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# Optimal control for a class of mechanical thermoviscoelastic frictional contact problems[1]

by

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**Abstract:** We study optimal control of systems governed by a coupled system of hemivariational inequalities, modeling a dynamic thermoviscoelastic problem, which describes frictional contact between a body and a foundation. We employ the Kelvin-Voigt viscoelastic law, include the thermal effects and consider the general nonmonotone and multivalued subdifferential boundary conditions. We consider optimal control problem for boundary and distributed parameter control systems, time optimal control problem and maximum stay control problem. We deliver conditions that guarantee the existence of optimal solutions.

**Keywords:** control problem, hemivariational inequality, thermoviscoelasticity, friction, nonconvexity, nonmonotonnicity, semicontinuity, multifunction.

## 1. Introduction

The aim of this paper is to present new existence results for optimal control problems for models governed by the system of two coupled evolution hemivariational inequalities: one of hyperbolic type for the displacement and the other of parabolic hemivariational inequality for the temperature. This system describes the dynamic contact between a linear thermoviscoelastic body and a foundation. Our efforts are of importance in the development of control theory for a large class of mechanical and engineering problems involving nonmonotone and multivalued relations. In this paper main interest lies in general nonmonotone and multivalued subdifferential boundary conditions. More precisely, it is supposed that on the boundary of the body under consideration, the subdifferential relations hold, the first one between the normal component of the velocity and

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the normal component of the stress, the second one between the tangential components of these quantities and the third one between temperature and the heat flux vector. These three subdifferential boundary conditions are natural generalizations of the normal damped response condition, the accociated friction law and the well known Fourier law of heat conduction, respectively. In order to formulate the mechanical problems we use the notion of hemivariational inequality which allows to treat situations with the nonsmooth, nondifferentiable and nonconvex energy functionals. The notion of hemivariational inequality was introduced in the early 1980s by Panagiotopoulos as a generalization of variational inequality. For examples, applications and detailed explanations concerning the boundary conditions we refer the Reader to Panagiotopoulos (1993) and Naniewicz and Panagiotopoulos (1995).

In this paper the state of the system is described by a fully dynamic coupled system of two hemivariational inequalities. The body is supposed to satisfy the Kelvin-Voigt constitutive law with added thermal effects. The existence and uniqueness results for such hemivariational inequalities have been obtained only recently, see Denkowski and Migórski (2005). Because of the multivalued multidimensional boundary conditions, the aformentioned problem is embedded into a more general class of problems for second order evolution inclusions. In this paper all subdifferentials, considered for locally Lipschitz functions, are understood in the sense of Clarke. This allows to incorporate in our model several types of boundary conditions considered earlier e.g. in Awbi et al. (2000), Panagiotopoulos (1993) and Naniewicz and Panagiotopoulos (1995).

It should be mentioned that in applications, control of frictional processes is one of the main issues. In several mechanical dynamical systems, friction has to be minimized, so that the wear is reduced and the lifetime and the efficiency of the system is increased. The economic loss caused by friction and wear is estimated at five percent of the US gross national product (see Sextro, 2002). This loss is caused by insufficient control of contact processes in machines, cars, mechanical equipment (e.g. brakes, machine tools, bearings, motors, turbines, etc.) and is mainly due to frictional wear, frictional heat losses, softening and damage of contacting surfaces. Therefore, accurate prediction of the evaluation of frictional contact processes and their control is of major economic importance.

Optimal control of contact frictional problems has been considered in Barbu (1993), who dealt with optimal control of variational inequalities, and in Amassad et al. (2002), who considered quasistatic problem with Tresca friction law. The optimal control problems for hemivariational inequalities have been studied in Chapter 8 of Panagiotopoulos (1993), Miettinen (1993) and Panagiotopoulos and Haslinger (1992) (elliptic problems), Denkowski and Migórski (1998) (shape design for elliptic hemivariational inequalities), Migórski and Ochal (2000) (parabolic problems), Ochal (2001), Migórski (2002) (hyperbolic hemivariational inequalities), Denkowski (2002), Denkowski and Migórski (2004) and the referenes therein.

In this paper we examine optimal control problems governed by a coupled system of evolution inclusions. For such systems we deal with the Bolza problem. We give conditions under which they admit optimal solutions. We remark that since the system of evolution inclusions has generally many solutions, the state of the control problem is not uniquely determined. The results on control problems cover in particular that of Lions (1971) (for control problems for partial differential equations) and Barbu (1993) (for control problems for variational inequalities). For more material on hemivariational inequalities we refer the Reader to Naniewicz and Panagiotopoulos (1995), Migórski and Ochal (2004, 2005), Migórski (2005) and the references therein.

The content of the paper is as follows. After the preliminary material of Section 2, in Section 3 we present the physical setting and the variational formulation of the problem. In Section 4 we formulate the problem in terms of a coupled system of evolution inclusions and recall the results on existence and uniqueness of solutions to such inclusions. The main results on existence of solutions to optimal control problems are delivered in Sections 5, 6 and 7. These sections are devoted to the boundary and distributed parameter control system, the time optimal control problem and the maximum stay control problem, respectively.

## 2. Notation and preliminaries

In this section we introduce notation (see Clarke, 1983; Han and Sofonea, 2002; Denkowski et al., 2003) and recall some definitions and auxiliary results needed in the sequel.

We denote by $\mathcal{S}_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$, $d = 2, 3$ (in applications), or equivalently, the space $\mathbb{R}_s^{d\times d}$ of symmetric matrices of order $d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathcal{S}_d$ by

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\|_{\mathcal{S}_d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma, \tau \in \mathcal{S}_d.$$

The summation convention over repeated indices is used.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $n$ denote the outward unit normal vector to $\Gamma$. The assumption that $\Gamma$ is Lipschitz continuous ensures that $n$ is defined a.e. on $\Gamma$. The deformation operator $\varepsilon\colon H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathcal{S}_d)$ is defined by $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$, where $\partial_j = \partial/\partial x_j$, $i, j = 1, \ldots, d$. The spaces $L^2(\Omega;\mathbb{R}^d)$, $L^2(\Omega;\mathcal{S}_d)$ and $H^1(\Omega;\mathbb{R}^d)$ are Hilbert spaces endowed with the corresponding inner products

$$\langle u, v\rangle_{L^2(\Omega;\mathbb{R}^d)} = \int_\Omega u \cdot v\, dx, \ \langle \sigma, \tau\rangle_{L^2(\Omega;\mathcal{S}_d)} = \int_\Omega \sigma : \tau\, dx,$$
$$\langle u, v\rangle_{H^1(\Omega;\mathbb{R}^d)} = \langle u, v\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \varepsilon(u), \varepsilon(v)\rangle_{L^2(\Omega;\mathcal{S}_d)}.$$

For every $v \in H^1(\Omega;\mathbb{R}^d)$ we denote by $\gamma_0 v$ its trace on $\Gamma$, where $\gamma_0\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ is the trace map. If $d = 1$, then the trace operator from $H^1(\Omega)$ into $L^2(\Gamma)$ is denoted by $\gamma_0^s$. Given $v \in L^2(\Gamma;\mathbb{R}^d)$ we denote by $v_N$ and $v_T$ the usual normal and tangential components of $v$ on the boundary $\Gamma$, i.e. $v_N = v \cdot n$ and $v_T = v - v_N n$. Similarly, for a regular tensor field $\sigma\colon \Omega \to \mathcal{S}_d$, we define its normal and tangential components by $\sigma_N = (\sigma n)\cdot n$ and $\sigma_T = \sigma n - \sigma_N n$, respectively. With no confusion the letter $T$ will appear in the time interval and will be used as the subscript in the tangential components of vectors and tensors.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $h\colon X \to \mathbb{R}$, where $X$ is a Banach space (see Clarke, 1983). The generalized directional derivative of $h$ at $x \in X$ in the direction $v \in X$, denoted by $h^0(x;v)$, is defined by $h^0(x;v) = \limsup_{y\to x,\ \lambda\downarrow 0} \dfrac{h(y+\lambda v) - h(y)}{\lambda}$. The generalized gradient of $h$ at $x$, denoted by $\partial h(x)$, is a subset of a dual space $X^*$ given by $\partial h(x) = \{\zeta \in X^* : h^0(x;v) \ge \langle \zeta, v\rangle_{X^*\times X}$ for all $v \in X\}$. The locally Lipschitz function $h\colon X \to \mathbb{R}$ is called regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in X$.

## 3. Evolution hemivariational inequalities

In this section we first describe the classical model of thermoviscoelasticity and then we present its variational formulation.

