Control and Cybernetics

vol. **36** (2007) No. 4

Generalized directional derivatives for locally Lipschitz functions which satisfy Leibniz rule

by

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Dedicated to the 75th Birthday of Professor Stefan Rolewicz[*]

Abstract: In this paper a concept of a generalized directional derivative, which satisfies Leibniz rule is proposed for locally Lipschitz functions, defined on an open subset of a Banach space. Although Leibniz rule is of less importance for a subdifferential calculus, it is of course of some theoretical interest to know about the existence of generalized directional derivatives which satisfy Leibniz rule. The proposed concept of generalized directional derivatives is adopted from the work of D. R. Sherbert (1964) who determined all point derivations for the Banach algebra of Lipschitz functions over a complete metric space.

Keywords: generalized directional derivatives, point derivations, Lipschitz functions.

1. Introduction

A derivative concept for a class of functions $f : X \to \mathbb{R}$ defined on a subset X of a real Banach space E may be viewed as an operator $\mathcal{D}$, which assigns reals $\mathcal{D}(f, x, v)$ to triples (f, x, v), where f is an element of a function class, $x \in X$, and $v \in E$. In applications, derivative concepts are mostly used as approximation tools for a function f in a neighborhood of a point x. Therefore studies of derivative concepts in nonsmooth analysis usually focus on the approximation

[*]This paper has not appeared due to purely technical reasons in the special issue of *Control and Cybernetics*, published in honour of Stefan Rolewicz (No. 3, 2007).

properties of the function $\mathcal{D}(f,.,.)$ rather than on its algebraic properties. However, if one is interested in characterizing the algebraic properties of a derivative concept, Leibniz rule is of particular importance.

We begin with a short review of some properties of the algebra of Lipschitz functions and on Sherbert's (1964) construction of point derivations. Then we propose a concept of a generalized directional derivative for locally Lipschitz functions which are defined on open subsets of real Banach spaces and study their properties. The case of convex functions is particularly considered.

2. The Lipschitz algebra

The following brief description of the Banach algebra of Lipschitz functions follows the presentation given in Sherbert (1964). Let (X,d) be a metric space. A function $f : X \longrightarrow \mathbb{R}$ is called a *Lipschitz function* if there exists a constant $K \geq 0$ such that for all $x, y \in X$ the inequality

$$|f(x) - f(y)| \leq K d(x,y)$$

holds. The set of all bounded Lipschitz functions defined on (X,d) is a real algebra and will be denoted by $\mathrm{Lip}(X,d)$. For $f \in \mathrm{Lip}(X,d)$ the following two constants are finite:

$$\|f\|_d := \sup\{\frac{|f(x) - f(y)|}{d(x,y)} \ : \ x, y \in X, \ x \neq y \ \} \ \text{ and}$$
$$\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X \ \}.$$

A norm on the space $\mathrm{Lip}(X,d)$ is defined by

$$\|f\| := \|f\|_d \ + \ \|f\|_\infty.$$

Arens and Eells (1956) show that $(\mathrm{Lip}(X,d), \|\cdot\|)$ is always the dual space of some normed linear space and hence complete. Moreover, $(\mathrm{Lip}(X,d), \|\cdot\|)$ is a Banach algebra, i.e, for all $f, g \in \mathrm{Lip}(X,d)$ the inequality $\|fg\| \leq \|f\| \ \|g\|$ holds. This follows from the inequality

$$\frac{|(fg)(x) - (fg)(y)|}{d(x,y)} \leq |f(x)| \frac{|(g)(x) - (g)(y)|}{d(x,y)} \ + \ |g(y)| \frac{|(f)(x) - (f)(y)|}{d(x,y)}$$

which implies

$$\|fg\|_d \leq \|f\|_\infty \cdot \|g\|_d \ + \ \|g\|_\infty \cdot \|f\|_d$$

and thus yields

$$\begin{aligned} \|fg\| &= \|fg\|_\infty \ + \ \|fg\|_d \\ &\leq \|f\|_\infty \cdot \|g\|_\infty \ + \ \|f\|_\infty \cdot \|g\|_d \ + \ \|g\|_\infty \cdot \|f\|_d \\ &= \|f\|_\infty \cdot (\|g\|_\infty \ + \ \|g\|_d) \ + \ \|g\|_\infty \cdot \|f\|_d \\ &\leq \|f\| \cdot \|g\|. \end{aligned}$$

The Banach algebra $(\mathrm{Lip}(X,d), \|\cdot\|)$ will be called the *Lipschitz algebra* on (X,d). The unit element is the characteristic function on X, which is denoted by $\mathbf{1}$.

3. Point derivations

Point derivations are linear functionals on $\mathrm{Lip}(X,d)$ which satisfy Leibniz rule.

DEFINITION 3.1 *Let (X,d) be a metric space and $x_0 \in X$. A continuous linear functional $l \in \mathrm{Lip}(X,d)^*$ is said to be a* point derivation *at $x_0 \in X$, if for all $f,g \in \mathrm{Lip}(X,d)$ Leibniz rule*

$$l(fg) \;=\; f(x_0)\cdot l(g) \;+\; g(x_0)\cdot l(f)$$

holds.

