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Some summations formulae in commutative Leibniz algebras with logarithms

by

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Abstract: A survey of summation formulae in commutative Leibniz algebras with logarithms is given. New results concerning generic functions and related summation formulae, which generalize well-known properties of the Bessel functions, are demonstrated.

Keywords: algebra with unit, Leibniz condition, logarithmic mapping, antilogarithmic mapping, true shift, binomial formula, harmonic logarithm, umbral calculus, generic function.

Some summations formulae in commutative Leibniz algebras with logarithms are presented. Sections 1, 2 and 3 consist of basic facts concerning Algebraic Analysis, true shifts (in particular, multiplicative true shifts) and Leibniz algebras with logarithms, all without proofs (which can be found in the cited papers). In Section 4 a generalization of the binomial theorem of the Umbral Calculus for harmonic logarithms in commutative Leibniz algebras with logarithms and a theorem about the corresponding invariant class of solutions to linear equations are given. This section also contains binomial theorems for other elements and for shifted binomials (all without proofs). In Section 5 the properties of some generic functions and related summation formulae (with proofs), which generalize well-known properties of the Bessel functions, are studied.

1. Basic notions of Algebraic Analysis

We recall here the following notions and theorems (without proofs; see Przeworska-Rolewicz, 1988, 1998, 2000).

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of the characteristic zero.

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- dom A is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \text{dom } A : Ax = 0\}$ is the kernel of an $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D.

By $R(X)$ we denote the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $\mathcal{R}_D \subset L_0(X)$ be the set of all right inverses for D, i.e. $DR = I$ whenever $R \in \mathcal{R}_D$. We have

$$\text{dom } D = RX \oplus \ker D, \quad \textit{independently of the choice of an } R \in \mathcal{R}_D.$$

Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants*. We should point out that, in general, constants are different than scalars, since they are elements of the space X. If two right inverses commute both one with another, then they are equal. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F; FX = \ker D \textit{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that **any** projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$.

If two initial operators commute both one with another, then they are equal. Thus, this theory is essentially **noncommutative**. An operator F is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that

$$F = I - RD \quad \textit{on } \text{dom } D. \tag{1.1}$$

Even more. Write $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$. Then, by (1.1), we conclude that $\mathcal{R}_D$ induces in a unique way the family $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$ of the corresponding initial operators defined by means of the equality $F_\gamma = I - R_\gamma D$ on dom D $\quad (\gamma \in \Gamma)$.

Formula (1.1) yields (by a two-line induction) the *Taylor Formula*:

$$I = \sum_{k=0}^{n} R^n F D^n + R^n D^n \ \textit{on } \text{dom } D^n \ (n \in \mathbb{N}). \tag{1.2}$$

It is enough to know one right inverse in order to determine all right inverses and all initial operators. Note that a superposition (if it exists) of a finite number of right invertible operators is again a right invertible operator.

The equation $Dx = y \ \ (y \in X)$ has the general solution $x = Ry + z$, where $R \in \mathcal{R}_D$ is arbitrarily fixed and $z \in \ker D$ is arbitrary. However, if we put an *initial condition*: $Fx = x_0$, where $F \in \mathcal{F}_D$ and $x_0 \in \ker D$, then this equation has a unique solution $x = Rx + x_0$.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If D is invertible, i.e. $D \in \mathcal{I}(X) = R(X) \cap \Lambda(X)$, then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

If $P(t) \in \mathbb{F}[t]$ (i.e. $P(t)$ is a polynomial with scalar coefficients, where $\mathbb{F}$ is the field of scalars under consideration) then all solutions of the equation

$$P(D)x = y, \qquad y \in X, \tag{1.3}$$

can be obtained by a decomposition of a rational function $1/P(t)$ into vulgar fractions. One can distinguish subspaces of X with the property that all solutions of Equation (1.3) belong to a subspace Y whenever $y \in Y$ (see von Trotha, 1981; Przeworska-Rolewicz, 1996).