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. Let $\Gamma$ be divided into three mutually disjoint measurable parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ such that $m(\Gamma_D) > 0$. We consider a viscoelastic body, occupying volume $\Omega$, which is supposed to be stress free and at a constant temperature, conveniently set as zero. We assume that the temperature changes accompanying the deformations are small and they do not produce any changes in the material parameters, which are regarded temperature independent. We confine ourselves to the classical linear thermoviscoelasticity theory and study the evolution of the system state in a time interval $[0,T]$ with $T > 0$. We suppose that the body is clamped on $\Gamma_D$, the volume forces of density $f_1(x,t) = (f_1^1(x,t),\ldots,f_1^d(x,t))$ act in $\Omega$ and the surface tractions of density $f_2(x,t) = (f_2^1(x,t),\ldots,f_2^d(x,t))$ are applied on $\Gamma_N$. Moreover, the body is subjected to a heat source term per unit volume $g = g(x,t)$ and it comes in contact with a rigid foundation over the potential contact surface $\Gamma_C$. We also put $Q = \Omega \times (0,T)$.

We denote by $u(x,t) = (u_1(x,t),\ldots,u_d(x,t))$ the displacement vector, by $\sigma = \{\sigma_{ij}\}$ the stress tensor, by $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ the linearized (small) strain tensor, where $i,j = 1,\ldots,d$ and by $\theta = \theta(x,t)$ the temperature. We assume with no loss of generality that the material density and the specific heat at constant deformation are constants, both set equal to one. The system of equations of motion assuming small displacements and the law of conservation of energy takes the form

$$u_i'' - \partial_j \sigma_{ij} = f_{1i} \quad \text{in } Q, \quad i = 1, \ldots, d,$$
$$\theta' + \partial_i q_i = -c_{ij} \partial_j u_i' + g \quad \text{in } Q.$$

Further, the heat flux vector $q = (q_1, \ldots, q_d)$ satisfies the Fourier law of heat conduction $q_i = -k_{ij} \partial_j \theta$ in $Q$. The behavior of the body is governed by the thermoviscoelastic constitutive equation of the Kelvin-Voigt type

$$\sigma_{ij} = a_{ijkl} \partial_l u_k' + b_{ijkl} \partial_l u_k - c_{ij} \theta \quad \text{in } Q, \ i, j = 1, \ldots, d,$$

where $\{a_{ijkl}\}$ and $\{b_{ijkl}\}$, $i$, $j$, $k$, $l = 1, \ldots, d$ are the viscosity and elasticity tensors, respectively, $\{c_{ij}\}$, $i$, $j = 1, \ldots, d$ being the so-called coefficients of thermal expansion.

Our main interest lies in the contact and friction boundary conditions on the surface $\Gamma_C$. As concerns the contact condition we assume that the normal stress $\sigma_N$ and the normal velocity $u_N'$ satisfy the nonmonotone normal damped response condition of the form

$$-\sigma_N \in \partial j_N(x, t, u_N') \quad \text{on } \Gamma_C \times (0, T).$$

The friction relation is given by

$$-\sigma_T \in \partial j_T(x, t, u_T') \quad \text{on } \Gamma_C \times (0, T)$$

and describes the multivalued law between the tangential force $\sigma_T$ on $\Gamma_C$ and the tangential velocity $u_T'$, see also Migórski (2005). Moreover, we suppose that between the boundary temperature and the heat flux vector the following relation holds: $q \cdot n \in \partial j(x, t, \theta)$, which we write as

$$-\frac{\partial \theta}{\partial n_K} \in \partial j(x, t, \theta) \quad \text{on } \Gamma_C \times (0, T),$$

where $\frac{\partial \theta}{\partial n_K} = k_{ij} \, \partial_j \theta \, n_i$. Here $j_N \colon \Gamma_C \times (0, T) \times \mathbb{R} \to \mathbb{R}$, $j_T \colon \Gamma_C \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ and $j \colon \Gamma_C \times (0, T) \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions in their last variables and $\partial j_N$, $\partial j_T$, $\partial j$ represent their Clarke subdifferentials. Finally, for the sake of simplicity we assume that $\theta = 0$ on $(\Gamma_D \cup \Gamma_N) \times (0, T)$.

Denoting by $u_0$, $u_1$ and $\theta_0$ the initial displacement, the initial velocity and the initial temperature, respectively, the classical formulation of the problem can be stated as follows (P):

find a displacement field $u\colon Q \to \mathbb{R}^d$ and a temperature $\theta\colon Q \to \mathbb{R}$ such that

$$\begin{cases} u_i'' - \partial_j\left(a_{ijkl}\partial_l u_k'\right) - \partial_j\left(b_{ijkl}\partial_l u_k\right) + \partial_j\left(c_{ij}\theta\right) = f_{1i} \ \text{ in } Q \\ \theta' - \partial_i\left(k_{ij}\partial_j\theta\right) + c_{ij}\partial_j u_i' = g \ \text{ in } Q \\ u = 0 \ \text{ on } \Gamma_D \times (0,T) \\ \sigma n = f_2 \ \text{ on } \Gamma_N \times (0,T) \\ -\sigma_N \in \partial j_N(x,t,u_N') \ \text{ on } \Gamma_C \times (0,T) \\ -\sigma_T \in \partial j_T(x,t,u_T') \ \text{ on } \Gamma_C \times (0,T) \\ -\dfrac{\partial\theta}{\partial n_K} \in \partial j(x,t,\theta) \ \text{ on } \Gamma_C \times (0,T) \\ \theta = 0 \ \text{ on } (\Gamma_D \cup \Gamma_N) \times (0,T) \\ u(0) = u_0, \ \ u'(0) = u_1, \ \ \theta(0) = \theta_0 \ \text{ in } \Omega. \end{cases}$$

We pass to the variational formulation of the above problem. We introduce the following spaces

$$E = \left\{ v \in H^1(\Omega;\mathbb{R}^d) \,:\, v = 0 \ \text{ on } \Gamma_D \right\},$$
$$V = \left\{ \eta \in H^1(\Omega) \,:\, \eta = 0 \ \text{ on } \Gamma_D \cup \Gamma_N \right\}.$$

On $E$ we consider the inner product and the corresponding norm given by

$$(u,v)_E = \langle \varepsilon(u), \varepsilon(v) \rangle_{L^2(\Omega;\mathcal{S}_d)}, \ \ \|v\|_E = \|\varepsilon(v)\|_{L^2(\Omega;\mathcal{S}_d)}.$$

From the Korn inequality, it follows that $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_E$ are the equivalent norms on $E$. Let $H = L^2(\Omega;\mathbb{R}^d)$ and $Z = H^\delta(\Omega;\mathbb{R}^d)$ with a fixed $\delta \in (1/2,1)$. Denoting by $i\colon E \to Z$ the embedding injection and by $\gamma\colon Z \to L^2(\Gamma;\mathbb{R}^d)$ the trace operator, for all $v \in E$, we have $\gamma_0 v = \gamma(iv)$. For simplicity we omit the notation of the embedding and write $\gamma_0 v = \gamma v$ for $v \in E$. Moreover, by $\gamma^*\colon L^2(\Gamma;\mathbb{R}^d) \to Z^*$ we denote the adjoint operator to $\gamma$. Identifying $H$ with its dual, we have the following evolution fivefold of spaces with dense, continuous and compact embeddings

$$E \subset Z \subset H \subset Z^* \subset E^*.$$

We also introduce the following spaces of vector valued functions $\mathcal{E} = L^2(0,T;E)$, $\mathcal{Z} = L^2(0,T;Z)$, $\mathcal{H} = L^2(0,T;H)$ and $\mathbb{E} = \{v \in \mathcal{E} \,:\, v' \in \mathcal{E}^*\}$, where the time derivative is understood in the sense of vector valued distributions. Endowed with the norm $||v||_{\mathbb{E}} = ||v||_{\mathcal{E}} + ||v'||_{\mathcal{E}^*}$, the space $\mathbb{E}$ becomes a separable reflexive Banach space. We have

$$\mathbb{E} \subset \mathcal{E} \subset \mathcal{Z} \subset \mathcal{H} \subset \mathcal{Z}^* \subset \mathcal{E}^*$$

with dense and continuous embeddings. The duality for the pair $(\mathcal{E},\mathcal{E}^*)$ is denoted by $\langle w,z\rangle_{\mathcal{E}^*\times\mathcal{E}} = \int_0^T \langle w(s), z(s)\rangle_{E^*\times E}\, ds$. It is well known (see, e.g., Denkowski et al., 2003) that the embeddings $\mathbb{E} \subset C(0,T;H)$ and $\{v \in \mathcal{E} : v' \in \mathbb{E}\} \subset C(0,T;E)$ are continuous and $\mathbb{E} \subset \mathcal{Z}$ is compact.