A well known algebraic characterization of point derivations goes as follows (see I. Singer and J. Wermer, 1955). Let us denote by

$$\mathbf{m}(x_0) := \{f \in \mathrm{Lip}(X,d) \;\mid\; f(x_0) = 0\}$$

the closed ideal in $\mathrm{Lip}(X,d)$ of functions which vanish in $x_0 \in X$. Then the ideal of functions in $\mathrm{Lip}(X,d)$ with a root of second order is given by:

$$\mathbf{m}^2(x_0) := \mathrm{cl}(\{f := \sum_{j=1}^{k} f_j g_j \;\mid\; f_j, g_j \in \mathbf{m}(x_0),\; j = 1,..,k, \quad k \in \mathbb{N}\}),$$

where $''\mathrm{cl}''$ denotes the closure in $\mathrm{Lip}(X,d)$. Then one has:

PROPOSITION 3.1 *Let (X,d) be a metric space and $x_0 \in X$.*
Then for a continuous linear functional $l \in \mathrm{Lip}(X,d)^*$ there hold

i) $l(\mathbf{1}) \;=\; 0,$

ii) $l\Big|\mathbf{m}^2(x_0) \;=\; 0$

if and only if for all $f,g \in \mathrm{Lip}(X,d)$ the Leibniz rule

$$l(fg) \;=\; f(x_0)\cdot l(g) \;+\; g(x_0)\cdot l(f)$$

is satisfied at the point $x_0 \in X$.

Proof. $''\Longleftarrow''$ Let us assume that the functional $l \in \mathrm{Lip}(X,d)^*$ satisfies the Leibniz rule.
Since $\mathbf{1}^2 = \mathbf{1}$, Leibniz rule implies for $f = g = \mathbf{1}$ that $l(\mathbf{1}) = 2l(\mathbf{1})$, which means that $l(\mathbf{1}) = 0$.
Now, assume that $f,g \in \mathbf{m}(x_0)$. Then, $l(fg) \;=\; f(x_0)\cdot l(g) \;+\; g(x_0)\cdot l(f) = 0$, and since the functional l is continuous, it follows that $l\Big|\mathbf{m}^2(x_0) \;=\; 0$.

$''\Longrightarrow''$ Let us assume, that the continuous linear functional l satisfies conditions i) and ii). Then, for every $f, g \in \mathrm{Lip}(X, d)$ there holds

$$\begin{aligned}
l(fg) &= l(fg - f(x_0)g(x_0)\mathbf{1}) \\
&= l((f-f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) + f(x_0)(g - g(x_0)\mathbf{1}) + g(x_0)(f - f(x_0)\mathbf{1})) \\
&= l((f-f(x_0)\mathbf{1}) \cdot (g-g(x_0)\mathbf{1}))+f(x_0) \cdot l(g-g(x_0)\mathbf{1})+g(x_0) \cdot l(f-f(x_0)\mathbf{1}) \\
&= f(x_0) \cdot l(g - g(x_0)\mathbf{1}) + g(x_0) \cdot l(f - f(x_0)\mathbf{1}) \\
&= f(x_0) \cdot l(g) + g(x_0) \cdot l(f),
\end{aligned}$$

since $(f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) \in \mathbf{m}^2(x_0)$. ∎

Less well known is D.R. Sherbert's (1964) construction of point derivations for the Lipschitz algebra. We briefly outline his construction:

Consider the real Banach space

$$\mathbf{l}^\infty := \{x := (x_n)_{n\in\mathbb{N}} \;\mid\; (x_n)_{n\in\mathbb{N}} \text{ bounded sequence } \}$$

endowed with the supremum norm

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

Let $\mathbf{c} \subset \mathbf{l}^\infty$ denote the closed linear subspace of all convergent sequences and let $\lim\limits_{n\to\infty} : \mathbf{c} \longrightarrow \mathbb{R}$ be the continuous linear functional which assigns to every convergent sequence its limit. Consider a norm-preserving Hahn-Banach extension "$\underset{n\to\infty}{\mathrm{LIM}}$" of the functional "$\lim\limits_{n\to\infty}$" to $\mathbf{l}^\infty$ as indicated:

$$\begin{tikzcd}
\mathbf{c} \arrow[r, "\subset"] \arrow[dr, "\lim\limits_{n\to\infty}"'] & \mathbf{l}^\infty \arrow[d, "\underset{n\to\infty}{\mathrm{LIM}}"] \\
 & \mathbb{R}
\end{tikzcd}$$

with the following additional properties:

i) $\underset{n\to\infty}{\mathrm{LIM}}\; x_n = \underset{n\to\infty}{\mathrm{LIM}}\; x_{n+1}$,

ii) $\liminf\limits_{n\to\infty} x_n \leq \underset{n\to\infty}{\mathrm{LIM}}\; x_n \leq \limsup\limits_{n\to\infty} x_n$.

Such a functional "$\underset{n\to\infty}{\mathrm{LIM}}$" is called a *translation invariant (discrete) Banach limit.* For more details we refer to Dunford and Schwartz (1957, Chapter II.4, Exercise 22).

Now let $x_0 \in X$ be a non-isolated point and $w := (x_n, y_n)_{n\in\mathbb{N}} \subset \{(s,t) \in X \times X \mid s \neq t\}$ which converges to the point (x_0, x_0). Then

$$T_w : \mathrm{Lip}(X,d) \longrightarrow \mathbf{l}^\infty \quad \text{with} \quad T_w(f) := \left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right)_{n\in\mathbb{N}}$$

is a continuous linear operator, since $\|T_w(f)\|_\infty \leq \|f\|_d \leq \|f\|$.

