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A generalization of the Opial's theorem

by

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Abstract: Opial presented in 1967 a theorem, which can be applied in order to prove the weak convergence of sequences (x_k) in a Hilbert space, generated by iterative schemes of the form $x_{k+1} = Ux_k$ for a nonexpansive and asymptotically regular operator U with nonempty $\operatorname{Fix} U$. Several iterative schemes have, however, the form $x_{k+1} = U_k x_k$, where (U_k) is a sequence of operators with a common fixed point. We show that under some conditions on the sequence (U_k) the sequence (x_k) converges weakly to a common fixed point of operators U_k. We show also that the Opial's theorem and the Krasnoselskii–Mann theorem are the corollaries descending from the obtained results. Finally, we present some applications of the results to the convex feasibility problems.

Keywords: nonexpansive operators, asymptotically regular operators, fixed points, weak convergence.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped with a scalar product $\langle \cdot, \cdot \rangle$ which induces the norm $\| \cdot \|$. Let $C \subset \mathcal{H}$ be a closed and convex subset. Several optimization problems have the form: find an element $x^* \in C$ which satisfies some additional condition leading to a solution set being a closed and convex subset, e.g. the convex minimization problem, the convex feasibility problem (see, e.g., Bauschke, Borwein, 1996), the split feasibility problem (see, e.g., Censor, Elfving, 1994). Iterative methods, which try to solve such problems consist very often in adaptive construction of the operator $U : C \to \mathcal{H}$ with $\operatorname{Fix} U$ being the solution set of the optimization problem with the property that $U^k x$ converges (at least weakly) to an element $x^* \in \operatorname{Fix} U$, for arbitrary $x \in C$. The Opial's theorem (Opial, 1967) gives the conditions for the operator U, under which the weak convergence holds (nonexpansivity and asymptotic regularity). Several iterative methods have, however, the form $x_{k+1} = U_k x_k$ where $U_k : C \to \mathcal{H}$ is a sequence of operators with a common fixed point. Furthermore, the nonexpansivity of U_k sometimes fails or is hard to prove. Therefore, we cannot apply the Opial's theorem in order to prove the weak convergence of the sequence (x_k) in this case.

2. Main result

Let $C \subset \mathcal{H}$ and $D \subset C$ be a closed and convex subsets. We say that an operator $U : C \to \mathcal{H}$ is *nonexpansive* if $\|Ux - Uy\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$, and *Fejér monotone* with respect to D if $\|Ux - z\| \leq \|x - z\|$ for all $x \in C$ and all $z \in D$. We say that a sequence $(x_k) \subset C$ is *Fejér monotone* with respect to D if $\|x_{k+1} - z\| \leq \|x_k - z\|$ for all $z \in D$, $k = 1, 2, ...$

Denote $S_k = U_k U_{k-1} ... U_1$ for a sequence of operators $U_k : C \to C$, $k = 1, 2, ...$. We say that a sequence of operators $U_k : C \to C$ is *asymptotically regular* if $\lim_{k\to\infty} \|S_{k+1}x - S_k x\| = 0$ for any $x \in C$. This definition coincides with the definition of an asymptotically regular sequence (x_k) (see Bauschke, Borwein, 1996, Definition 6) and with the definition of an asymptotically regular operator (see Opial, 1967, Section 1, or Stark, Yang, 1998, Section 2.5), i.e. an operator $U : C \to C$ is asymptotically regular if the constant sequence of operators $U_k = U$ is asymptotically regular, i.e. $\lim_{k\to\infty} \|U^{k+1}x - U^k x\| = 0$ for any $x \in C$.

Let $x \in C$. Consider a sequence $(x_k) \subset C$, defined by the recurrence

$$\begin{aligned} &x_1 = x \\ &x_{k+1} = U_k x_k \end{aligned} \tag{1}$$

for a sequence of operators $U_k : C \to C$, $k = 1, 2, ...$.