Similarly, we introduce $Y = H^\delta(\Omega)$ with the same $\delta \in (1/2, 1)$ and we have the evolution fivefold of spaces

$$V \subset Y \subset L^2(\Omega) \subset Y^* \subset V^*$$

with dense, continuous and compact embeddings. Let $\mathcal{V} = L^2(0,T;V)$, $\mathcal{Y} = L^2(0,T;Y)$ and $\mathcal{W} = \{\eta \in \mathcal{V} : \eta' \in \mathcal{V}^*\}$. We obtain

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{Y} \subset L^2(0,T;L^2(\Omega)) \subset \mathcal{Y}^* \subset \mathcal{V}^*,$$

where all the embeddings are dense and continuous. Furthermore, we know that the embeddings $\mathcal{W} \subset C(0,T;L^2(\Omega))$ and $\{\eta \in \mathcal{V} : \eta' \in \mathcal{W}\} \subset C(0,T;V)$ are continuous and $\mathcal{W} \subset \mathcal{Y}$ is compact.

Similarly as before, we denote by $\gamma_s \colon Y \to L^2(\Gamma)$ the trace operator for scalar valued functions, by $\gamma_s^* \colon L^2(\Gamma) \to Y^*$ its adjoint and we write $\gamma_0^s v = \gamma_s v$ for $v \in V$. We assume the following regularity for the density of heat sources, the body forces and surface tractions: $g \in \mathcal{V}^*$, $f_1 \in L^2(0,T;E^*)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, and we define $f \in \mathcal{E}^*$ by

$$\langle f(t), v\rangle_{E^*\times E} = \langle f_1(t), v\rangle_{E^*\times E} + (f_2(t), \gamma_0 v)_{L^2(\Gamma_N;\mathbb{R}^d)} \tag{1}$$

for $v \in E$ and a.e. $t \in (0,T)$. The fact that $f \in \mathcal{E}^*$ follows easily from the continuity of the trace operator. To obtain the variational formulation of the problem, we use a standard technique multiplying the first equation in (P) by $v \in E$ and using the Green formula. Then, taking into account the boundary condition on $\Gamma_N$ we have

$$\begin{aligned}&\langle u''(t) + Au'(t) + Bu(t) + C_1\theta(t), v\rangle_{E^*\times E} + \\ &+ \int_{\Gamma_C} \big(j_N^0(x,t,u_N'; v_N) + j_T^0(x,t,u_T'; v_T)\big)\, d\Gamma \geq \langle f(t), v\rangle_{E^*\times E}\end{aligned}$$

for all $v \in E$ and a.e. $t \in (0,T)$, where $A \colon E \to E^*$, $B \colon E \to E^*$ and $C_1 \colon V \to E^*$ are given by

$$\begin{aligned}\langle Aw, v\rangle_{E^*\times E} &= \int_\Omega a_{ijkl} \frac{\partial w_k}{\partial x_l} \frac{\partial v_i}{\partial x_j}\, dx,\\ \langle Bw, v\rangle_{E^*\times E} &= \int_\Omega b_{ijkl} \frac{\partial w_k}{\partial x_l} \frac{\partial v_i}{\partial x_j}\, dx,\\ \langle C_1\theta, v\rangle_{E^*\times E} &= -\int_\Omega c_{ij}\theta \frac{\partial v_i}{\partial x_j}\, dx\end{aligned}$$

for all $w$, $v \in E$ and $\theta \in V$.

Next, let $\eta \in V$. Multiplying the second equation in (P) by $\eta$, using again the Green formula and the heat flux boundary condition, we arrive at the following inequality

$$\langle \theta'(t) + C_2\theta(t) + C_3 u'(t), \eta\rangle_{V^*\times V} + \int_{\Gamma_C} j^0(x,t,\theta(t);\eta)\, d\Gamma \geq \langle g(t), \eta\rangle_{V^*\times V}$$

for all $\eta \in V$ and a.e. $t \in (0,T)$, where the operators $C_2 \colon V \to V^*$ and $C_3 \colon E \to V^*$ are defined by

$$\langle C_2\theta, \eta\rangle_{V^*\times V} = \int_\Omega k_{ij} \frac{\partial \theta}{\partial x_i}\frac{\partial \eta}{\partial x_j}\, dx,$$
$$\langle C_3 w, \eta\rangle_{V^*\times V} = \int_\Omega c_{ij} \frac{\partial w_i}{\partial x_j}\eta\, dx$$

for all $\theta$, $\eta \in V$ and $w \in E$. Hence we obtain the following variational formulation of the problem (P), which consists of two inequalities which are called hemivariational inequalities: find a displacement field $u \colon (0,T) \to E$ with $u' \colon (0,T) \to E$ and $u'' \colon (0,T) \to E^*$ and a temperature $\theta \colon (0,T) \to V$ with $\theta' \colon (0,T) \to V^*$ such that

$$\begin{cases} \langle u''(t) + Au'(t) + Bu(t) + C_1\theta(t), v\rangle_{E^*\times E} + \\ \quad + \displaystyle\int_{\Gamma_C} \left(j_N^0(x,t,u_N'(t);v_N) + j_T^0(x,t,u_T'(t);v_T)\right) d\Gamma \geq \langle f(t), v\rangle_{E^*\times E} \\ \qquad\qquad\qquad\qquad \text{for } v \in E, \text{a.e. } t \\ \langle \theta'(t) + C_2\theta(t) + C_3u'(t), \eta\rangle_{V^*\times V} + \\ \quad + \displaystyle\int_{\Gamma_C} j^0(x,t,\theta(t);\eta)\, d\Gamma \geq \langle g(t), \eta\rangle_{V^*\times V} \quad \text{for } \eta \in V, \text{a.e. } t \\ u(0) = u_0,\ u'(0) = u_1,\ \theta(0) = \theta_0. \end{cases} \tag{2}$$

In the next section we embed the inequality problem (2) into a more general class of problems for evolution inclusions for which we prove an existence result.

## 4. Existence and uniqueness of weak solutions to hemivariational inequalities

The goal of this section is to formulate a system of coupled evolution inclusions related to the system (2) of hemivariational inequalities. We start with the formulation of the hypotheses on the data of the problem (2).

$\underline{H(a)}$ : $a_{ijkl} \in L^\infty(\Omega)$, $a_{ijkl} = a_{jikl} = a_{lkij}$, $a_{ijkl}(x)\chi_{ij}\chi_{kl} \geq \alpha_1 \chi_{ij}\chi_{ij}$, for all $\chi = \{\chi_{ij}\} \in \mathcal{S}_d$, a.e. $x \in \Omega$ with $\alpha_1 > 0$.

$\underline{H(b)}$ : $b_{ijkl} \in L^\infty(\Omega)$, $b_{ijkl} = b_{jikl} = b_{lkij}$, $b_{ijkl}(x)\chi_{ij}\chi_{kl} \geq 0$, for all $\chi = \{\chi_{ij}\} \in \mathcal{S}_d$, a.e. $x \in \Omega$.

$\underline{H(c)}$ : $c_{ij} \in L^\infty(\Omega)$, $c_{ij} = c_{ji}$.

<u>$H(k)$</u> : $k_{ij} \in L^\infty(\Omega)$, $k_{ij} = k_{ji}$, $k_{ij}(x)\xi_i\xi_j \geq \alpha_2\xi_i\xi_j$ for all $\xi = \{\xi_i\} \in \mathbb{R}^d$, a.e. $x \in \Omega$ with $\alpha_2 > 0$.

<u>$H(j_N)$</u> : $j_N : \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $j_N(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;
(ii) $j_N(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $|\eta| \leq c_N\,(1+|r|)$ for all $\eta \in \partial j_N(x,t,r), r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $c_N > 0$;
(iv) $j_N^0(x,t,r;-r) \leq d_N\,(1+|r|)$ for all $r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $d_N \geq 0$.

<u>$H(j_T)$</u> : $j_T : \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j_T(\cdot,\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}^d$;
(ii) $j_T(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $\|\eta\| \leq c_T\,(1+\|\xi\|)$ for all $\eta \in \partial j_T(x,t,\xi), \xi \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $c_T > 0$;
(iv) $j_T^0(x,t,\xi;-\xi) \leq d_T\,(1+\|\xi\|)$ for all $\xi \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $d_T \geq 0$.

<u>$H(j)$</u> : $j : \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $j(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;
(ii) $j(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
(iii) $|\eta| \leq c\,(1+|r|)$ for all $\eta \in \partial j(x,t,r), r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $c > 0$;
(iv) $j^0(x,t,r;-r) \leq d\,(1+|r|)$ for all $r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $d \geq 0$.

<u>$H(f)$</u> : $f_1 \in L^2(0,T;E^*)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, $g \in \mathcal{V}^*$, $u_0 \in E$, $u_1 \in H$ and $\theta_0 \in L^2(\Omega)$.

In the hypotheses $H(j_N)$, $H(j_T)$ and $H(j)$ the Clarke subdifferentials are taken with respect to the last variables of $j_N$, $j_T$ and $j$, respectively.