D. R. Sherbert showed that for any translation invariant (discrete) Banach limit $\underset{n\to\infty}{\mathrm{LIM}}$ the continuous linear functional

$$D_w : \mathrm{Lip}(X,d) \longrightarrow \mathbb{R} \quad \text{with} \quad D_w(f) = \underset{n\to\infty}{\mathrm{LIM}} \ (T_w(f))$$

is a point derivation at $x_0 \in X$. For abbreviation let us put $\Delta := \{(s,t) \in X \times X \mid s = t\}$.

Proposition 3.2 (Sherbert, 1964, Lemma 9.4) *Let $x_0 \in X$ be a non-isolated point of a metric space (X,d) and $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta$ be a sequence which converges to the point (x_0, x_0). Then, for every translation invariant (discrete) Banach limit $\underset{n\to\infty}{\mathrm{LIM}} : \mathbf{l}^\infty \longrightarrow \mathbb{R}$ the continuous linear functional*

$$D_w : \mathrm{Lip}(X,d) \longrightarrow \mathbb{R} \quad \text{with} \quad D_w(f) = \underset{n\to\infty}{\mathrm{LIM}} \ (T_w(f))$$

is a point derivation at $x_0 \in X$.

Proof. First observe that for every convergent sequence $(a_n)_{n\in\mathbb{N}} \in \mathbf{c}$ and every bounded sequence $(b_n)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ the formula $\underset{n\to\infty}{\mathrm{LIM}} \ (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \underset{n\to\infty}{\mathrm{LIM}} \ b_n$ holds.
Namely put $\alpha := \lim_{n\to\infty} a_n$. Then $\underset{n\to\infty}{\mathrm{LIM}} \ (a_n \cdot b_n - \alpha b_n) = 0$, since $(a_n \cdot b_n - \alpha b_n)_{n\in\mathbb{N}}$ is a sequence converging to zero and hence $\underset{n\to\infty}{\mathrm{LIM}} \ (a_n \cdot b_n) = \alpha \underset{n\to\infty}{\mathrm{LIM}} \ b_n$.

Now, let $f, g \in \mathrm{Lip}(X,d)$ be given. From the above observation it follows that

$$\begin{aligned}
D_w(fg) &= \underset{n\to\infty}{\mathrm{LIM}} \ (T_w(fg)) \\
&= \underset{n\to\infty}{\mathrm{LIM}} \ \left(\frac{(fg)(y_n) - (fg)(x_n)}{d(y_n, x_n)}\right) \\
&= \underset{n\to\infty}{\mathrm{LIM}} \ \left(f(y_n)\frac{g(y_n) - g(x_n)}{d(y_n, x_n)} + g(x_n)\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\underset{n\to\infty}{\mathrm{LIM}} \ \left(\frac{g(y_n) - g(x_n)}{d(y_n, x_n)}\right) + g(x_0)\underset{n\to\infty}{\mathrm{LIM}} \ \left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\underset{n\to\infty}{\mathrm{LIM}} \ (T_w(g)) + g(x_0)\underset{n\to\infty}{\mathrm{LIM}} \ (T_w(f)) \\
&= f(x_0)D_w(g) + g(x_0)D_w(f).
\end{aligned}$$

Since D_w is continuous, it is a point derivation at $x_0 \in X$. ■

The relation between Clarke's generalized derivative (Clarke, 1989)

$$\mathcal{D}_{cl}(f, x_0, v) := f^0(x_0, v) = \limsup_{\substack{x \to x_0 \\ t \downarrow 0}} \frac{f(x+tv) - f(x)}{t}$$

and point derivations has been investigated in Demyanov and Pallaschke (1997). Note that Clarke's generalized derivative is continuous and sublinear, not only as a function of the direction v, but also as a function on the Banach space of Lipschitz functions f. Continuous sublinear functions on Banach spaces are completely characterized by their support sets which are subsets of the dual space. In Demyanov and Pallaschke (1997) it was shown that the support set of the sublinear function $\mathcal{D}_{cl}\Big|_{x_0, v} = \mathcal{D}_{cl}(., x_0, v)$ consists of point derivations in the sense of Sherbert (1964).

4. Generalized directional derivatives

Following D. R. Sherbert's construction we will now propose a concept of a *generalized directional derivative*, which satisfies Leibniz rule for locally Lipschitz functions, defined on an open subset of a Banach space. Therefore we proceed as follows:

Consider the real Banach space

$$\mathbf{C}[1, \infty] := \{x \;\mid\; x : [1, \infty) \longrightarrow \mathbb{R} \text{ is a bounded continuous function } \}$$

endowed with the supremum norm $\|x\|_\infty := \sup\limits_{t \geq 1} |x(t)|$ and let

$$\mathbf{C}_\bullet[1, \infty] = \{x \in \mathbf{C}[1, \infty] \;\mid\; \lim_{t \to \infty} x(t) \quad \text{exists } \}$$

denote the closed linear subspace of all bounded continuous functions having a limit in ∞. Then

$$\lim_{t \to \infty} : \mathbf{C}_\bullet[1, \infty] \longrightarrow \mathbb{R} \quad \text{with} \quad \lim_{t \to \infty}(x) = \lim_{t \to \infty} x(t)$$

is a continuous linear functional. Consider a norm-preserving Hahn-Banach extension "$\underset{t \to \infty}{\mathrm{LIM}}$" of the functional "$\lim\limits_{t \to \infty}$" to $\mathbf{C}[1, \infty]$ as indicated:

