step [42]

[44] Determine vector vec $(\tilde{\mathbf{H}}^{\mathrm{T}})$, choosing any of vectors from basis calculated in step [43], and then transform it into matrix $\tilde{\mathbf{H}}^{\mathrm{T}}$

[45] Compute vec $(\tilde{\mathbf{V}}^{\mathrm{T}})$, using equation (121) and then transform it into matrix $\tilde{\mathbf{V}}^{\mathrm{T}}$

[46] Check condition (83), substituting $\mathbf{V} \to \tilde{\mathbf{V}}$ and $\mathbf{H} \to \tilde{\mathbf{H}}$. If true, then go to next step, else jump to step [41]

[47] Compute matrix $\tilde{\mathbf{V}}^{\dagger}$, using equation (101), and then matrix $\mathbf{K}$, using equation (100). Quit.

5. Computational example

The following example illustrates the derived formulae and the proposed algorithm. Linear time-invariant system is characterized by matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & -6 & 68 & 4 \\ 0 & -13 & -19 & -21 \\ -25 & -61 & -16 & 1 \\ -2 & -30 & 12 & 13 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -10 & -21 \\ 15 & 3 \\ 48 & 23 \\ 36 & -1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 68 & 17 & 5 & -12 \\ -11 & 31 & -43 & -69 \\ -28 & -16 & -23 & 46 \end{bmatrix}.$$

Its roots of characteristic polynomial are:

$$\Lambda_o = \{31.7311, -2.08944 + 29.1279j, -2.08944 - 29.1279j, -38.5523\}.$$

Required values of close-loop roots are:

$$\Lambda = \{-2 + 1j, -2 - 1j, -2 + 1j, -2 - 1j\}.$$

For this system all posed assumptions are valid, i.e. condition (3) holds, matrices $\mathbf{B}$ and $\mathbf{C}$ have full rank, and the system is fully controllable and observable.

According to the proposed algorithm the following matrix $\mathbf{Q}$ was (randomly) chosen

$$\mathbf{Q} = \begin{bmatrix} 59 & 6 & -12 & -43 \end{bmatrix}^{\mathrm{T}}$$

and then matrix

$$\tilde{\mathbf{L}} = \begin{bmatrix} -4336.72 & 3508.72 & -306.133 & 277.317 & 1111.79 & -847.015 & 3121.41 & -2555.44 \\ 3677.2 & -983.372 & 513.572 & -61.5167 & -513.123 & 264.406 & -2695.66 & 716.128 \\ -304.706 & -1808.4 & -122.105 & -208.081 & -92.0386 & 326.533 & 237.603 & 1325.98 \\ 59 & 0 & 6 & 0 & -12 & 0 & -43 & 0 \\ 0 & 59 & 0 & 6 & 0 & -12 & 0 & -43 \end{bmatrix}$$

was computed. Its null-space has basis

$$\text{null } \tilde{\mathbf{L}} = \begin{bmatrix} -0.0309995 & 0.500434 & -0.131405 \\ 0.218891 & -0.214713 & 0.240127 \\ 0.130391 & -0.422021 & -0.722815 \\ -0.826464 & -0.17031 & -0.0211482 \\ -0.0772983 & 0.304622 & 0.388822 \\ 0.492791 & 0.0353569 & 0.0700755 \\ -0.00276844 & 0.542745 & -0.389666 \\ 0.0474948 & -0.328237 & 0.30697 \end{bmatrix}.$$

Taking into consideration the subsequent columns of null $\tilde{\mathbf{L}}$ three variants of matrix $\tilde{\mathbf{F}}$ and corresponding matrix $\tilde{\mathbf{W}}$ were computed, which eventually brought to three feedback matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ assigning characteristic roots to required positions. Partial and final results of those computations are given below:

$$\tilde{\mathbf{F}}_1 = \begin{bmatrix} -0.0309995 & 0.130391 & -0.0772983 & -0.00276844 \\ 0.218891 & -0.826464 & 0.492791 & 0.0474948 \end{bmatrix}$$

$$\tilde{\mathbf{W}}_1 = \begin{bmatrix} 15.8136 & -58.2195 & 34.8337 & 3.85307 \\ -5.65982 & 21.794 & -12.993 & -1.09882 \\ -6.49622 & 24.6665 & -14.7286 & -1.36128 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} -0.00558095 & -0.00822899 & -0.011188 \\ -0.0193672 & 0.101137 & -0.101566 \end{bmatrix} \Rightarrow$$
$$\Rightarrow \Lambda = \{-2 + 1.00033j, -2 - 1.00033j, -2 + 0.999671j, -2 - 0.999671j\}$$

$$\tilde{\mathbf{F}}_2 = \begin{bmatrix} 0.500434 & -0.422021 & 0.304622 & 0.542745 \\ -0.214713 & -0.17031 & 0.0353569 & -0.328237 \end{bmatrix}$$