Let us consider the functionals $J_1 : (0,T)\times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ and $J_2 : (0,T)\times L^2(\Gamma_C) \to \mathbb{R}$ defined by

$$J_1(t,v)=\int_{\Gamma_C}\Big(j_N(x,t,v_N(x))+j_T(x,t,v_T(x))\Big)\,d\Gamma,\quad v \in L^2(\Gamma_C;\mathbb{R}^d),\ t \in (0,T),$$

$$J_2(t,\theta) = \int_{\Gamma_C} j(x,t,\theta(x))\,d\Gamma,\quad \theta \in L^2(\Gamma_C),\ t \in (0,T).$$

Under the assumptions $H(j_N)$, $H(j_T)$ and $H(j)$ from Lemmas 4 and 5 of Denkowski and Migórski (2005) we know that $J_1$ and $J_2$ are well defined, $J_1(t,\cdot)$, $J_2(t,\cdot)$ are Lipschitz on every bounded subsets and satisfy suitable growth and sign conditions. We now formulate the following system of evolution inclusions:

find $u \in \mathcal{E}$ with $u' \in \mathbb{E}$ and $\theta \in \mathcal{W}$ such that

$$\begin{cases} u''(t) + Au'(t) + Bu(t) + C_1\theta(t) + \gamma^* \partial J_1(t, \gamma u'(t)) \ni f(t) \ \text{ a.e. } t \\ \theta'(t) + C_2\theta(t) + C_3 u'(t) + \gamma_s^* \partial J_2(t, \gamma_s \theta(t)) \ni g(t) \ \text{ a.e. } t \\ u(0) = u_0, \ u'(0) = u_1, \ \theta(0) = \theta_0. \end{cases} \tag{3}$$

By Proposition 6 in Denkowski and Migórski (2005), we have

PROPOSITION 1 *Under hypotheses $H(j_N)$, $H(j_T)$ and $H(j)$, every solution to the problem (3) is a solution to the system (2) of hemivariational inequalities. The converse also holds, provided $j_N(x,t,\cdot)$, $j_T(x,t,\cdot)$ and $j(x,t,\cdot)$ (or $-j_N(x,t,\cdot)$, $-j_T(x,t,\cdot)$ and $-j(x,t,\cdot)$) are regular in the sense of Clarke.*

The following is the main existence result for the problem (3). For the proof we refer the Reader again to Denkowski and Migórski (2005).

THEOREM 1 *Under the hypotheses $H(a)$, $H(b)$, $H(c)$, $H(k)$, $H(j_N)$, $H(j_T)$, $H(j)$ and $H(f)$ the problem (3) has a solution.*

Under some additional assumptions on the subdifferential terms we can get (see Theorem 15 of Denkowski and Migórski, 2005) the uniqueness of solutions to the problem (3).

$\underline{H(J)}$ : $J_1 \colon (0,T) \times L^2(\Gamma_C; \mathbb{R}^d) \to \mathbb{R}$ and $J_2 \colon (0,T) \times L^2(\Gamma_C) \to \mathbb{R}$ satisfy the following relaxed monotonicity conditions

$$(z_1 - z_2, v_1 - v_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \geq -m_1 \|v_1 - v_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$$

for all $z_i \in \partial J_1(t, v_i)$, $v_i \in L^2(\Gamma_C; \mathbb{R}^d)$, $i = 1, 2$ and a.e. $t$ with $m_1 > 0$,

$$(\eta_1 - \eta_2, \theta_1 - \theta_2)_{L^2(\Gamma_C)} \geq -m_2 \|\theta_1 - \theta_2\|^2_{L^2(\Gamma_C)}$$

for all $\eta_i \in \partial J_2(t, \theta_i)$, $\theta_i \in L^2(\Gamma_C)$, $i = 1, 2$ and a.e. $t$ with $m_2 > 0$.

$\underline{(H_1)}$ : $\alpha_1 > m_1 \|\gamma\|^2$ and $\alpha_2 > m_2 \|\gamma_s\|^2$, where $\|\gamma\| = \|\gamma\|_{\mathcal{L}(E, L^2(\Gamma;\mathbb{R}^d))}$ and $\|\gamma_s\| = \|\gamma_s\|_{\mathcal{L}(V, L^2(\Gamma))}$.

For the examples of functionals, which satisfy $H(J)$ we refer the Reader to Migórski (2005) and Migórski and Ochal (2005).

THEOREM 2 *Besides the hypotheses of Theorem 1, we assume additionally $H(J)$ and $(H_1)$. Then the problem (3) admits a unique solution.*

From Proposition 1 and Theorem 2 we have

COROLLARY 1 *If the hypotheses of Theorem 2 are satisfied, and $j_N$, $j_T$ and $j$ or $-j_N$, $-j_T$ and $-j$ are regular with respect to their third variables, then the system (2) of hemivariational inequalities has a unique solution $u \in \mathcal{E}$ with $u' \in \mathbb{E}$ and $\theta \in \mathcal{W}$.*

## 5. Optimal control problem

In this section we shall study an optimal control problem for a system described by the evolution inclusions (3). We suppose that in the problem (3) the control variable is denoted by $\varphi = (f, g, u_0, u_1, \theta_0) \in \Phi$ and $\Phi = \mathcal{E}^* \times \mathcal{V}^* \times E \times H \times L^2(\Omega)$ represents the space of controls. Recall that $f$ is given by (1) and corresponds to the density of volume forces $f_1 \in L^2(0,T;E^*)$ and the density of surface tractions $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$. For every $\varphi \in \Phi$ we introduce the solution set $S(\varphi) = \{y \in \mathcal{S} : y = y(\varphi) \text{ is a solution of } (3)\}$, where $\mathcal{S} = \mathcal{E} \times \mathbb{E} \times \mathcal{W}$ and $y = y(\varphi) = (u(\varphi), u'(\varphi), \theta(\varphi))$ denotes the solution to (3) corresponding to $\varphi$. It is known from the previous section that under the hypotheses of Theorem 2, for every $\varphi \in \Phi$, the problem (3) admits a unique solution $y = y(\varphi) \in \mathcal{S}$, while under assumptions of Theorem 1 the set $\mathcal{S}(\varphi)$ can contain more than one element.

The control problem is formulated as follows. Given $\Phi_{ad}$, a nonempty subset of $\Phi$ (representing the set of admissible controls), and an objective functional $F \colon \Phi \times \mathcal{S} \to \mathbb{R}$, $F = F(\varphi, y)$ find a control $\varphi^* \in \Phi_{ad}$ and a state $y^* = y(\varphi^*) \in \mathcal{S}$ such that

$$F(\varphi^*, y^*) = \inf\{F(\varphi, y) : \varphi \in \Phi_{ad}, y = y(\varphi)\}. \tag{4}$$

We remark that since $f$ depends on the density of surface tractions on the boundary $\Gamma_N \times (0,T)$, the boundary control problems are incorporated in the formulation of (4). Moreover, the problem (4) also covers the control via initial conditions, see Lions (1971).

The proof of existence of optimal solutions for (4) is based on a result on the continuous dependence on the data in the weak topologies for solutions of (3).

PROPOSITION 2 *Assume that the hypotheses of Theorem 1 hold. Then the multivalued map $\Phi \ni \varphi \mapsto S(\varphi) \subset \mathcal{S}$ is upper semicontinuous, where both $\Phi$ and $\mathcal{S}$ are endowed with their weak topologies.*

*Proof.* Let $\{(f_k, g_k, u_0^k, u_1^k, \theta_0^k)\}$ be a sequence in $\mathcal{E}^* \times \mathcal{V}^* \times E \times H \times L^2(\Omega)$ which converges weakly in this space to $(f, g, u_0, u_1, \theta_0)$. From Theorem 1 we know that for every $k \in \mathbb{N}$ there exists a solution $(u_k, u_k', \theta_k) \in \mathcal{E} \times \mathbb{E} \times \mathcal{W}$ of the problem

$$\begin{cases} u_k''(t) + Au_k'(t) + Bu_k(t) + C_1\theta_k(t) + \gamma^*\partial J_1(t, \gamma u_k'(t)) \ni f_k(t) \quad \text{a.e. } t \\ \theta_k'(t) + C_2\theta_k(t) + C_3 u_k'(t) + \gamma_s^*\partial J_2(t, \gamma_s\theta_k(t)) \ni g_k(t) \quad \text{a.e. } t \\ u_k(0) = u_0^k, \ u_k'(0) = u_1^k, \ \theta_k(0) = \theta_0^k. \end{cases} \tag{5}$$

By Lemma 14 of Denkowski and Migórski (2005), we have the following estimate

$$\begin{aligned} &\|u_k\|_{C(0,T;E)} + \|u_k'\|_{\mathbb{E}} + \|\theta_k\|_{C(0,T;L^2(\Omega))} + \|\theta_k\|_{\mathcal{W}} \le \\ &\le c\left(1 + \|u_0^k\|_E + \|u_1^k\|_H + \|\theta_0^k\|_{L^2(\Omega)} + \|f_k\|_{\mathcal{E}^*} + \|g_k\|_{\mathcal{V}^*}\right) \end{aligned} \tag{6}$$

with a positive constant $c$ independent of $k$. Since $\{(f_k, g_k, u_0^k, u_1^k, \theta_0^k)\}$ is bounded in $\mathcal{E}^* \times \mathcal{V}^* \times E \times H \times L^2(\Omega)$, by passing to a subsequence, if necessary, we may assume that

$$\begin{cases} u_k \to u \quad \text{weakly in} \quad \mathcal{E} \\ u_k' \to u' \quad \text{weakly in} \quad \mathcal{E} \\ u_k'' \to u'' \quad \text{weakly in} \quad \mathcal{E}^* \\ \theta_k \to \theta \quad \text{weakly in} \quad \mathcal{W}. \end{cases} \tag{7}$$