$$\begin{tikzcd}
\mathbf{C}_\bullet[1, \infty] \arrow[r, "\subset"] \arrow[dr, "\lim\limits_{t \to \infty}"'] & \mathbf{C}[1, \infty] \arrow[d, "\underset{t \to \infty}{\mathrm{LIM}}"] \\
& \mathbb{R}
\end{tikzcd}$$

with the following additional properties:

i) $\underset{t\to\infty}{\mathrm{LIM}}\ x(t) \geq 0$ if $x \geq 0$,

ii) $\underset{t\to\infty}{\mathrm{LIM}}\ x(t) = \underset{t\to\infty}{\mathrm{LIM}}\ x_s(t)$, where $x_s(t) = x(t+s)$ and $s > 0$,

iii) $\liminf_{t\to\infty} x(t) \leq \underset{t\to\infty}{\mathrm{LIM}}\ x(t) \leq \limsup_{t\to\infty} x(t)$.

Such a functional "$\underset{t\to\infty}{\mathrm{LIM}}$" is called a *translation invariant (continuous) Banach limit* (see Dunford and Schwartz, 1957, Chapter II.4, Exercise 22).

Now let $(X, \|\ \|)$ be a real Banach space, $U \subset X$ an open subset and $x_0 \in U$. Let us denote by $\mathrm{Lip}_{\mathrm{loc}}(U)$ the linear space of all locally Lipschitz functions $f : U \longrightarrow \mathbb{R}$. Then we define for $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ and a vector $u \in X$ the *generalized directional derivative* of f at x_0 in the direction $u \in X$ by

$$\mathrm{D}_u(f)\Big|_{x_0} = \underset{t\to\infty}{\mathrm{LIM}}\ t\left(f(x_0 + \frac{1}{t}\ u) - f(x_0)\right).$$

First, note that the difference quotient $\left(\frac{f(x_0+\tau u)-f(x_0)}{\tau}\right)$ is transformed into the expression $t\left(f(x_0 + \frac{1}{t}\ u) - f(x_0)\right)$ under $\tau = \frac{1}{t}$ for $t > 0$. Then, observe that for every $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ the inequality $\left|t\left(f(x_0 + \frac{1}{t}\ u) - f(x_0)\right)\right| \leq L\|u\|$ holds for every $u \in X$ and $t \geq 1$, where $L \geq 0$ is a local Lipschitz constant of f. Hence, the generalized directional derivative $\mathrm{D}_u(f)\Big|_{x_0}$ exists for every $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ and $u \in X$. Now we show:

THEOREM 4.1 *Let $(X, \|\ \|)$ be a real Banach space, $U \subset X$ an open subset and $x_0 \in U$. Then the linear mapping*

$$\mathrm{D}_u\Big|_{x_0} : \mathrm{Lip}_{\mathrm{loc}}(U) \longrightarrow \mathbb{R} \qquad \textit{with} \quad f \mapsto \mathrm{D}_u(f)\Big|_{x_0}$$

satisfies Leibniz rule. If the directional derivative

$$\frac{df}{du}\Big|_{x_0} = \lim_{\tau\downarrow 0}\left(\frac{f(x_0+\tau u) - f(x_0)}{\tau}\right)$$

exists, then $\frac{df}{du}\Big|_{x_0} = \mathrm{D}_u(f)\Big|_{x_0}$.

Proof. Let $u \in X$ and $x_0 \in U$ be given. First, observe that $\underset{\sigma\to\infty}{\mathrm{LIM}}\ (f(x_0 + \frac{1}{\sigma}\ u) - f(x_0)) = 0$, for every $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$. This implies

$$\underset{t\to\infty}{\mathrm{LIM}}\, t\left(f(x_0+\frac{1}{t}\, u)\left(f(x_0+\frac{1}{t}\, u)-f(x_0)\right)\right) = f(x_0)\underset{t\to\infty}{\mathrm{LIM}}\ t\left(f(x_0+\frac{1}{t}\, u)-f(x_0)\right).$$

Now let $f, g \in \mathrm{Lip}_{\mathrm{loc}}(U)$ be given. From the above observation it follows that

$$\begin{aligned}
\mathrm{D}_u(fg)\Big|_{x_0} &= \operatorname*{LIM}_{t\to\infty} t\left((fg)(x_0+\frac{1}{t}\,u)-(fg)(x_0)\right)\\
&= \operatorname*{LIM}_{t\to\infty}\ t\left(f(x_0+\frac{1}{t}\,u)\left(g(x_0+\frac{1}{t}\,u)-g(x_0)\right)+g(x_0)\left(f(x_0+\frac{1}{t}\,u)-f(x_0)\right)\right)\\
&= \operatorname*{LIM}_{t\to\infty}\ t\left(f(x_0+\frac{1}{t}\,u)\left(g(x_0+\frac{1}{t}\,u)-g(x_0)\right)\right)+\operatorname*{LIM}_{t\to\infty}\ t\,g(x_0)\left(f(x_0+\frac{1}{t}\,u)-f(x_0)\right)\\
&= f(x_0)\operatorname*{LIM}_{t\to\infty} t\left(g(x_0+\frac{1}{t}\,u)-g(x_0)\right)+g(x_0)\operatorname*{LIM}_{t\to\infty} t\left(f(x_0+\frac{1}{t}\,u)-f(x_0)\right)\\
&= f(x_0)\mathrm{D}_u(g)\Big|_{x_0}+g(x_0)\mathrm{D}_u(f)\Big|_{x_0},
\end{aligned}$$

which proves Leibniz rule.