THEOREM 1 *Let $C \subset \mathcal{H}$ be a closed and convex subset, $U : C \to \mathcal{H}$ be a nonexpansive operator with nonempty* Fix U*, let $x \in C$ be arbitrary and let the sequence $(x_k) \subset C$ be generated by the recurrence* (1)*, where the sequence (U_k) is asymptotically regular. Further, let (x_k) be Fejér monotone with respect to* Fix U *and let the inequality*

$$\|U_k x_k - x_k\| \geq \alpha \|U x_k - x_k\| \tag{2}$$

be satisfied for a constant $\alpha > 0$, $k = 1, 2, ...$. Then, (x_k) converges weakly to an element of Fix U.

REMARK 1 Bauschke and Borwein (1996, Example 7), have obtained sufficient conditions for asymptotic regularity of a sequence (x_k), which is an assumption in Theorem 1. Roughly speaking, they have supposed that U_k are nonexpansive and strongly attracting (see Bauschke, Borwein, 1996, Definition 2.1). Note that we have only supposed the nonexpansivity of the operator U but not of operators U_k.

The proof of Theorem 1 is a modification of the original Opial's proof (Opial, 1967, Theorem 1). First we recall some facts from Opial (1967).

LEMMA 1 *If x_k converges weakly to an element $y \in \mathcal{H}$, then for any $y' \in \mathcal{H}$, $y' \neq y$, the following inequality holds:*

$$\liminf_k \|x_k - y'\| > \liminf_k \|x_k - y\|. \tag{3}$$

Proof. See Opial (1967, Lemma 1) ■

LEMMA 2 *Let $U : C \to \mathcal{H}$ be a nonexpansive operator, $z \in C$ be a weak cluster point of a sequence $(x_k) \subset C$. If $\|Ux_k - x_k\| \to 0$ then $z \in \operatorname{Fix} U$.*

Proof. The Lemma is a special case of Opial (1967, Lemma 2). ■

LEMMA 3 *Let $D \subset \mathcal{H}$ be a convex and closed subset and let a sequence $(x_k) \subset \mathcal{H}$ be Fejér monotone with respect to D. Then there exists the unique element $y^* \in D$ such that*

$$\lim_k \|x_k - y^*\| = \inf_{y \in D} \lim_k \|x_k - y\|. \tag{4}$$

Proof. The Lemma can be proved using similar techniques as in the first part of the proof of Opial (1967, Theorem 1). See also Stark, Yang (1998, Section 2.8), where the Lemma was proved similarly to the proof of the existence and the uniqueness of the metric projection of a given point onto a closed and convex subset. ■

Proof of Theorem 1. The sequence (x_k) is bounded since it is Fejér monotone with respect to $\operatorname{Fix} U$, consequently, there exists a weak cluster point $y \in C$ of (x_k) (see, e.g., Rolewicz, 1984, Theorem 5.3.1). Let $(x_{n_k}) \subset (x_k)$ be a subsequence which is weakly convergent to y. Since the sequence (U_k) is asymptotically regular we have by assumptions that

$$\alpha\|Ux_k - x_k\| \leq \|U_k x_k - x_k\| = \|S_k x - S_{k-1} x\| \to 0 \tag{5}$$

for a nonexpansive operator $U : C \to \mathcal{H}$ and for a constant $\alpha > 0$. Hence, $\|Ux_k - x_k\| \to 0$. It follows from Lemma 2 that $y \in \operatorname{Fix} U$. Since U is nonexpansive, $\operatorname{Fix} U$ is closed and convex. Let $y^* \in \operatorname{Fix} U$ be such that

$$\lim_k \|x_k - y^*\| = \inf_{u \in \operatorname{Fix} U} \lim_k \|x_k - u\|.$$

The existence and the uniqueness of y^* follows from Lemma 3. We show that x_{n_k} converges weakly to y^*. Suppose that $y^* \neq y$. Since (x_k) is Fejér monotone with respect to $\operatorname{Fix} U$, we have by Lemmas 1 and 3

$$\lim_k \|x_k - y^*\| = \lim_k \|x_{n_k} - y^*\| > \lim_k \|x_{n_k} - y\| \geq \lim_k \|x_k - y^*\|.$$

The contradiction shows that $y = y^*$. We have shown that y^* is the unique weak cluster point of (x_k). Consequently, x_k converges weakly to y^*. ■

Remark 2 The assumptions in Theorem 1 can be weakened. It follows from the proof of the Theorem that it is enough to suppose that

$$\|Ux_k - x_k\| \to 0 \tag{6}$$

instead of the asymptotic regularity of (U_k) and of inequality (2). Note that condition (6) does not denote here the asymptotic regularity of the operator U because the sequence (U_k) is not constant in general.