We will show that $(u, u', \theta)$ is a solution to the limit problem. Due to the continuity of the embedding $\mathbb{E} \subset C(0, T; H)$ from the convergences $u_k \to u$ and $u_k' \to u'$ both weakly in $\mathbb{E}$, we obtain

$$u_k(t) \to u(t), \quad u_k'(t) \to u'(t) \ \text{ both weakly in } H \text{ for all } t \in [0, T].$$

Hence $u_k(0) \to u(0)$ weakly in $H$ gives $u(0) = u_0$. From the convergences $u_1^k \to u_1$ weakly in $H$ and $u_k'(0) \to u'(0)$ weakly in $H$ we obtain $u'(0) = u_1$. Analogously, exploiting the continuity of the embedding $\mathcal{W} \subset C(0, T; L^2(\Omega))$ we deduce $\theta_k(t) \to \theta(t)$ weakly in $L^2(\Omega)$ for all $t \in [0, T]$ which implies $\theta(0) = \theta_0$.

Next, from the compactness of the embeddings $\mathbb{E} \subset \mathcal{Z}$ and $\mathcal{W} \subset \mathcal{Y}$, we may assume that $u_k' \to u'$ in $\mathcal{Z}$ and $\theta_k \to \theta$ in $\mathcal{Y}$. Let us define the operator $\mathcal{N}_1 : \mathcal{E} \to 2^{\mathcal{E}^*}$ by

$$\mathcal{N}_1 v = \{w \in \mathcal{Z}^* : w(t) \in \gamma^* \partial J_1(t, \gamma v(t)) \text{ a.e. } t\}, \quad v \in \mathcal{E}.$$

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be the Nemitsky operators corresponding to $A$, $B$, $C_1$, $C_2$ and $C_3$, respectively. The first relation in (5) can be written as

$$u_k'' + \mathcal{A}u_k' + \mathcal{B}u_k + \mathcal{C}_1\theta_k + w_k = f_k \quad \text{with} \quad w_k \in \mathcal{N}_1 u_k'. \tag{8}$$

Using the boundedness of $\mathcal{N}_1$ (see Denkowski and Migórski, 2005, Lemma 11(i)), we may suppose

$$w_k \to w \quad \text{weakly in} \quad \mathcal{Z}^*.$$

Since $u_k' \to u'$ in $\mathcal{Z}$, from Lemma 11(iii) of Denkowski and Migórski (2005), we easily get $w \in \mathcal{N}_1 u'$. Next, since the operators $\mathcal{A}$, $\mathcal{B} \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ and $\mathcal{C}_1 \in \mathcal{L}(\mathcal{V}, \mathcal{E}^*)$ are weakly-weakly continuous, by (7) we pass to the limit in (8) and obtain

$$u''(t) + \mathcal{A}u'(t) + \mathcal{B}u(t) + \mathcal{C}_1\theta(t) + w = f(t) \quad \text{with} \quad w \in \mathcal{N}_1 u'. \tag{9}$$

Subsequently, we consider the operator $\mathcal{N}_2 : \mathcal{V} \to 2^{\mathcal{V}^*}$ defined by

$$\mathcal{N}_2 \theta = \{\eta \in \mathcal{Y}^* : \eta(t) \in \gamma_s^* \partial J_2(t, \gamma_s \theta(t)) \text{ a.e. } t\}, \quad \theta \in \mathcal{V}.$$

The second inclusion in (5) can be written as follows

$$\theta_k' + \mathcal{C}_2\theta_k + \mathcal{C}_3 u_k' + \eta_k = g_k \quad \text{with} \quad \eta_k \in \mathcal{N}_2\theta_k. \tag{10}$$

From the boundedness of $\mathcal{N}_2$ (see Denkowski and Migórski, 2005, Lemma 12(i)), we may assume that $\eta_k \to \eta$ weakly in $\mathcal{Y}^*$. Using the convergence $\theta_k \to \theta$ in $\mathcal{Y}$ and Lemma 12(iii) of Denkowski and Migórski (2005), we deduce $\eta \in \mathcal{N}_2\theta$. Similarly as before, we use the weak-weak continuity of $\mathcal{C}_2 \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and $\mathcal{C}_3 \in \mathcal{L}(\mathcal{E}, \mathcal{V}^*)$ and pass to the weak-$\mathcal{V}^*$ limit in (10). We have

$$\theta' + \mathcal{C}_2\theta + \mathcal{C}_3 u' + \eta = g \quad \text{with} \quad \eta \in \mathcal{N}_2\theta. \tag{11}$$

The equations (9) and (11), together with the initial conditions $u(0) = u_0$, $u'(0) = u_1$ and $\theta(0) = \theta_0$, imply that $(u, u', \theta)$ is a solution to the limit problem corresponding to (5). This proves the theorem. ∎

We need two more hypotheses:

$\underline{H(\Phi_{ad})}$ : $\Phi_{ad}$ is a weakly compact subset of $\Phi$.

$\underline{H(F)}$ : $F$ is lower semicontinuous with respect to weak-$(\Phi \times \mathcal{S})$ topology.

We are now in a position to deliver an existence result for the optimal control problem (4).

THEOREM 3 *If the hypotheses of Theorem 1 hold and $H(\Phi_{ad})$ and $H(F)$ are satisfied, then the problem (4) has an optimal solution.*

*Proof.* Let $\{(\varphi_n, y_n)\}$ be a minimizing sequence for the problem (4), that is, $\varphi_n \in \Phi_{ad}$, $y_n \in \mathcal{S}(\varphi_n)$ and

$$\lim_{n\to\infty} F(\varphi_n, y_n) = \inf\{F(\varphi, y) : \varphi \in \Phi_{ad}, y = y(\varphi)\} = m \in [-\infty, +\infty).$$

By the compactness of $\Phi_{ad}$ we may choose a subsequence $\{\varphi_n\}$ such that $\varphi_n \to \varphi^*$ weakly in $\Phi$ with $\varphi^* \in \Phi_{ad}$. From Proposition 2 we obtain $y_n \to y^*$ weakly in $\mathcal{S}$ with $y^* = y(\varphi^*)$. Thus, due to $H(F)$ we have $m \leq F(\varphi^*, y^*) \leq \liminf_{n\to\infty} F(\varphi_n, y_n) = m$, so $m \in (-\infty, +\infty)$ which completes the proof. ∎

As examples, we may consider the following cost functionals or their combinations:

$$F_1(\varphi, y) = \int_0^T L(t, u(t), u'(t), \theta(t))\, dt + G(\varphi)$$

with prescribed functions $L$ and $G$,

$$F_2(\varphi, y) = \sum_{i=1}^{r} \left( \left\| u(t_i) - w_i^1 \right\|_E^2 + \left\| u'(t_i) - w_i^2 \right\|_E^2 \right),$$

$$F_3(\varphi, y) = \int_0^T \int_\Gamma \left( \|u(x,t) - w^3\|^2 + \|u'(x,t) - w^4\|^2 \right) dxdt,$$

$$F_4(\varphi, y) = \sum_{i=1}^{r} \|\theta(t_i) - w_i^5\|_{L^2(\Omega)}^2,$$

where $\varphi = (f, g, u_0, u_1, \theta_0)$, $y = (u, u', \theta)$, $0 < t_1 < t_2 < \ldots t_r \le T$ are points of measurements and $w_i^1$, $w_i^2$, $w^3$, $w^4$ and $w_i^5$ are fixed targets,

$$F_5(\varphi, y) = \int_0^T \int_\Omega \varrho(x,t) |\sigma^D(u)(x,t) - \sigma_g(x,t)|^2 \, dxdt,$$

where $\sigma^D = \sigma - \frac{1}{d}(tr\,\sigma)I$ is the stress deviator, $tr\,\sigma$ is the trace of $\sigma$ and $I$ is the identity matrix, $\varrho$ is a smooth weight function and $\sigma_g$ is a given target.

## 6. Time optimal control problem

In this section we consider a time optimal control problem. The task is to reach a moving target set in minimal time. The target set is moving independently of the solution set to the controlled system of evolution inclusions.