The second statement, $\dfrac{df}{du}\Big|_{x_0} = \mathrm{D}_u(f)\Big|_{x_0}$, follows from condition iii) of the translation invariant (continuous) Banach limit. ■

For a suitable definition of a generalized second-order directional derivative we impose the following condition on a locally Lipschitz function.

Let $(X, \|\ \|)$ be a real Banach space, $U \subset X$ an open subset, $x_0 \in U$, $u \in X$ and $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$. Then the imposed condition is:

for every $\varepsilon > 0$ there exists an $M \geq 0$ such that for all $0 < \tau \leq \varepsilon$

$(\mathbf{SOD})_{\mathbf{u}}$
$$\left|\frac{f(x_0+2\tau u)-2f(x_0+\tau u)+f(x_0)}{\tau^2}\right| \leq M.$$

holds.

If for $u \in X$ and $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ the function $\tau \mapsto f(x_0+\tau u)$, $0 \leq \tau \leq \varepsilon$, is of class $C^{1,1}$ for some $\varepsilon > 0$, then condition $(\mathbf{SOD})_{\mathbf{u}}$ is satisfied.

Proposition 4.1 *Let $(X, \|\ \|)$ be a real Banach space, $U \subset X$ an open subset, $x_0 \in U$ and let $f, g \in \mathrm{Lip}_{\mathrm{loc}}(U)$ satisfy condition $(\mathbf{SOD})_{\mathbf{u}}$. Then the functions $f+g$ and fg also satisfy $(\mathbf{SOD})_{\mathbf{u}}$.*

Proof. Let $f, g \in \mathrm{Lip}_{\mathrm{loc}}(U)$ and let us assume that for some $u \in X$ condition $(\mathbf{SOD})_{\mathbf{u}}$ is satisfied. Then the condition $(\mathbf{SOD})_{\mathbf{u}}$ is obviously satisfied for $f+g$.

Now let us consider the case of fg. Then we have the following estimation:

$$
\begin{aligned}
&\Big| \left(\frac{(fg)(x_0+2\tau u) - 2(fg)(x_0+\tau u) + (fg)(x_0)}{\tau^2} \right) \Big| \\
&= \Big| \left(\frac{1}{\tau} \left(\frac{(fg)(x_0+2\tau u) - (fg)(x_0+\tau u)}{\tau} - \frac{(fg)(x_0+\tau u) - (fg)(x_0)}{\tau} \right) \right) \Big| \\
&= \Big| \left(\frac{1}{\tau} \left(f(x_0+2\tau u) \frac{g(x_0+2\tau u) - g(x_0+\tau u)}{\tau} \right.\right. \\
&\quad \left.\left. + \ g(x_0+\tau u) \frac{f(x_0+2\tau u) - f(x_0+\tau u)}{\tau} \right) \right) \\
&\quad - \left(\frac{1}{\tau} \left(f(x_0+\tau u) \frac{g(x_0+\tau u) - g(x_0)}{\tau} + g(x_0) \frac{f(x_0+\tau u) - f(x_0)}{\tau} \right) \right) \Big| \\
&= \Big| \left(\frac{1}{\tau} \left(f(x_0+2\tau u) \frac{g(x_0+2\tau u) - g(x_0+\tau u)}{\tau} \right.\right. \\
&\quad \left.\left. + g(x_0+\tau u) \frac{f(x_0+2\tau u) - f(x_0+\tau u)}{\tau} \right) \right) \\
&\quad - \left(\frac{1}{\tau} \left(f(x_0+2\tau u) \frac{g(x_0+\tau u) - g(x_0)}{\tau} + g(x_0+\tau u) \frac{f(x_0+\tau u) - f(x_0)}{\tau} \right) \right) \\
&\quad + \left(\frac{1}{\tau} \left(f(x_0+2\tau u) \frac{g(x_0+\tau u) - g(x_0)}{\tau} + g(x_0+\tau u) \frac{f(x_0+\tau u) - f(x_0)}{\tau} \right) \right) \\
&\quad - \left(\frac{1}{\tau} \left(f(x_0+\tau u) \frac{g(x_0+\tau u) - g(x_0)}{\tau} + g(x_0) \frac{f(x_0+\tau u) - f(x_0)}{\tau} \right) \right) \Big| \\
&\le \Big| \ f(x_0+2\tau u) \frac{g(x_0+2\tau u) - 2g(x_0+\tau u) + g(x_0)}{\tau^2} \\
&\quad + g(x_0+\tau u) \frac{f(x_0+2\tau u) - 2f(x_0+\tau u) + f(x_0)}{\tau^2} \Big| \\
&\quad + \Big| \left(\frac{f(x_0+2\tau u) - f(x_0+\tau u)}{\tau} \right) \left(\frac{g(x_0+\tau u) - g(x_0)}{\tau} \right) \\
&\quad + \left(\frac{g(x_0+\tau u) - g(x_0)}{\tau} \right) \left(\frac{f(x_0+\tau u) - f(x_0)}{\tau} \right) \Big| .
\end{aligned}
$$