Corollary 1 (Opial's Theorem) *Let $C \subset \mathcal{H}$ be a closed and convex subset, $U : C \to C$ be a nonexpansive and asymptotically regular operator with $\operatorname{Fix} U \neq \varnothing$. Then for any $x \in C$ the sequence $(U^k x)$ converges weakly to an element of $\operatorname{Fix} U$.*

Proof. Let $x \in C$ and let x_k be defined by $x_k = U^k x$. We have for any $z \in \operatorname{Fix} U$

$$\|x_k - z\| = \|Ux_{k-1} - Uz\| \leq \|x_{k-1} - z\|,$$

consequently, (x_k) is Fejér monotone with respect to $\operatorname{Fix} U$. Now we see that all assumptions of Theorem 1 are satisfied for $U_k = U$, $k = 1, 2, ...$, and for $\alpha = 1$. Hence, $U^k x = U_k x_k$ and the sequence (x_k) converges weakly to an element of $\operatorname{Fix} U$, for any $x \in C$. ■

Corollary 2 *Let $U : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator with nonempty $\operatorname{Fix} U$, let $x \in \mathcal{H}$ be arbitrary and let the sequence $(x_k) \subset \mathcal{H}$ be generated by the recurrence*

$$\begin{aligned} x_1 &= x \\ x_{k+1} &= U_k x_k = x_k + \mu_k \lambda_k (T_k x_k - x_k), \end{aligned}$$

where $\mu_k \in [\varepsilon, 2-\varepsilon]$ for some small $\varepsilon > 0$, $T_k : \mathcal{H} \to \mathcal{H}$, is such that $\operatorname{Fix} T_k \supset \operatorname{Fix} U$ and for all $z \in \operatorname{Fix} U$

$$\langle z - x_k, T_k x_k - x_k \rangle \geq \lambda_k \|T_k x_x - x_k\|^2 \tag{7}$$

with $\lambda_k \geq \gamma > 0$, $k = 1, 2, ...$. Furthermore, suppose that

$$\|T_k x_k - x_k\| \geq \beta \|Ux_k - x_k\| \tag{8}$$

for a constant $\beta > 0$, $k = 1, 2, ...$. Then (x_k) converges weakly to an element $y \in \operatorname{Fix} U$.

Proof. Let $z \in \operatorname{Fix} U$. Then we have by (7)

$$\begin{aligned} \|x_{k+1} - z\|^2 &= \|x_k + \mu_k \lambda_k (T_k x_k - x_k) - z\|^2 \\ &= \|x_k - z\|^2 + \mu_k^2 \lambda_k^2 \|T_k x_k - x_k\|^2 - 2\mu_k \lambda_k \langle z - x_k, T_k x_k - x_k \rangle \\ &\leq \|x_k - z\|^2 + \mu_k^2 \lambda_k^2 \|T_k x_k - x_k\|^2 - 2\mu_k \lambda_k^2 \|T_k x_k - x_k\|^2 \\ &= \|x_k - z\|^2 - \mu_k (2 - \mu_k) \lambda_k^2 \|T_k x_k - x_k\|^2, \end{aligned}$$

i.e. (x_k) is Fejér monotone with respect to $\operatorname{Fix} U$ and $\mu_k\lambda_k\|T_kx_k - x_k\| \to 0$. Since

$$\|S_kx - S_{k-1}x\| = \|U_kx_k - x_k\| = \mu_k\lambda_k\|T_kx_k - x_k\|$$

we see that (U_k) is an asymptotically regular sequence. By assumption (8) we have for $\alpha = \varepsilon\gamma\beta$

$$\|U_kx_k - x_k\| = \mu_k\lambda_k\|T_kx_k - x_k\| \geq \varepsilon\gamma\|T_kx_k - x_k\| \geq \alpha\|Ux_k - x_k\|.$$