We consider a system described by the following inclusions

$$\begin{cases} u''(t) + Au'(t) + Bu(t) + C_1\theta(t) + \\ \qquad\qquad + \gamma^* \partial J_1(t, \gamma u'(t)) \ni f(t) + G_1(t)\psi_1(t) \quad \text{a.e. } t \\ \theta'(t) + C_2\theta(t) + C_3 u'(t) + \\ \qquad\qquad + \gamma_s^* \partial J_2(t, \gamma_s \theta(t)) \ni g(t) + G_2(t)\psi_2(t) \quad \text{a.e. } t \\ u(0) = u_0, \ u'(0) = u_1, \ \theta(0) = \theta_0. \end{cases} \tag{12}$$

The data $A$, $B$, $C_1$, $C_2$, $C_3$, $J_1$, $J_2$, $f$, $g$, $u_0$, $u_1$ and $\theta_0$ are as in Section 4 and they are fixed. We admit $(\psi_1, \psi_2)$ as control parameters which provide "source-like" densities of volume forces and heat sources by means of operators $G_1$ and $G_2$. We denote by $y = y(\psi_1, \psi_2) = (u(\psi_1, \psi_2), u'(\psi_1, \psi_2), \theta(\psi_1, \psi_2))$ a solution correspoding to a control $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2 = L^2(0, T; Y_1 \times Y_2)$, where $Y_i$ is a space of control variables and $G_i$ represents a controller, $i = 1, 2$. Thus we consider the solution map

$$S \colon L^2(0, T; Y_1 \times Y_2) \ni (\psi_1, \psi_2) \mapsto S(\psi_1, \psi_2) \subset \mathcal{E} \times \mathbb{E} \times \mathcal{W},$$

where $S(\psi_1, \psi_2)$ is the solution set of the controlled system (12) corresponding to $(\psi_1, \psi_2)$. We observe that if $(u, u', \theta) \in S(\psi_1, \psi_2)$, then $(u, u', \theta) \in C(0, T; E \times$

$H \times L^2(\Omega)$). We introduce the following assumption concerning the controllers and spaces of controls.

<u>$H(G)$</u> : $G_1 \in L^\infty(0,T;\mathcal{L}(Y_1,Z^*)), G_2 \in L^\infty(0,T;\mathcal{L}(Y_2,Y^*))$, $Y_1$ and $Y_2$ are separable reflexive Banach spaces of controls.

Moreover, an additional hypothesis on the elasticity tensor is needed.

<u>$H(b)_1$</u> the coefficients $b_{ijkl}$ satisfy $H(b)$ and $b_{ijkl}(x)\chi_{ij}\chi_{kl} \geq \beta\chi_{ij}\chi_{ij}$, for all $\chi = \{\chi_{ij}\} \in \mathcal{S}_d$, a.e. $x \in \Omega$ with $\beta > 0$.

Proposition 3 *Suppose the hypotheses $H(a)$, $H(b)_1$, $H(c)$, $H(k)$, $H(j_N)$, $H(j_T)$, $H(j)$, $H(f)$, $H(G)$ hold. If $\{(\psi_1^n,\psi_2^n)\} \subset \Psi_1 \times \Psi_2$, $\psi_i^n \to \psi_i$ weakly in $\Psi_i$, $i = 1, 2$, $y_n \in S(\psi_1^n,\psi_2^n)$, then there exists a subsequence of $\{y_n\}$, denoted by the same symbol, such that $y_n \to y$ in $C(0,T;E \times H \times L^2(\Omega))$ and $y \in S(\psi_1,\psi_2)$.*

*Proof.* Let $\{(\psi_1^n,\psi_2^n)\} \subset \Psi_1 \times \Psi_2$, $\psi_i^n \to \psi_i$ weakly in $\Psi_i$, $i = 1, 2$ and $y_n \in S(\psi_1^n,\psi_2^n) \neq \emptyset$ (by Theorem 1). From the a priori estimate (6), we obtain

$$\begin{aligned}&\|u_n\|_{C(0,T;E)} + \|u_n'\|_{\mathbb{E}} + \|\theta_n\|_{C(0,T;L^2(\Omega))} + \|\theta_n\|_{\mathcal{W}} \leq\\ &\leq c_2(1 + \|u_0\|_E + \|u_1\|_H + \|\theta_0\|_{L^2(\Omega)} + \|f\|_{\mathcal{E}^*} + \|g\|_{\mathcal{V}^*} +\\ &+\|G_1\|_{L^\infty(0,T;\mathcal{L}(Y_1,Z^*))}\|\psi_1^n\|_{L^2(0,T;Y_1)} + \|G_2\|_{L^\infty(0,T;\mathcal{L}(Y_2,Y^*))}\|\psi_2^n\|_{L^2(0,T;Y_2)}),\end{aligned}$$

where a positive constant $c_2$ is independent of $n$. Hence $\{(u_n, u_n', \theta_n)\}$ is bounded in reflexive Banach space $\mathcal{E}\times\mathbb{E}\times\mathcal{W}$, so, by passing to a subsequence if necessary, we have

$$(u_n, u_n', \theta_n) \to (u, u', \theta) \text{ weakly in } \mathcal{E} \times \mathbb{E} \times \mathcal{W}.$$

We introduce the linear continuous Nemitsky operators $\mathcal{G}_1 \colon L^2(0,T;Y_1) \to \mathcal{Z}^*$ and $\mathcal{G}_2 \colon L^2(0,T;Y_2) \to \mathcal{Y}^*$ given by $(\mathcal{G}_1\psi_1)(t) = G_1(t)\psi_1(t)$ for $\psi_1 \in \Psi_1$ and $(\mathcal{G}_2\psi_2)(t) = G_2(t)\psi_2(t)$ for $\psi_2 \in \Psi_2$. We have $\mathcal{G}_1\psi_1^n \to \mathcal{G}_1\psi_1$ weakly in $\mathcal{Z}^*$ and hence also weakly in $\mathcal{E}^*$, and $\mathcal{G}_2\psi_2^n \to \mathcal{G}_2\psi_2$ weakly in $\mathcal{Y}^*$ and hence weakly in $\mathcal{V}^*$. Similarly as in Proposition 2 we can show that $y \in S(\psi_1,\psi_2)$. In order to prove the assertion, first we notice that from the following formulae

$$\begin{cases} u_n''(t) + Au_n'(t) + Bu_n(t) + C_1\theta_n(t) + w_n(t) = f(t) + G_1(t)\psi_1^n(t) \text{ a.e. } t\\ w_n(t) \in \gamma^*\partial J_1(t, \gamma u_n'(t)) \text{ a.e. } t\\ \theta_n'(t) + C_2\theta_n(t) + C_3u_n'(t) + \eta_n(t) = g(t) + G_2(t)\psi_2^n(t) \text{ a.e. } t\\ \eta_n(t) \in \gamma_s^*\partial J_2(t, \gamma_s\theta_n(t)) \text{ a.e. } t\\ u_n(0) = u_0,\ u_n'(0) = u_1,\ \theta_n(0) = \theta_0 \end{cases}$$

and

$$\begin{cases} u''(t) + Au'(t) + Bu(t) + C_1\theta(t) + w(t) = f(t) + G_1(t)\psi_1(t) \text{ a.e. } t \\ w(t) \in \gamma^*\partial J_1(t, \gamma u'(t)) \text{ a.e. } t \\ \theta'(t) + C_2\theta(t) + C_3u'(t) + \eta(t) = g(t) + G_2(t)\psi_2(t) \text{ a.e. } t \\ \eta(t) \in \gamma_s^*\partial J_2(t, \gamma_s\theta(t)) \text{ a.e. } t \\ u(0) = u_0,\ u'(0) = u_1,\ \theta(0) = \theta_0, \end{cases}$$

we get for $t \in [0, T]$

$$\begin{aligned} &\int_0^t \langle u_n''(s) - u''(s), u_n'(s) - u'(s)\rangle\, ds + \int_0^t \langle A(u_n'(s) - u'(s)), u_n'(s) - u'(s)\rangle\, ds + \\ &+ \int_0^t \langle B(u_n(s) - u(s)), u_n'(s) - u'(s)\rangle\, ds + \int_0^t \langle C_1(\theta_n(s) - \theta(s)), u_n'(s) - u'(s)\rangle\, ds + \\ &+ \int_0^t \langle w_n(s) - w(s), u_n'(s) - u'(s)\rangle\, ds + \int_0^t \langle \theta_n'(s) - \theta'(s), \theta_n(s) - \theta(s)\rangle\, ds + \\ &+ \int_0^t \langle C_2(\theta_n(s) - \theta(s)), \theta_n(s) - \theta(s)\rangle\, ds + \int_0^t \langle C_3(u_n'(s) - u'(s)), \theta_n(s) - \theta(s)\rangle\, ds + \\ &+ \int_0^t \langle \eta_n(s) - \eta(s), \theta_n(s) - \theta(s)\rangle\, ds = \\ &= \int_0^t \langle G_1(s)\psi_1^n(s) - G_1(s)\psi_1(s), u_n'(s) - u'(s)\rangle\, ds + \\ &+ \int_0^t \langle G_2(s)\psi_2^n(s) - G_2(s)\psi_2(s), \theta_n(s) - \theta(s)\rangle\, ds. \end{aligned}$$