For every $u \in X$ the above expression is bounded in a neighborhood of $x_0 \in U$. Since f, g satisfy condition $(\mathbf{SOD})_{\mathbf{u}}$ the first summand is bounded and since f, g are locally Lipschitz the second summand can be estimated from above by the product of the local Lipschitz constants of f and g and $\|u\|$. Hence the product of fg satisfies condition $(\mathbf{SOD})_{\mathbf{u}}$. ∎

For the definition of a *generalized second-order directional derivative* at $x_0 \in U$ of $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ in the direction $u \in X$, we transform the second-order difference quotient $\left(\frac{f(x_0+2\tau u) - 2f(x_0+\tau u) + f(x_0)}{\tau^2} \right)$ under $\tau = \frac{1}{t}$ into the function

$t \mapsto t^2 \left(f(x_0 + \frac{2}{t}\ u) - 2f(x_0 + \frac{1}{t}\ u) + f(x_0)\right) \in C[1, \infty]$ and define:

$$\mathrm{D}_u^2(f)\Big|_{x_0} = \underset{t\to\infty}{\mathrm{LIM}}\ t^2 \left(f(x_0 + \frac{2}{t}\ u) - 2f(x_0 + \frac{1}{t}\ u) + f(x_0)\right).$$

If condition $(\mathbf{SOD})_{\mathbf{u}}$ is satisfied then $\mathrm{D}_u^2(f)\Big|_{x_0}$ exists.

Now we prove:

THEOREM 4.2 *Let* $(X, \|\ \|)$ *be a real Banach space,* $U \subset X$ *an open subset,* $u \in X,\ x_0 \in U$ *and*

$$\mathcal{L}_{\mathrm{loc}}(U, x_0) = \{f \mid f \in \mathrm{Lip}_{\mathrm{loc}}(U) \text{ and } \sigma \mapsto f(x_0 + \sigma u) \text{ is } C^{1,1} \text{ on } [0, \varepsilon] \text{ for some } \varepsilon > 0\ \}.$$

Then, for all $f, g \in \mathcal{L}_{\mathrm{loc}}(U, x_0)$ *the generalized second-oder directional derivative* $\mathrm{D}_u^2\Big|_{x_0}$ *satisfies the Leibniz rule*

$$\mathrm{D}_u^2(fg)\Big|_{x_0} \;=\; f(x_0)\mathrm{D}_u^2(g)\Big|_{x_0} + 2\frac{df}{du}\Big|_{x_0}\frac{dg}{du}\Big|_{x_0} + g(x_0)\mathrm{D}_u^2(f)\Big|_{x_0},$$

where $\frac{df}{du}\Big|_{x_0}$ *(resp.* $\frac{dg}{du}\Big|_{x_0}$*) is the directional derivative of* f *(resp.* g*) at* $x_0 \in U$ *in the direction* $u \in X$.

Proof. By definition we have:

$$\begin{aligned}
&\mathrm{D}_u^2(fg)\Big|_{x_0} = \underset{t\to\infty}{\mathrm{LIM}}\ t^2 \left((fg)(x_0 + \frac{2}{t}u) - 2(fg)(x_0 + \frac{1}{t}u) + (fg)(x_0)\right)\\
&= \underset{t\to\infty}{\mathrm{LIM}}\ t \left(t\left((fg)(x_0 + \frac{2}{t}u) - (fg)(x_0 + \frac{1}{t}u)\right) - t\left((fg)(x_0 + \frac{1}{t}u) - (fg)(x_0)\right)\right)\\
&= \underset{t\to\infty}{\mathrm{LIM}}\ t \left(tf(x_0 + \frac{2}{t}u)\left(g(x_0 + \frac{2}{t}u) - g(x_0 + \frac{1}{t}u)\right) + tg(x_0 + \frac{1}{t}u)\left(f(x_0 + \frac{2}{t}u) - f(x_0 + \frac{1}{t}u)\right)\right)\\
&\quad - \underset{t\to\infty}{\mathrm{LIM}}\ t\left(t\, f(x_0 + \frac{1}{t}u)\left(g(x_0 + \frac{1}{t}u) - g(x_0)\right) + t\, g(x_0)\left(f(x_0 + \frac{1}{t}u) - f(x_0)\right)\right)\\
&= \underset{t\to\infty}{\mathrm{LIM}}\ t \left(tf(x_0 + \frac{2}{t}u)\left(g(x_0 + \frac{2}{t}u) - g(x_0 + \frac{1}{t}u)\right) + tg(x_0 + \frac{1}{t}u)\left(f(x_0 + \frac{2}{t}u) - f(x_0 + \frac{1}{t}u)\right)\right)\\
&\quad - \underset{t\to\infty}{\mathrm{LIM}}\ t\left(t\, f(x_0 + \frac{2}{t}u)\left(g(x_0 + \frac{1}{t}u) - g(x_0)\right) + t\, g(x_0 + \frac{1}{t})\left(f(x_0 + \frac{1}{t}u) - f(x_0)\right)\right)\\
&\quad + \underset{t\to\infty}{\mathrm{LIM}}\ t\left(t\, f(x_0 + \frac{2}{t}u)\left(g(x_0 + \frac{1}{t}u) - g(x_0)\right) + t\, g(x_0 + \frac{1}{t}u)\left(f(x_0 + \frac{1}{t}u) - f(x_0)\right)\right)\\
&\quad - \underset{t\to\infty}{\mathrm{LIM}}\ t\left(t\, f(x_0 + \frac{1}{t}u)\left(g(x_0 + \frac{1}{t}u) - g(x_0)\right) + t\, g(x_0)\left(f(x_0 + \frac{1}{t}u) - f(x_0)\right)\right)\\
&= \underset{t\to\infty}{\mathrm{LIM}}\ t\left(t\, f(x_0 + \frac{2}{t}u)\left(g(x_0 + \frac{2}{t}u) - 2g(x_0 + \frac{1}{t}u) + g(x_0)\right)\right.\\
&\quad \left. +\ t\, g(x_0 + \frac{1}{t}u)\left(f(x_0 + \frac{2}{t}u) - 2f(x_0 + \frac{1}{t}u) + f(x_0)\right)\right)
\end{aligned}$$