We see that all assumptions of Theorem 1 are satisfied. Consequently, (x_k) converges weakly to a point $y \in \operatorname{Fix} U$. ■

Now we apply Corollary 2 to a sequence of (U_k) being relaxations of firmly nonexpansive operators. Let $T : C \to \mathcal{H}$, where $C \subset \mathcal{H}$ is a closed and convex subset. Recall that T is called *firmly nonexpansive* if for all $x, y \in C$

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2.$$

Furthermore, $T_\mu = (1-\mu)I + \mu T$, where $\mu \in [0, 2]$, is called a *relaxation* of the operator T. If $\mu \in (0, 2)$ then we say that T_μ is a *strict relaxation* of T. First, we present a useful property of a firmly nonexpansive operator.

LEMMA 4 *Let $C \subset \mathcal{H}$ be a closed and convex subset and let $T : C \to \mathcal{H}$ be a firmly nonexpansive operator with nonempty* $\operatorname{Fix} T$. *Then for all $z \in \operatorname{Fix} T$ and for all $x \in C$*

$$\langle z - x, Tx - x\rangle \geq \|Tx - x\|^2. \tag{9}$$

Proof. Let $z \in \operatorname{Fix} T$ and $x \in C$. Then we have by the firm nonexpansivity of T

$$\begin{aligned}\langle z - x, Tx - x\rangle &= \langle z - Tx, Tx - x\rangle + \|Tx - x\|^2\\ &= \langle Tz - Tx, z - x\rangle + \langle Tz - Tx, Tx - Tz\rangle + \|Tx - x\|^2\\ &\geq \|Tx - x\|^2,\end{aligned}$$

i.e. condition (9) is satisfied. ■

COROLLARY 3 *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator with nonempty* $\operatorname{Fix} T$, *let $x \in \mathcal{H}$ be arbitrary and let the sequence $(x_k) \subset \mathcal{H}$ be generated by the recurrence*

$$\begin{aligned}x_1 &= x\\ x_{k+1} &= U_kx_k = x_k + \mu_k(T_kx_k - x_k),\end{aligned}$$

where $\mu_k \in [\varepsilon, 2-\varepsilon]$ for some small $\varepsilon > 0$, and (T_k) is a sequence of firmly nonexpansive operators $T_k : \mathcal{H} \to \mathcal{H}$, $k = 1, 2, \ldots$ with $\bigcap_k \operatorname{Fix} T_k \supset \operatorname{Fix} T$. Furthermore, suppose that

$$\|T_kx_k - x_k\| \geq \beta\|Tx_k - x_k\| \tag{10}$$

for a constant $\beta > 0$, $k = 1, 2, ...$. Then (x_k) converges weakly to an element $y \in \operatorname{Fix} T$.

Proof. Let $z \in \operatorname{Fix} T$. Since T_k is firmly nonexpansive, we have by Lemma 4 that inequality (7) is satisfied for $\lambda_k = \gamma = 1$. By Corollary 2, (x_k) converges weakly to an element $y \in \operatorname{Fix} T$. ■

COROLLARY 4 *Let $T : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator with nonempty* $\operatorname{Fix} T$*, $x \in \mathcal{H}$ and let the sequence $(x_k) \subset \mathcal{H}$ be generated by the recurrence*

$$\begin{aligned} x_1 &= x \\ x_{k+1} &= x_k + \mu(Tx_k - x_k) \end{aligned}$$

where $\mu \in (0, 2)$. Then (x_k) converges weakly to an element $y \in \operatorname{Fix} T$.