Exploiting $H(j_N)$(iii) and $H(j_T)$(iii), we have $\|w_n - w\|_{\mathcal{Z}^*} \le \|w_n\|_{\mathcal{Z}^*} + \|w\|_{\mathcal{Z}^*} \le c_3\,(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})$ with $c_3 > 0$ and subsequently

$$\begin{aligned} &\int_0^t \langle w_n(s) - w(s), u_n'(s) - u'(s)\rangle\, ds \le \int_0^t \|w_n(s) - w(s)\|_{Z^*}\|u_n'(s) - u'(s)\|_Z\, ds \le \\ &\le \|w_n - w\|_{\mathcal{Z}^*}\|u_n' - u'\|_{\mathcal{Z}} \le c_3\,(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\ \|u_n' - u'\|_{\mathcal{Z}} \quad \text{for } t \in [0, T]. \end{aligned}$$

Analogously, from $H(j)$(iii), we get

$$\begin{aligned} &\int_0^t \langle \eta_n(s) - \eta(s), \theta_n(s) - \theta(s)\rangle\, ds \le \int_0^t \|\eta_n(s) - \eta(s)\|_{Y^*}\|\theta_n(s) - \theta(s)\|_Y\, ds \le \\ &\le \|\eta_n - \eta\|_{\mathcal{Z}^*}\|\theta_n - \theta\|_{\mathcal{Y}} \le c_4\,(1 + \|\theta_n\|_{\mathcal{Y}} + \|\theta\|_{\mathcal{Y}})\ \|\theta_n - \theta\|_{\mathcal{Y}} \end{aligned}$$

for $t \in [0, T]$ with $c_4 > 0$. Next, using the last two inequalities, the following relations

$$\int_0^t \langle u_n''(s) - u''(s), u_n'(s) - u'(s)\rangle_{E^*\times E}\, ds = \frac{1}{2}\|u_n'(t) - u'(t)\|_H^2$$
$$\int_0^t \langle A(u_n'(s) - u'(s)), u_n'(s) - u'(s)\rangle_{E^*\times E}\, ds \geq \alpha_1 \|u_n' - u'\|^2_{L^2(0,t;E)}$$
$$\int_0^t \langle B(u_n(s) - u(s)), u_n'(s) - u'(s)\rangle_{E^*\times E}\, ds \geq \beta \|u_n(t) - u(t)\|_E^2$$
$$\int_0^t \langle \theta_n'(s) - \theta'(s), \theta_n(s) - \theta(s)\rangle_{V^*\times V}\, ds = \frac{1}{2}\|\theta_n(t) - \theta(t)\|^2_{L^2(\Omega)}$$
$$\int_0^t \langle C_2(\theta_n(s) - \theta(s)), \theta_n(s) - \theta(s)\rangle_{V^*\times V}\, ds \geq \alpha_2 \|\theta_n - \theta\|^2_{L^2(0,t;V)}$$

and $\langle C_1\eta, v\rangle_{E^*\times E} + \langle C_3 v, \eta\rangle_{V^*\times V} = 0$ for all $v \in E$, $\eta \in V$ (see Denkowski and Migórski, 2005, Lemma 3(vi)), we obtain, for all $t \in [0, T]$

$$\begin{aligned}
&\|u_n(t) - u(t)\|_E^2 + \|u_n'(t) - u'(t)\|_H^2 + \|\theta_n(t) - \theta(t)\|^2_{L^2(\Omega)} \leq \\
&\leq c_3\,(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\ \|u_n' - u'\|_{\mathcal{Z}} + c_4\,(1 + \|\theta_n\|_{\mathcal{Y}} + \|\theta\|_{\mathcal{Y}})\ \|\theta_n - \theta\|_{\mathcal{Y}} + \\
&+ \|\mathcal{G}_1(\psi_1^n - \psi_1)\|_{\mathcal{Z}^*}\|u_n' - u'\|_{\mathcal{Z}} + \|\mathcal{G}_2(\psi_2^n - \psi_2)\|_{\mathcal{Y}^*}\|\theta_n - \theta\|_{\mathcal{Y}} \leq \\
&\leq c_5\,(1 + \|u_n'\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}} + \|\mathcal{G}_1(\psi_1^n - \psi_1)\|_{\mathcal{Z}^*})\ \|u_n' - u'\|_{\mathcal{Z}} + \\
&+ c_5\,(1 + \|\theta_n\|_{\mathcal{Y}} + \|\theta\|_{\mathcal{Y}} + \|\mathcal{G}_2(\psi_2^n - \psi_2)\|_{\mathcal{Y}^*})\ \|\theta_n - \theta\|_{\mathcal{Y}}
\end{aligned}$$

with a positive constant $c_5$ independent of $n$. Since $\mathcal{G}_1\psi_1^n \to \mathcal{G}_1\psi_1$ weakly in $\mathcal{Z}^*$, $u_n' \to u'$ weakly in $\mathbb{E}$ and strongly in $\mathcal{Z}$, $\mathcal{G}_2\psi_2^n \to \mathcal{G}_2\psi_2$ weakly in $\mathcal{Y}^*$, $\theta_n \to \theta$ weakly in $\mathcal{W}$ and strongly in $\mathcal{Y}$ (recall that $\mathbb{E} \subset \mathcal{Z}$ and $\mathcal{W} \subset \mathcal{Y}$ compactly), we deduce

$$(u_n, u_n', \theta_n) \to (u, u', \theta) \ \text{ in } C(0, T; E \times H \times L^2(\Omega)),$$

which concludes the proof. ∎

Corollary 2 *Under the assumptions of Proposition 3 it is obvious that for every fixed* $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$, $S(\psi_1, \psi_2)$ *is a compact subset of* $C(0, T; E \times H \times L^2(\Omega))$.

We are now in a position to consider a time optimal control problem. We need the following hypotheses concerning the target set $N$ (e.g. changing in time a set of desirable states for $(u, u', \theta)$) and the control constraint sets $U_i$, $i = 1, 2$.

H(N) : $N\colon [0, T] \to 2^{E\times H\times L^2(\Omega)} \setminus \{\emptyset\}$ is a measurable multifunction with closed graph in $[0, T] \times E \times H \times L^2(\Omega)$.

H(U) : $U_i\colon [0, T] \to 2^{Y_i} \setminus \{\emptyset\}$, $i = 1, 2$ is a multifunction such that for all $t \in [0, T]$, $U_i(t)$ is a closed convex subset of $Y_i$ and $t \mapsto \|U_i(t)\|_{Y_i} := \sup\{\|u\|_{Y_i} : u \in U_i(t)\}$ belongs to $L^\infty(0, T)$.

We assume the following controllability type hypothesis:

$\underline{(H_c)}$ There exists $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2 = L^2(0, T; Y_1 \times Y_2)$ such that $\psi_i(t) \in \overline{U_i(t)}$, $i = 1, 2$ and for some appropriate $\tau \in (0, T)$ we have $y(\tau) \in N(\tau)$, where $y = y(\psi_1, \psi_2)$ is a solution to (12) corresponding to $(\psi_1, \psi_2)$.

A control-state triple $(\psi_1, \psi_2, y)$ is called admissible if $\psi_i \in L^2(0, T; Y_i)$, $\psi_i(t) \in U_i(t)$, $i = 1, 2$ for a.e. $t$ and $y \in S(\psi_1, \psi_2)$. We define the optimal time as follows

$$t_0 = \inf\{\tau \in (0, T) : \tau \text{ is such that } (\mathrm{H_c}) \text{ holds}\} \tag{13}$$

and we consider the following time optimal control problem

$$\begin{cases} \text{find an admissible triple } (\psi_1, \psi_2, y) \text{ such that } y(t_0) \in N(t_0), \\ \text{where } t_0 \text{ is the optimal time given by (13).} \end{cases} \tag{14}$$

An admissible triple $(\psi_1, \psi_2, y)$ satisfying (14) is called the time optimal solution.

Now we establish the existence result for the problem (14).

THEOREM 4 *If assumptions $H(a)$, $H(b)_1$, $H(c)$, $H(k)$, $H(j_N)$, $H(j_T)$, $H(j)$, $H(f)$, $H(G)$, $H(N)$, $H(U)$ and $(H_c)$ hold, then the problem (14) admits a time optimal solution.*

*Proof.* First we note that from Theorem 1 for every $(\psi_1, \psi_2) \in \Psi_1 \times \Psi_2$ the solution set $S(\psi_1, \psi_2)$ of (12) is nonempty.