$$+\operatorname*{LIM}_{t\to\infty} t \left(t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \left(g(x_0+\frac{1}{t}u) - g(x_0) \right) \right.$$
$$\left. +\ t \left(g(x_0+\frac{1}{t}u) - g(x_0) \right) \left(f(x_0+\frac{1}{t}u) - f(x_0) \right) \right)$$
$$= \operatorname*{LIM}_{t\to\infty} t^2 \left(\left(f(x_0+\frac{2}{t}u) \left(g(x_0+\frac{2}{t}u) - 2g(x_0+\frac{1}{t}u) + g(x_0) \right) \right.\right.$$
$$\left.\left. +\ g(x_0+\frac{1}{t}u) \left(f(x_0+\frac{2}{t}u) - 2f(x_0+\frac{1}{t}u) + f(x_0) \right) \right) \right)$$
$$+\operatorname*{LIM}_{t\to\infty} \left(\left[t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \right] t \left(g(x_0+\frac{1}{t}u) - g(x_0) \right) \right.$$
$$\left. +\ t \left(g(x_0+\frac{1}{t}u) - g(x_0) \right)\ t \left(f(x_0+\frac{1}{t}u) - f(x_0) \right) \right)$$
$$= f(x_0)\mathrm{D}_u^2(g)\Big|_{x_0} + g(x_0)\mathrm{D}_u^2(f)\Big|_{x_0} + 2\frac{df}{du}\Big|_{x_0}\frac{dg}{du}\Big|_{x_0}.$$

The last equality can be seen as follows. Since $\lim_{t\to\infty}\left(f(x_0+\frac{2}{t}u) - f(x_0) \right) = 0$ we have

$$\operatorname*{LIM}_{t\to\infty} t^2 \left(f(x_0+\frac{2}{t}u) \left(g(x_0+\frac{2}{t}u) - 2g(x_0+\frac{1}{t}u) + g(x_0) \right) \right) \ = \ f(x_0)\mathrm{D}_u^2(g)\Big|_{x_0}.$$

Analogously we obtain the term $g(x_0)\mathrm{D}_u^2(f)\Big|_{x_0}$.

Since $\tau \mapsto f(x_0+\tau u)$ and $\tau \mapsto g(x_0+\tau u)$ are $C^{1,1}$-functions in τ, we have:

$$\operatorname*{LIM}_{t\to\infty} t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \ = \ \lim_{t\to\infty} t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \ = \ \frac{df}{du}\Big|_{x_0}.$$

Hence

$$\operatorname*{LIM}_{t\to\infty} \left(\left[t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \ - \ \frac{df}{du}\Big|_{x_0} \right] t \left(g(x_0+\frac{1}{t}u) - g(x_0) \right) \right) \ = \ 0,$$

and this gives

$$\operatorname*{LIM}_{t\to\infty} \left(\left[t \left(f(x_0+\frac{2}{t}u) - f(x_0+\frac{1}{t}u) \right) \right] t \left(g(x_0+\frac{1}{t}u) - g(x_0) \right) \right) \ = \ \frac{df}{du}\Big|_{x_0}\frac{dg}{du}\Big|_{x_0}.$$

The rest follows analogously and this completes the proof. ■

Algebraic characterizations of second order point derivations, which are similar to that one given in Proposition 3.1 were proven in Pallaschke, Recht and Urbański (1987, 1991), Pallaschke and Rolewicz (1997). A finite difference scheme which satisfies the Leibniz rule in the discrete case was studied in Bouguenaya and Fairlie (1986).

Example 4.1 Let us consider the following function

$$f:\mathbb{R}\longrightarrow\mathbb{R} \quad \text{given by} \quad f(x) \ = \ \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if} \quad x\neq 0 \\ 0 & \text{if} \quad x=0 \end{cases}.$$

For $x_0 = 0$ and $u = 1$ we have $\mathrm{D}_u(f)\Big|_{x_0} = 0$, because f is differentiable at $x_0 = 0$. Moreover, for $u = 1$ and $x_0 = 0$ the fraction

$$\left|\frac{f(x_0 + 2\tau u) - 2f(x_0 + \tau u) + f(x_0)}{\tau^2}\right| = \left|4\sin\left(\frac{1}{2\tau}\right) - 2\sin\left(\frac{1}{\tau}\right)\right| \leq 6$$

is bounded, so that $\mathrm{D}_u^2(f)\Big|_{x_0}$ exists. In particular, $\mathrm{D}_u^2(f)\Big|_{x_0} \in [-6,\ 6]$.

5. Estimations

The concept of generalized convexity, in particular Φ-convexity (see Pallaschke and Rolewicz, 1997, and Rolewicz, 2003) can be used to determine upper and lower bounds for generalized directional derivatives.