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \operatorname{dom} D$ implies $xy, yx \in \operatorname{dom} D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\mathsf{A}(X)$. If $D \in \mathbf{A}(X)$ then

$$f_D(x,y) = D(xy) - c_D[xDy + (Dx)y] \quad \textit{for} \quad x, y \in \operatorname{dom} D, \tag{1.4}$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component* (see Przeworska-Rolewicz, 1988). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x,y) \quad \textit{for} \quad x, y \in \operatorname{dom} D.$$

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*:

$$D(xy) = xDy + (Dx)y \quad \textit{for } x, y \in \operatorname{dom} D, \tag{1.5}$$

then X is said to be a *Leibniz algebra*. It means that in Leibniz algebras $c_D = 1$ and $f_D = 0$. The Leibniz condition implies that $xy \in \operatorname{dom} D$ whenever $x, y \in \operatorname{dom} D$. If X is a Leibniz algebra with unit e then $e \in \ker D$, i.e. D is not left invertible.

Non-Leibniz components for powers of $D \in \mathbf{A}(X)$ are determined by recurrence formulae (see Przeworska-Rolewicz, 1988, 1998).

Suppose that $D \in \mathbf{A}(X)$ and $p \neq 0$ is an arbitrarily fixed scalar. Then $pD \in \mathbf{A}(X)$ and $c_{pD} = c_D,\ f_{pD} = pf_D$.

If $D_1, D_2 \in \mathbf{A}(X)$, the superposition $D = D_1D_2$ exists and $D_1D_2 \in \mathbf{A}(X)$, then

$$c_{D_1D_2} = c_{D_1}c_{D_2} \quad \textit{and for } x, y \in \operatorname{dom} D = \operatorname{dom} D_1 \cap D_2 \tag{1.6}$$
$$f_{D_1D_2}(x,y) = f_{D_1}(x,y) + D_1 f_{D_2}(x,y) + +c_{D_1}c_{D_2}[(D_1x)D_2y + (D_2x)D_1y].$$

For higher powers of D in Leibniz algebras, by an easy induction from Formulae (1.6) and the Leibniz condition, we obtain *the Leibniz formula*:

$$D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (D^k x)D^{n-k}y \quad \textit{for } x, y \in \operatorname{dom} D^n \quad (n \in \mathbb{N}). \tag{1.7}$$

By $M(X)$ we shall denote the set of all multiplicative mappings in X, i.e.

$$M(X) = \{A : X \to X : A(xy) = (Ax)(Ay) \ for \ x, y \in X\} \tag{1.8}$$

Let X be an algebra with unit e. Then A is an *algebra isomorphism* if it is a structure preserving invertible mapping, i.e. $A \in L_0(X) \cap \mathcal{I}(X) \cap M(X)$. If it is the case, then A^{-1} is also an algebra isomorphism. Moreover, $Ae = e$.

By $V(X)$ we denote the set of all *Volterra operators* belonging to $L(X)$, i.e. the set of all operators $A \in L(X)$ such that $I - \lambda A$ is invertible for all scalars λ. Clearly, $A \in V(X)$ if and only if $v_{\mathbb{F}}A = \mathbb{F} \setminus \{0\}$, where

$$v_{\mathbb{F}}A = \{0 \neq \lambda \in \mathbb{F} : I - \lambda A \quad is \quad invertible\} \quad for \ A \in L(X). \tag{1.9}$$

It means that $0 \neq \lambda \in v_{\mathbb{F}}A$ if and only if $1/\lambda$ is a regular value of A.

Let X be a Banach space. Denote by $QN(X)$ the set of all quasinilpotent operators belonging to $L(X)$, i.e. the set of all bounded operators $A \in L_0(X)$ such that

$$\lim_{n\to\infty} \sqrt[n]{\|A^n x\|} \ = \ 0 \qquad for \quad x \in X.$$

It is well-known that $QN(X) \subset V(X)$. If $\mathbb{F} = \mathbb{C}$ then $QN(X) = V(X) \cap B(X)$, where $B(X)$ is the set of all bounded operators belonging to $L(X)$.