Proof. Set $\mu_k = \mu$ and $T_k = T$, $k = 1, 2, ...$, in Corollary 3. Since a firmly nonexpansive operator is nonexpansive we see that all assumptions of Corollary 3 are satisfied. Consequently, (x_k) converges weakly to a point $y \in \operatorname{Fix} T$. ■

REMARK 3 Let $\lambda \in (0, 2)$. An operator $T_\mu = (1-\mu)I + \mu T$ is a strict relaxation of a firmly nonexpansive operator T if and only if T_μ is averaged, i.e. $T_\mu = (1 - \alpha)I + \alpha U$, where $\alpha \in (0, 1)$ and a U is a nonexpansive operator (see, e.g., Byrne, 2004, Lemma 2.3 or Goebel, Kirk, 1990, Theorem 12.1). Therefore, Corollary 4 is equivalent to the Krasnoselskii–Mann Theorem (see, e.g., Byrne, 2004, Theorem 2.1, or the original paper of Krasnoselskii, 1955, Theorem 1).

3. Applications to convex feasibility problems

Let $C_i \subset \mathcal{H}$, $i \in I = \{1, ..., m\}$, be closed and convex subsets (constraints). The *convex feasibility problem* (CFP) is to find $x \in C = \bigcap_{i \in I} C_i$ if such an element exists (the system $\{C_i\}_{i \in I}$ is consistent). However, in several applications the system $\{C_i\}_{i \in I}$ is not supposed to be consistent. In these cases one introduces a proximity function, which measures the grade of violation of the constraints and one minimizes this proximity function.

EXAMPLE 1 Let us introduce a proximity function for the CFP defined by the equality $f(x) = \frac{1}{2} \sum_{i \in I} \omega_i \|P_{C_i} x - x\|^2$, where $w = (\omega_1, ..., \omega_m)^\top \in \Delta_m$. The subset Δ_m denotes here the standard simplex, i.e. $\Delta_m = \{u \in \mathbb{R}^m : u \geq 0,\ e^\top u = 1\}$, where $e = (1, ..., 1)^\top$. The CFP can be formulated as the minimization of $f(x)$ with respect to $x \in \mathcal{H}$. Since f is convex and differentiable, we easily obtain that $\operatorname{Argmin}_{x \in \mathcal{H}} f(x) = \operatorname{Fix} P_w$, where $P_w = \sum_{i \in I} \omega_i P_{C_i}$. Now we see that the CFP is equivalent to finding a fixed point of P_w. One of important methods for the CFP is the *simultaneous projection method* (SP-method) which has the form:

$$x_{k+1} = x_k + \mu_k(P_{w_k} x_k - x_k), \tag{11}$$

where $\mu_k \in [0,2]$ and $w_k \in \Delta_m$, $k = 1, 2, \ldots$. Under mild assumptions on the sequence of weights (w_k) and on the sequence of relaxation parameters (μ_k) the sequence (x_k) generated by (11) converges weakly to an element $x^* \in C$ if $C \neq \varnothing$ (see, e.g., Combettes, 1997, Theorem 4.1, or Censor, Zenios, 1997, Theorems 5.6.1 and 10.4.1 for finite dimensional case). If we do not suppose that $C \neq \varnothing$ then the convergence requires more restrictive assumptions. It is known that in this case the sequence (x_k) converges weakly to $\operatorname{Fix} P_w$ for $\mu_k \in [\varepsilon, 2-\varepsilon]$, where $\varepsilon > 0$, and for constant $w_k = w \in \operatorname{ri}\Delta_m = \{u \in \mathbb{R}^m : u > 0, e^\top u = 1\}$ if $\operatorname{Fix} P_w \neq \varnothing$ (see Combettes, 1994, Theorem 4, or Stark, Yang, 1998, Corollary 2.10-1). We show that the convergence follows also simply from Corollary 3. Set $T_k = T = P_w$ in Corollary 3 and suppose that $\operatorname{Fix} T \neq \varnothing$. Observe that T is firmly nonexpansive as a convex combination of firmly nonexpansive operators P_{C_i}, $i \in I$, T is nonexpansive as a relaxation of a firmly nonexpansive operator (see, e.g., Goebel, Kirk, 1990, Theorem 12.1, or Combettes, 1994, Proposition 3ii)) and that $\operatorname{Fix} T_k = \operatorname{Fix} T$. We see that all assumptions of Corollary 3 are satisfied. Therefore, the sequence (x_k) generated by (11) converges weakly to an element $x^* \in \operatorname{Fix} P_w$.