Therefore, let X be a non empty set and Φ a set of real valued functions defined on X with the following properties:

i) for all $c \in \mathbb{R}$ and all $\phi \in \Phi$ there holds $\phi + c \in \Phi$,

ii) for all $\lambda \geq 0$ and all $\phi \in \Phi$ there holds $\lambda\phi \in \Phi$.

Natural examples for Φ are for instance the affine functions or the quadratic convex functions defined on a Banach space (see Pallaschke and Rolewicz, 1997, Dolecki and Kurcyusz, 1978, and Eberhard and Nyblom, 1998).

Now a function $f : X \longrightarrow \mathbb{R}$ is called Φ-subdifferentiable at $x_0 \in X$ if there exists a $\varphi \in \Phi$ with $\varphi(x) - \varphi(x_0) \leq f(x) - f(x_0)$ for all $x \in X$. The set

$$\partial_\Phi f(x_0) = \{\phi \in \Phi \mid \phi(x) - \phi(x_0) \leq f(x) - f(x_0) \text{ for all } x \in X\}$$

is called the Φ-subdifferential of f at x_0.

THEOREM 5.1 *Let $(X, \| \ \|)$ be a real Banach space, $U \subset X$ an open subset, $x_0 \in U$ and $u \in X$. Then the following estimates hold true:*

i) For a locally Lipschitz function $f : U \longrightarrow \mathbb{R}$ there holds:

$$\liminf_{\tau\downarrow 0}\left(\frac{f(x_0 + \tau u) - f(x_0)}{\tau}\right) \leq \mathrm{D}_u(f)\Big|_{x_0} \leq \limsup_{\tau\downarrow 0}\left(\frac{f(x_0 + \tau u) - f(x_0)}{\tau}\right) \leq f^0(x_0, u)$$

where $f^0(x_0, u)$ is Clarke's generalized directional derivative. If, in addition, f is Φ-subdifferentiable for some differentiable $\varphi_0 \in \partial_\Phi f(x_0)$, then

$$\langle \nabla\varphi_0\Big|_{x_0}, u\rangle \leq \mathrm{D}_u(f)\Big|_{x_0}$$

ii) If $U \subset X$ is open and convex and $f : U \longrightarrow \mathbb{R}$ is a convex function, for which $\mathrm{D}_u^2(f)\Big|_{x_0}$ exists, then $\mathrm{D}_u^2(f)\Big|_{x_0} \geq 0$.

iii) For $U = X$ and $p : X \longrightarrow \mathbb{R}$ sublinear there holds:

$$\mathrm{D}_u(p)\Big|_0 = p(u) \quad \text{and} \quad \mathrm{D}_u^2(p)\Big|_0 = 0 \quad \text{for arbitrary } u \in X.$$

Proof. i) The inequality

$$\liminf_{\tau\downarrow 0}\left(\frac{f(x_0+\tau u)-f(x_0)}{\tau}\right) \le \mathrm{D}_u(f)\Big|_{x_0} \le \limsup_{\tau\downarrow 0}\left(\frac{f(x_0+\tau u)-f(x_0)}{\tau}\right) \le f^0(x_0,u)$$

follows from the fact that the (continuous) Banach limit lies between limes-inferior and limes-superior and from the definition of Clarke's generalized directional derivative. If $\varphi_0 \in \partial_\Phi f(x_0)$, then for all $\tau > 0$

$$\frac{\varphi_0(x_0+\tau u)-\varphi_0(x_0)}{\tau} \le \frac{f(x_0+\tau u)-f(x_0)}{\tau}$$

holds, which implies by the monotonicity of the (continuous) Banach limit that

$$\mathrm{D}_u(\varphi_0)\Big|_{x_0} \le \mathrm{D}_u(f)\Big|_{x_0}.$$

Since φ_0 is differentiable, we have

$$\langle \nabla\varphi_0\Big|_{x_0}, u\rangle = \mathrm{D}_u(\varphi_0)\Big|_{x_0} \le \mathrm{D}_u(f)\Big|_{x_0}.$$

ii) Since f is assumed to be convex, we have for $\tau > 0$ that

$$f(x_0+\tau u) \le \frac{1}{2}\left(f(x_0+2\tau u)+f(x_0)\right)$$

which implies $\mathrm{D}^2_u(f)\Big|_{x_0} \ge 0$.

iii) This follows immediately from sublinearity. ■

Example 5.1 In the book of B. Mordukhovich (2006, page 123) the second codifferential of the $C^{1,1}$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \frac{1}{2}(sign\ x)x^2$ at $0 \in \mathbb{R}$ is considered which is:

$$\partial^2 f(0)(u) = \begin{cases} [-u,u] & : \quad u \ge 0 \\ \{u,-u\} & : \quad u < 0 \end{cases}.$$

Observe that the second order directional derivative reflects the properties of the second order codifferential, because

$$\mathrm{D}_u(f)\Big|_x = \begin{cases} xu & : \quad x \ge 0 \\ -xu & : \quad x < 0 \end{cases}$$

which gives:

$$\mathrm{D}^2_u(f)\Big|_x = \begin{cases} u^2 sign\ u & : \quad x = 0 \\ u^2 & : \quad x > 0 \\ -u^2 & : \quad x < 0 \end{cases}.$$

Acknowledgement

The authors would like to thank an anonymous referee for his insightful comments on the first draft of this paper. His suggestions led to a substantial improvement of the paper.

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