Definition 1.1 *(see Przeworska-Rolewicz, 1998). Let X be a complete linear metric space over a field $\mathbb{F}$ of scalars. Let $A \in L(X)$ be continuous. Let $E \subset$ dom $A \subset X$ be a subspace. Let ω be a non-empty subset of $v_{\mathbb{F}}A$. The operator $A \in L(X)$ is said to be ω-*almost quasinilpotent *on E if*

$$\lim_{n\to\infty} \ \lambda^n A^n x = 0 \quad for \ all \ \lambda \in \omega, \ x \in E. \tag{1.10}$$

The set of all operators ω-almost quasinilpotent on the set E will be denoted by $AQN(E;\omega)$. If $\omega = v_{\mathbb{F}}A$ then we say that A is almost quasinilpotent *on E. The set of all almost quasinilpotent operators on E will be denoted by $AQN(E)$.*

Theorem 1.1 *(see Przeworska-Rolewicz, 1998). Let E be a subspace of a complete linear metric space X over $\mathbb{F}$. If $A \in L(X)$, $E \subset$ dom A and $\emptyset \neq \omega \subset v_{\mathbb{F}}A$, then the following conditions are equivalent:*

(i) A is ω-almost quasinilpotent on E;

(ii) for every $\lambda \in \omega$, $x \in E$ the series $\sum_{n=0}^{\infty} \ \lambda^n A^n x$ is convergent and

$$(I - \lambda A)^{-1}x = \sum_{n=0}^{\infty} \ \lambda^n A^n x \qquad (\lambda \in \omega, \ x \in E); \tag{1.11}$$

(iii) for every $\lambda \in \omega$, $x \in E$, $m \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} \ \binom{n+m-1}{m-1}\lambda^n A^n x$ is convergent and

$$(I-\lambda A)^{-m}x = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1}\lambda^n A^n x \quad (\lambda \in \omega, \ x \in E, \ m \in \mathbb{N}). \tag{1.12}$$

For given $D \in R(X)$, $R \in \mathcal{R}_D$ we shall consider (see von Trotha, 1981; Przeworska-Rolewicz, 1996) the following subspaces

• the space of *smooth* elements

$$D_\infty = \bigcap_{k\in\mathbb{N}_0} \text{dom } D^k, \quad \textit{where } \text{dom } D^0 = X;$$

• the space of *D-polynomials*

$$S = \bigcup_{n\in\mathbb{N}} \ker\ D^n; \quad S = P(R) = \text{lin } \{R^k z : z \in \ker\ D,\ k \in \mathbb{N}_0\} \subset D_\infty,$$

which, by definition, is independent of the choice of an $R \in \mathcal{R}_D$;

• the space of *exponentials*

$$E(R) = \bigcup_{\lambda\in v_{\mathbb{F}}R} \ker\ (D - \lambda I) =$$
$$= \text{lin } \{(I - \lambda R)^{-1} z : z \in \ker\ D,\ \lambda \in v_{\mathbb{F}}R \quad \textit{or } \lambda = 0\} \subset\ D_\infty,$$

which is independent of the choice of the right inverse R, provided that R is a Volterra operator,

• the space of *D-analytic* elements in a complete linear metric space X ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$)

$$A_R(D) = \{x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x\} = \{x \in D_\infty : \lim_{n\to\infty} R^n D^n x = 0\},$$

where F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$.

Clearly, by definitions, we have $S,\ E(R) \subset D_\infty$. If X is a complete linear metric space then $S \subset A_R(D) \subset D_\infty$.

2. True shifts

We begin with

Definition 2.1 *(see Przeworska-Rolewicz, 1998, also 2001). Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed,* $\ker D \neq \{0\}$ *and F is a continuous initial operator for D corresponding to a right inverse R, almost quasinilpotent on* $\ker D$. *Let $A(\mathbb{R}) = \mathbb{R}_+$ or $\mathbb{R}$. If $\{S_h\}_{h\in A(\mathbb{R})} \subset\ L_0(X)$ is a family of continuous linear operators such that $S_0 = I$ and for $h \in A(\mathbb{R})$ either*

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \textit{for } k \in \mathbb{N}_0$$

or

$$S_h(I-\lambda R)^{-1}F = e^{\lambda h}(I-\lambda R)^{-1}F \quad \textit{for } \lambda \in v_{\mathbb{F}}R,$$

then S_h *are said to be true shifts. The family* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a semigroup (or group) with respect to the superposition of operators as a structure operation.*