Example 2 Consider the following problem, known as the *split feasibility problem* (SFP): Let $C \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$ be nonempty, closed and convex subsets, and A be an $m \times n$ real matrix. Find $x \in C$ satisfying $Ax \in Q$, if such an element exists. In general, the problem has the form:

$$\begin{array}{ll} \text{minimize} & f(x) = \frac{1}{2}\|P_Q(Ax) - Ax\|^2 \\ \text{subject to} & x \in C \end{array}$$

and is equivalent to finding a fixed point of the operator $P_C(I - \gamma A^\top(I - P_Q)A)$, where $\gamma > 0$. This problem was introduced by Censor and Elfving (1994) and was studied by Byrne (2002, 2004). The SFP has many practical applications (see, e.g., Byrne, 2002). Byrne has proposed the following CQ-method for solving the split feasibility problem: $x_{k+1} = R_\mu x_k$, where $R_\mu = P_C(I - \frac{\mu}{L} A^\top(I - P_Q)A)$ for $\mu \in (0,2)$ and $L \geq \lambda_{\max}(A^\top A)$. The number $\lambda_{\max}(B)$ denotes here the maximal eigenvalue of a symmetric matrix B. Suppose that $\operatorname{Fix} R_\mu \neq \varnothing$. In this case the CQ-method converges to a fixed point of the operator R_μ (see Byrne, 2002, Theorem 2.1 or Byrne, 2004, Section 8). Now we consider the following generalization of the CQ-method for the SFP: $x_{x+1} = U_k x_k$, where $U_k = R_{\mu_k}$ for $\mu_k \in [\varepsilon, 2-\varepsilon]$ and for some small $\varepsilon > 0$. Denote $U = R_\varepsilon$. Of course, $\operatorname{Fix} R_\mu = \operatorname{Fix} U$ for all $\mu > 0$. For any $\mu \in (0,2)$ the operator R_μ is averaged (Byrne, 2004, Section 8), therefore R_μ is a strict relaxation of a firmly nonexpansive operator (see, e.g., Byrne, 2004, Lemma 2.3, or Goebel, Kirk, 1990, Theorem 12.1) and (x_k) is Fejér monotone with respect to $\operatorname{Fix} U$ (see, e.g., Bauschke, Borwein, 1996, Lemma 2.4(iv)). In particular, one can prove that

$$\|U_k x_k - z\|^2 \leq \|x_k - z\|^2 - \frac{2-\mu_k}{2+\mu_k}\|U_k x_k - x_x\|^2$$

(details are omitted). Consequently, (U_k) is asymptotically regular. Observe that the function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(t) = \|P_C(x + ts) - x\|$, where $x \in C$ and $s \in \mathbb{R}^n$ is nondecreasing. Hence,

$$\begin{aligned} \|U_k x_k - x_k\| &= \|R_{\mu_k} x_k - x_k\| \\ &= \|P_C(x_k - \frac{\mu_k}{L} A^\top (I - P_Q) A x_k) - x_k\| \\ &\geq \|P_C(x_k - \frac{\varepsilon}{L} A^\top (I - P_Q) A x_k) - x_k\| \\ &= \|U x_k - x_k\|. \end{aligned}$$

Of course, U is nonexpansive. Now we see that all assumptions of Theorem 1 are satisfied. Therefore, the sequence (x_k) converges to a fixed point of the operator S_μ.