From the definition of $t_0$, we know that there exists a sequence $\{t_n\} \subset (0, T)$ such that $t_n \to t_0$ as $n \to \infty$ and $t_n$ is as in $(H_c)$. By $(H_c)$ we know that for all $n \in \mathbb{N}$, there exist a control $(\psi_1^n, \psi_2^n) \in \Psi_1 \times \Psi_2$, $\psi_i^n \in U_i(t)$ for a.e. $t \in (0, T)$ and a state $y_n \in S(\psi_1^n, \psi_2^n)$ such that $y_n(t_n) \in N(t_n)$. From $H(U)$, we get that for $i = 1, 2$, the sequence $\{\psi_i^n\}$ is bounded in $L^2(0, T; Y_i)$, thus we may assume that $\psi_i^n \to \psi_i$ weakly in $\Psi_i$, as $n \to \infty$. Since for $i = 1, 2$ and a.e. $t \in (0, T)$, $U_i(t)$ is a weakly compact subset of $Y_i$, by applying Proposition 4.7.44 of Denkowski et al. (2003), we have

$$\psi_i(t) \in \overline{\text{conv}}\ (w{-}Y_i)\text{–}\limsup_{n\to\infty}\ \{\psi_i^n(t)\} \subset \overline{\text{conv}}\ U_i(t) \ \text{ a.e. } t \in (0, T),$$

which, together with the fact that $U_i(t)$ is a closed and convex subset of $Y_i$, implies $\psi_i(t) \in U_i(t)$ for a.e. $t \in (0, T)$, $i = 1, 2$. From Proposition 3, at least for a subsequence, we have

$$y_n \to y \text{ in } \ C(0, T; E \times H \times L^2(\Omega)) \text{ and } y \in S(\psi_1, \psi_2).$$

Because $y \in C(0, T; E \times H \times L^2(\Omega))$ and $t_n \to t_0$, we immediately get

$$\begin{aligned}
&\|y_n(t_n) - y(t_0)\|_{E\times H\times L^2(\Omega)} \le \\
&\le \|y_n(t_n) - y(t_n)\|_{E\times H\times L^2(\Omega)} + \|y(t_n) - y(t_0)\|_{E\times H\times L^2(\Omega)} \le \\
&\le \sup_{t\in[0,T]} \|y_n(t) - y(t)\|_{E\times H\times L^2(\Omega)} + \|y(t_n) - y(t_0)\|_{E\times H\times L^2(\Omega)} \to 0,
\end{aligned}$$

as $n \to \infty$. Hence, $y_n(t_n) \to y(t_0)$ in $E \times H \times L^2(\Omega)$. Since $y_n(t_n) \in N(t_n)$, from $H(N)$, it follows that $y(t_0) \in N(t_0)$. This means that $t_0$ is the optimal time for (14) and $(\psi_1, \psi_2, y)$ is a required time optimal solution. ■

## 7. Maximum stay control problem

In this section our task is to regulate the system governed by evolution inclusions (12) so as to maximize its stay in a preferred region $N(\cdot)$ of the state space. We consider the following function

$$\vartheta\colon C(0,T;E\times H\times L^2(\Omega)) \ni y \mapsto \vartheta(y) = m\left(\{t \in [0,T] : y(t) \in N(t)\}\right) \in \mathbb{R}_+,$$

where $m$ is the Lebesgue measure on $\mathbb{R}$.

A maximum stay problem is as follows: find an admissible triple $(\psi_1, \psi_2, y)$ such that it solves the following maximization problem

$$\sup\{\vartheta(y) : y \in S(\psi_1,\psi_2),\ (\psi_1,\psi_2) \in \Psi_1 \times \Psi_2,\ \psi_i(t) \in U_i(t), i = 1,2\}. \tag{15}$$

Theorem 5 *Under the hypotheses of Theorem 4, the problem (15) has a solution.*

*Proof.* Let us introduce the function

$$\rho\colon \Psi_1 \times \Psi_2 \ni (\psi_1,\psi_2) \mapsto \sup\{\vartheta(y) \,:\, y \in S(\psi_1,\psi_2)\} \in \mathbb{R}_+.$$

Then the problem (15) can be written as

$$\sup\{\rho(\psi_1,\psi_2) \,:\, (\psi_1,\psi_2) \in L^2(0,T;Y_1 \times Y_2) \text{ and } \psi_i(t) \in U_i(t), i = 1,2\}.$$

First we show that $\rho$ is upper semicontinuous on $L^2(0,T;Y_1\times Y_2)$ endowed with the weak topology. To this end, we will use Theorem 5 of Chapter 1 in Aubin and Cellina (1984) on the upper semicontinuity of marginal functions. We will prove that

(a) the solution map $S\colon L^2(0,T;Y_1 \times Y_2) \to 2^{C(0,T;E\times H\times L^2(\Omega))}$ is upper semicontinuous in $(w\!-\!L^2(0,T;Y_1\times Y_2)) \times C(0,T;E \times H \times L^2(\Omega))$–topology;

(b) the function $\vartheta\colon C(0,T;E\times H\times L^2(\Omega)) \to \mathbb{R}_+$ is upper semicontinuous.

Proof of (a). We show that the set

$$S^-(\Delta) = \{(\psi_1,\psi_2) \in L^2(0,T;Y_1\times Y_2) :\ S(\psi_1,\psi_2) \cap \Delta \neq \emptyset\}$$

is weakly closed in $L^2(0,T;Y_1 \times Y_2)$ for every closed set $\Delta \subset C(0,T;E \times H \times L^2(\Omega))$. Let $\{(\psi_1^n,\psi_2^n)\} \subset S^-(\Delta)$, $(\psi_1^n,\psi_2^n) \to (\psi_1,\psi_2)$ weakly in $L^2(0,T;Y_1 \times Y_2)$. Thus for all $n \in \mathbb{N}$, there is $y_n \in S(\psi_1^n,\psi_2^n) \cap \Delta$. From Proposition 3, by passing to a subsequence, if necessary, we have $y_n \to y$ in $C(0,T;E\times H\times L^2(\Omega))$ and $y \in S(\psi_1,\psi_2)$. Furthermore, since $\{y_n\} \subset \Delta$ and $\Delta$ is a closed set in

$C(0,T;E \times H \times L^2(\Omega))$, we get $y \in \Delta$. Hence, $(\psi_1, \psi_2) \in S^-(\Delta)$, which gives the upper semicontinuity of $S$ in the desired topology.

Proof of (b). Let $\{y_n\} \subset C(0,T;E \times H \times L^2(\Omega))$, $y_n \to y$ in $C(0,T;E \times H \times L^2(\Omega))$. Define

$$D_n = \{t \in [0,T] : y_n(t) \in N(t)\} = \{t \in [0,T] : d(y_n(t), N(t)) = 0\}.$$

By $H(N)$, the multifunction $N(\cdot)$ is measurable, so $t \mapsto d(y_n(t), N(t))$ is measurable on $[0,T]$ and also $D_n$ is a measurable subset of $[0,T]$. We observe that

$$K\text{–}\limsup_{n\to\infty} D_n \subset D, \tag{16}$$

where $D = \{t \in [0,T] : y(t) \in N(t)\}$ is also measurable. Indeed, by the definition of the Kuratowski limit of sets, for every $t \in K\text{–}\limsup_{n\to\infty} D_n$, there exists $t_{n_k} \in D_{n_k}$, $t_{n_k} \to t$, as $k \to \infty$. So $y_{n_k}(t_{n_k}) \in N(t_{n_k})$. From $H(N)$ and the convergence $y_n \to y$ in $C(0,T;E \times H \times L^2(\Omega))$, we obtain $y(t) \in N(t)$. This means that $t \in D$ and therefore (16) holds. Moreover, since

$$\begin{aligned}&\limsup_{n\to\infty} m(D_n) \le \limsup_{n\to\infty} m\left(\cup_{k\ge n} D_k\right) = \lim_{n\to\infty} m\left(\cup_{k\ge n} D_k\right)\\ &= m\left(\cap_{n\ge 1} \cup_{k\ge n} D_k\right) \le m(K-\limsup_{n\to\infty} D_n) \le m(D),\end{aligned}$$

we deduce that $\vartheta$ is upper semicontinuous.

Applying now Theorem 5 of Aubin and Cellina (1984), by (a), (b) and Corollary 2 we deduce that $\rho$ is upper semicontinuous on $L^2(0,T;Y_1 \times Y_2)$ endowed with the weak topology. On the other hand, since $U_i\colon [0,T] \to 2^{Y_i} \setminus \{\emptyset\}$, $i = 1, 2$ is integrably bounded with closed, convex values (see $H(U)$), the set $U_i(t)$ is weakly compact in $Y_i$ for all $t \in [0,T]$, $i = 1, 2$. By Theorem 4.5.25 of Denkowski et al. (2003), we know that the set of selections $\{(\psi_1, \psi_2) \in L^2(0,T;Y_1 \times Y_2)\colon \psi_i(t) \in U_i(t) \text{ a.e. } t, \text{ for } i = 1, 2\}$ is a weakly compact subset of $L^2(0,T;Y_1 \times Y_2)$. Hence the Weierstrass theorem implies the existence of a solution to the problem (15). ■

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