THEOREM 2.1 *(see Przeworska-Rolewicz, 1998, also 2001). Suppose that all conditions of Definition 2.1 are satisfied,* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a strongly continuous semigroup (group) of true shifts and either* $\overline{P(R)} = X$ *or* $\overline{E(R)} = X$*. Then* D *is an infinitesimal generator for* $\{S_h\}_{h\in A(\mathbb{R})}$*, hence* $\overline{\mathrm{dom}\ D} = X$ *and* $S_hD = DS_h$ *on dom* D*. Moreover, the canonical mapping* κ *defined as*

$$\kappa x = \{x^\wedge(t)\}_{t\in A(\mathbb{R})}, \quad \textit{where} \quad x^\wedge(t) = FS_tx \quad (x\in X) \tag{2.1}$$

is a topological isomorphism (hence separate points) and

$$\kappa D = \frac{\mathrm{d}}{\mathrm{d}t}\kappa, \quad \kappa R = \int_0^t \kappa, \quad \kappa F x = \kappa x|_{t=0},$$
$$\textit{and} \quad (\kappa S_h x)(t) = x^\wedge(t+h) \quad \textit{for } x\in X,\ t,h\in A(\mathbb{R}).$$

THEOREM 2.2 *(see Przeworska-Rolewicz, 1998, also 2001). Suppose that all conditions of Definition 2.1 are satisfied and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Then, for all* $h\in A(\mathbb{R})$ *and* $x\in A_R(D)$ *the series*

$$\mathrm{e}^{hD}x = \sum_{n=0}^{\infty}\frac{h^n}{n!}D^nx$$

is convergent,

$$S_hx = \mathrm{e}^{hD}x \quad \textit{for } x\in A_R(D) \tag{2.2}$$

and e^{hD} *maps* $A_R(D)$ *into itself.*

This implies the *Lagrange-Poisson formula* for a right invertible operator D

$$\Delta_h = \mathrm{e}^{hD} - I \quad \textit{on } A_R(D), \quad \textit{where} \quad \Delta_h = S_h - I \qquad (h\in A(\mathbb{R})) \tag{2.3}$$

(see Przeworska-Rolewicz, 1998).

Note that (under assumptions of Theorem 2.1) $v_{\mathbb{F}}(R - FS_hR) = v_{\mathbb{F}}R$ whenever F is an initial operator for D corresponding to R and S_h are true shifts. This means that the family $\{R_h\}_{h\in A(\mathbb{R})} = \{R - FS_hR\}_{h\in A(\mathbb{R})}$ of right inverses induced by shifts have the same regular values as R (see Przeworska-Rolewicz, 1998).

We shall consider now true shifts in commutative algebras.

Theorem 2.3 *(see Przeworska-Rolewicz, 2001) Let all conditions of Definition 2.1 be satisfied and let* $D \in \mathsf{A}(X)$. *Let* $\{S_h\}_{h\in A(\mathbb{R})}$ *be a family of true shifts. Let* $\mathcal{A}_R(D) = \{x, y \in A_R(D) : xy \in A_R(D)\}$. *Then* S_h *are multiplicative on* $\mathcal{A}_R(D)$ *for all* $h \in A(\mathbb{R})$*:* $S_h(xy) = (S_h x)(S_h y)$ *for all* $x, y \in \mathcal{A}_R(D)$*, if and only if* $D|_{\mathcal{A}_R(D)}$ *satisfies the Leibniz condition, i.e.* $D(xy) = xDy + yDx$.

Note that in Leibniz algebras $xy \in A_R(D)$ whenever $x, y \in A_R(D)$. Thus in this case $\mathcal{A}_R(D) = A_R(D)$ and we have

Corollary 2.1 *(see Przeworska-Rolewicz, 2001). Let all conditions of Definition 2.1 be satisfied and let* $D \in \mathsf{A}(X)$. *Let* $\{S_h\}_{h\in A(\mathbb{R})}$ *be a family of true shifts. If* $D|_{A_R(D)}$ *satisfies the Leibniz condition, then* S_h *are multiplicative on* $A_R(D)$ *for all* $h \in A(\mathbb{R})$.