EXAMPLE 3 Let $A, B \subset \mathcal{H}$ be closed and convex subsets. Consider the problem of finding an element of the intersection $A \cap B$ or, more general, to solve the following problem

$$\begin{array}{ll} \text{minimize} & f(x, y) = \frac{1}{2}\|x - y\|^2 \\ \text{subject to} & x \in A, y \in B. \end{array} \tag{12}$$

We suppose that this problem has a solution, i.e. f attains its minimum on $A \times B$. This problem has many practical applications (see, e.g., Stark, Yang, 1998, for details). It is known that (x^*, y^*) is a solution of problem (12) if and only if $x^* = P_A y^*$ and $y^* = P_B x^*$, i.e. $x^* \in \operatorname{Fix} P_A P_B$ (see, e.g., Bauschke, Borwein, 1994, Lemma 2.2.(i)). Of course, $x^* = y^*$ if and only if $A \cap B \neq \varnothing$. There are several methods generating sequences which converge weakly to a solution of problem (12). One of them is the von Neumann alternating projection method:

$$\begin{array}{l} x_1 \in A - \text{arbitrary} \\ x_{k+1} = P_A P_B x_k \end{array} \tag{13}$$

(see, e.g, Bauschke, Borwein, 1994, Section 4). Observe that $T = P_A P_B$ is nonexpansive as a product of nonexpansive operators P_A and P_B. Furthermore, one can show that T is asymptotically regular (see, e.g., Stark, Yang, 1998, Lemma 2.5-3). Therefore, the weak convergence of the sequence (x_k) generated by (13) follows from the Opial's theorem (Corollary 1). Consider the following generalization of method (13)

$$\begin{array}{l} x_1 \in A - \text{arbitrary} \\ x_{k+1} = P_A(x_k + \mu_k \lambda_k (T x_k - x_k)), \end{array} \tag{14}$$

where $T = P_A P_B$ with $\operatorname{Fix} T \neq \varnothing$, $\lambda_k \geq \gamma$ for some $\gamma > 0$ and $\mu_k \in [\varepsilon, 2 - \varepsilon]$ for $\varepsilon > 0$. If we set $\mu_k = \lambda_k = 1$, $k = 1, 2, ...$, in method (14) we obtain the alternating projection method (13).

Suppose that A is an affine subspace. Then the operator $T = P_A P_B$ restricted to A is firmly nonexpansive (see Combettes, 1994, Proposition 3i). If $\lambda_k \in [\varepsilon, 2-\varepsilon]$ for some small $\varepsilon > 0$ then it follows from Corollary 3 that the sequence (x_k) generated by (14) converges weakly to $\operatorname{Fix} T$ (see also Combettes, 1994, Theorem 1). We obtain an even better result if we are able to find a good upper bound $\bar{\delta}$ of $\delta = \inf_{x\in A, y\in B} \|x-y\|$. Let $z \in \operatorname{Fix} T$ and $x_k \in A \setminus \operatorname{Fix} T$ and set $\lambda_k = 1 + \frac{(\|Tx_k - P_B x_k\| - \bar{\delta}_k)^2}{\|Tx_k - x_k\|^2}$, where $\bar{\delta}_k \in [\delta, \|Tx_k - P_B x_k\|]$ and $\mu_k \in [\varepsilon, 2-\varepsilon]$ for $\varepsilon > 0$, in recurrence (14). Of course, $\lambda_k \geq 1$. One can prove that inequality (7) for $T_k = T$ is satisfied (see Cegielski, Suchocka, 2007). If we now apply Corollary 2 for $U = T$, $\beta = \gamma = 1$, we obtain the weak convergence of (x_k) to $\operatorname{Fix} T$.

Let us come back to the general case ($A \subset \mathcal{H}$ is closed and convex). Let $z \in \operatorname{Fix} T$ and $x_k \in A \setminus \operatorname{Fix} T$. Denote $a_k = P_B x_k - x_k$, $b_k = Tx_k - x_k$ and $c_k = P_B x_k - Tx_k$ and set $\lambda_k = (\|c_k\|^2 - \|a_k\| \cdot \|c_k\| + \langle a_k, b_k\rangle)/\|b_k\|^2$ in recurrence (14). Then $\lambda_k \geq \frac{1}{2}$ and inequality (7) for $T_k = T$ is satisfied (see Cegielski, Suchocka, 2007). Using similar techniques as in the proof of Corollary 2 one can prove the weak convergence of a sequence (x_k) generated by (14) to $\operatorname{Fix} T$ (see Cegielski, Suchocka, 2007, for details).

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