$$\tilde{\mathbf{W}}_2 = \begin{bmatrix} -49.572 & 21.1556 & -18.557 & -59.8867 \\ 34.555 & -23.6016 & 19.5306 & 38.669 \\ 4.13779 & 7.45322 & -3.40226 & 7.66688 \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} 0.00972818 & -0.00290036 & 0.0198253 \\ 0.00858005 & 0.0135605 & 0.0414376 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \Lambda = \{-2.00001 + 1j, -2.00001 - 1j, -1.99999 + 0.999997j,$$
$$-1.99999 - 0.999997j\}$$

$$\tilde{\mathbf{F}}_3 = \begin{bmatrix} -0.131405 & -0.722815 & 0.388822 & -0.389666 \\ 0.240127 & -0.0211482 & 0.0700755 & 0.30697 \end{bmatrix}$$

$$\tilde{\mathbf{W}}_3 = \begin{bmatrix} 23.0971 & 51.1517 & -25.6237 & 45.9797 \\ -12.9863 & -45.4946 & 22.4482 & -30.4311 \\ -6.15837 & 4.85962 & -3.74814 & -6.72578 \end{bmatrix}$$

$$\mathbf{K}_3 = \begin{bmatrix} 0.00985002 & -0.00256779 & 0.0210199 \\ 0.00818552 & 0.0132078 & 0.0418405 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \Lambda = \{-2 + 1j, -2 - 1j, -2 + 0.999997j, -2 - 0.999997j\}.$$

6. Conclusion

In this paper effective formulae have been developed, enabling arbitrary pole placement by static output feedback matrix $\mathbf{K}$ for the system described by equations (1) and (2). All – real and complex, simple and multiple – poles distributions are covered. To accomplish this, generalized inverses and the Kronecker product were used. This enabled the development of equations for matrices $\mathbf{F}$ or $\mathbf{H}$ (called by other authors parameter matrices), that ensure existence of the feedback matrix $\mathbf{K}$ which is the solution of the problem. Computations can be carried out using standard procedures for matrix null-space calculation.

On the basis of the developed formulae, two new algorithms were proposed. Their specific feature is a one-stage structure, in opposition to the two-stage nature of most known algorithms. It should be noted here that the first algorithm should be used when $p \geqslant m$, and the second when $m \geqslant p$. This ensures smaller dimensions of matrices $\mathbf{L}$ or $\tilde{\mathbf{L}}$ in comparison to $\mathbf{D}$ or $\tilde{\mathbf{D}}$ in the first case and vice-versa in second. Moreover, in one cycle of computations one can obtain several feedback matrices $\mathbf{K}$, according to the number of column of matrices null $\mathbf{L}$ or null $\mathbf{D}$, which implies $(m + p - n)p$ variants in the first and $(m + p - n)m$ variants in the second case. Therefore it is possible to use some quality criterion e.g.

$$\|\mathbf{K}\| = \sqrt{\text{trace }(\mathbf{K}\mathbf{K}^{\mathrm{T}})} \tag{124}$$

to choose the best variant. Of course it is also possible to search for the best solutions, considering all linear combinations of columns of matrices null $\mathbf{L}$ or null $\mathbf{D}$.

Simultaneously it should be noted that in general, $mp-n$ degrees of freedom occur in the problem under consideration. This is reflected in the free choice of elements of matrices $\mathbf{Q}$ or $\mathbf{R}$ by application of the proposed methods.

The proposed method has the disadvantage that the set of desired poles Λ cannot contain roots of equation (12). This requirement can be eliminated by certain modifications of the formulae used in the presented method. For this purpose closed-loop eigenvectors must be taken into consideration, which – according to Fahmy and O'Reilly (1983) – are given by the formulae:

$$\mathbf{v}_i = (s_i \mathbf{1}_n - \mathbf{A})^{-1} \mathbf{B}\mathbf{f}_i \qquad \text{(right)} \tag{125}$$

$$\mathbf{g}_i{}^{\mathrm{T}} = \mathbf{h}_i{}^{\mathrm{T}} \mathbf{C} (s_i \mathbf{1}_n - \mathbf{A})^{-1} \qquad \text{(left)}. \tag{126}$$

In this particular case of limited practical importance, the methods introduced in Clarke et al., (2003), Clarke and Griffin (2004) and Askarpour and Owens (1997, 1998) can be applied.

The other potential disadvantage of the proposed method is the increase of problem dimensionality due to use of the Kronecker product. But the Kronecker product is becoming a commonly used tool in various areas, e.g. digital signal processing, image processing, semidefinite programming, quantum computing (Van Loan, 2000). One can expect that, due to increasing computational power of today's computers, this inconvenience will decline in importance.

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