Theorem 2.4 *Suppose that* X *is a complete linear metric locally convex space (*$\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$*) and a Leibniz* D_i*-algebra (*$i = 1, 2$*),* $D_i \in R(X)$ *are closed,* $\ker D_i \neq \{0\}$ *and* F_i *are continuous initial operators for* D_i *corresponding to a right inverse* R_i *almost quasinilpotent on* $\ker D_i$*, respectively. Let* $A(\mathbb{R}) = \mathbb{R}_+$ *or* $\mathbb{R}$. *Suppose that* $\{S_{i,h}\}_{h\in A(\mathbb{R})}$ *are strongly continuous semigroups (groups) of true shifts for* D_i $(i = 1, 2)$ *respectively, and either* $\overline{P(R_i)} = X$ *or* $\overline{E(R_i)} = X$ *for* $i = 1, 2$. *Let* κ_1, κ_2 *be the canonical mappings for* D_1, D_2*, respectively. Then* κ_i *are algebra isomorphisms on* $A_{R_i}(D_i)$ $(i = 1, 2)$ *and*

$$\kappa_1 D_1 \kappa_1^{-1} = \frac{\mathrm{d}}{\mathrm{d}t} = \kappa_2 D_2 \kappa_2^{-1} \quad \text{on } X. \tag{2.4}$$

Corollary 2.2 *Suppose that all assumptions of Theorem 2.4 are satisfied. Then the operators satisfying the Leibniz condition are uniquely determined as* $\frac{\mathrm{d}}{\mathrm{d}t}$ *up to isomorphisms determined by the canonical mappings.*

Definition 2.2 *Let* X *be a linear metric space. Let* $T \in L(X)$ *and* $x \in X$. *The set* $\mathcal{O}(T : x) = \{T^n x : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ *is said to be the* orbit *of* x *with respect to* T *(see Rolewicz, 1969). A continuous linear operator* T *acting in* X *is said to be* hypercyclic *if there is an element* $x \in X$ *(called later* hypercyclic vector*), such that its orbit* $\mathcal{O}(T : x)$ *is dense in* X *(see Shapiro, 1993).*

Theorem 2.5 *(see Przeworska-Rolewicz, 2001). Suppose that all conditions of Definition 3.1 are satisfied and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of true shifts. Let* $h \in A(\mathbb{R})$ *be arbitrarily fixed. Then the operator* e^{hD} *is hypercyclic and there is a* $\chi \in A_R(D)$ *which is a hypercyclic vector for* e^{hD}.

Properties enumerated in this sections show that true shifts are, indeed, true.

3. Algebras with logarithms

We start with

DEFINITION 3.1 *Suppose that* $D \in \mathsf{A}(X)$. *Let a multifunction* $\Omega : \text{dom } D \longrightarrow 2^{\text{dom } D}$ *be defined as follows:*

$$\Omega u = \{x \in \text{dom } D : Du = uDx\} \quad for\ u \in \text{dom } D. \tag{3.1}$$

The equation

$$Du = uDx \quad for\ (u, x) \in \text{graph } \Omega \tag{3.2}$$

is said to be the basic equation. *Clearly,*

$$\Omega^{-1}x = \{u \in \text{dom } D : Du = uDx\} \quad for\ x \in \text{ dom D}.$$

The multifunction Ω *is well-defined and* dom $\Omega \supset \ker D \setminus \{0\}$.

Suppose that $(u, x) \in$ graph Ω, *L is a selector of* Ω *and E is a selector of* Ω^{-1}. *By definitions,* $Lu \in \text{dom } \Omega^{-1}$, $Ex \in \text{dom } \Omega$ *and the following equations are satisfied:* $Du = uDLu$, $DEx = (Ex)Dx$.

Any invertible selector L of Ω *is said to be a* logarithmic mapping *and its inverse* $E = L^{-1}$ *is said to be a* antilogarithmic mapping. *By* $G[\Omega]$ *we denote the set of all pairs* (L, E), *where L is an invertible selector of* Ω *and* $E = L^{-1}$. *For any* $(u, x) \in \text{dom } \Omega$ *and* $(L, E) \in G[\Omega]$ *elements* Lu, Ex *are said to be* logarithm *of u and* antilogarithm *of x, respectively. The multifunction* Ω *is examined in Przeworska-Rolewicz (1998). The assumption that X is a commutative algebra is admitted here for simplicity and the sake of brevity only.*

Clearly, by definition, for all $(L, E) \in G[\Omega]$, $(u, x) \in$ graph Ω we have

$$ELu = u, \quad LEx = x; \qquad DEx = (Ex)Dx, \quad Du = uDLu. \tag{3.3}$$

A logarithm of zero is not defined. If $(L, E) \in G[\Omega]$ then $L(\ker D \setminus \{0\}) \subset \ker D$, $E(\ker D) \subset \ker D$. In particular, $E(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant.

Let $D \in \mathsf{A}(X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x + y) = (Ex)(Ey)$ for $x, y \in \text{dom } \Omega^{-1}$. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *Leibniz commutative algebra* (see Przeworska-Rolewicz, 1998). Moreover, $Le = 0$, i.e. $E(0) = e$. In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for $u \in$ dom Ω is that $u \in I(X)$ (see Przeworska-Rolewicz, 1998).

By $\mathbf{Lg}(D)$ we denote the class of these commutative algebras with $D \in R(X)$ and with unit $e \in$ dom Ω, for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$. By $\mathsf{L}(D)$ we denote the class of these commutative Leibniz algebras with unit $e \in \text{dom}\,\Omega$, for which there exist invertible selectors of Ω. By these definitions, $X \in \mathbf{Lg}(D)$ is a Leibniz algebra if and only if $X \in \mathsf{L}(D)$ and $D \in R(X)$. This class we shall denote by $L(D)$. It means that $L(D)$ is the class of these commutative Leibniz algebras with $D \in R(X)$ and with unit $e \in \text{dom}\,\Omega$, for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$.

Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Let $m \in \mathbb{N}$ be arbitrarily fixed. Let $X \in \mathbf{Lg}(D)$. Let $(L, E) \in G[\Omega]$. If

$$FD^j L = 0 \quad \mathit{for} \quad j = 0, 1, ..., m-1, \tag{3.4}$$

then (L, E) is said to be m-*normalized* by R and we write $(L, E) \in G_{R,m}[\Omega]$ (see Taylor Formula (1.2)).

If $\ker D = \{0\}$ then either X is not a Leibniz algebra or X has no unit. Thus, by our definition, if $X \in L(D)$ then $\ker D \neq \{0\}$, i.e. the operator D is right invertible but not invertible.

Theorem 3.1 *Suppose that* $X \in L(D)$*,* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$*,* $(L, E) \in G[\Omega]$ *and* A *is an algebra isomorphism of* X*. Let* $D' = A^{-1}DA$ *and let* $\Omega' : \text{dom}\ D' \longrightarrow 2^{\text{dom}\ D'}$ *be defined as follows:*

$$\Omega' u = \{x \in \text{dom}\ D' : D'u = uD'x\} \quad \mathit{for}\ u \in \text{dom}\ D'. \tag{3.5}$$

Then there are $(L', E') \in G[\Omega']$ *and* $L' = A^{-1}LA$*,* $E' = A^{-1}EA$*.*

Theorem 3.2 *Suppose that* X *is a complete linear metric locally convex space (*$\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$*),* $D \in R(X)$ *is closed,* $\ker D \neq \{0\}$ *and* F *is a continuous initial operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$*. Let* $A(\mathbb{R}) = \mathbb{R}_+$ *or* $\mathbb{R}$*,* $\{S_h\}_{h \in A(\mathbb{R})}$ *is a strongly continuous semigroup (group) of true shifts and either* $\overline{P(R)} = X$ *or* $\overline{E(R)} = X$*. Suppose, moreover, that* $X \in L(D)$*,* $(L, E) \in G[\Omega]$*. Write:* $D' = \kappa \frac{\mathrm{d}}{\mathrm{d}t} \kappa^{-1}$*, where* κ *is the canonical mapping defined by (2.1). Let* Ω' *be defined by (3.5). Then there are* $(L', E') \in G[\Omega']$ *such that* $L' = \ln$*,* $E'(\cdot) = \exp(\cdot)$*.*

Consider now some metric properties (see Przeworska-Rolewicz, 1998).

Definition 3.2 X *is said to be a complete* m-pseudoconvex algebra *if it is an algebra and a complete locally pseudoconvex space with the topology induced by a sequence* $\{\|\cdot\|_n\}$ *of* submultiplicative p_n*-homogeneous* F*-norms, i.e. such pseudonorms that* $\|xy\|_n \leq \|x\|_n \|y\|_n$ *for all* $x, y \in X$*,* $n \in \mathbb{N}$*.*

Theorem 3.3 *Suppose that either* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$*,* $X \in L(D)$ *with unit* $e \in \text{dom}\ \Omega^{-1}$ *is a complete* m*-pseudoconvex algebra and* $(L, E) \in G[\Omega]$*. Let* D *be closed. Let* $g = Re$ *and let* $\lambda g \in \text{dom}\ \Omega^{-1}$ *for an* $R \in \mathcal{R}_D$ *and a* $\lambda \in \mathbb{F}$*. Let the initial operator* F *corresponding to* R *be multiplicative. Write*

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3.6}$$

whenever this series is convergent. Then $\lambda \in v_{\mathbb{F}} R$ *and*

$$\mathrm{e}^{\lambda g} = (I - \lambda R)^{-1} e = E(\lambda g), \quad L\mathrm{e}^{\lambda g} = \lambda g. \tag{3.7}$$

THEOREM 3.4 *Suppose that all assumptions of Theorem 3.3 are satisfied and the series*

$$\sum_{n=0}^{\infty} g^{kn} \qquad (k \in \mathbb{N}) \tag{3.8}$$

is convergent. Then $e - g^k \in I(X)$ *and*

$$(e - g^k)^{-1} = \sum_{n=0}^{\infty} g^{kn} \quad (k \in \mathbb{N}). \tag{3.9}$$

COROLLARY 3.1 *Suppose that all assumptions of Theorem 3.3 are satisfied and* $e - g^k \in I(X)$ *for a* $k \in \mathbb{N}$. *Then* $(e - g^k)^2 \in I(X)$ *and*

$$(e - g^k)^{-2} = \sum_{n=1}^{\infty} (-1)^n kn g^{kn-1} \quad (k \in \mathbb{N}). \tag{3.10}$$

In particular, $DL(e - g) = -(e - g)^{-1}$, $D(e - g)^{-1} = -(e - g)^{-2}$ *and*

$$(e - g)^{-2} = \sum_{n=1}^{\infty} (-1)^n n g^{n-1}. \tag{3.11}$$

Write now $\Omega_1 = \Omega$, $E_1 = E$, $L_1 = L$. Similarly, for the operator D^2 in question, denote the corresponding multifunction and mappings by Ω_2, E_2, L_2. The mappings L_1, E_1 and L_2, E_2 will be called in the sequel logarithms and antilogarithms of *order one* and *order two*, respectively. Then we have the following summation formulae for antilogarithms of order one and order two.

THEOREM 3.5 *(see Przeworska-Rolewicz, 2004). Let* $X \in \mathsf{L}(D)$. *If* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$, $(L_1, E_1) \in G_{R,1}[\Omega]$, $(L_2, E_2) \in G_{R,2}[\Omega_2]$, $(u, x) \in \operatorname{graph} \Omega_2$, $a_1, ..., a_n \in \mathbb{C}$ *(*$n \in N$*), then* $u = E_1 x$ *and*

$$\sum_{j=1}^{n} a_j (E_1 jx) = \sum_{j=1}^{n} a_j (E_2 jx) E_2 [j^2 R^2 (Dx)^2] \quad (n \in \mathbb{N}). \tag{3.12}$$

THEOREM 3.6 *(see Przeworska-Rolewicz, 2004). Let* $X \in \mathsf{L}(D)$. *If* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$, $(L_1, E_1) \in G_{R,1}[\Omega]$, $(L_2, E_2) \in G_{R,2}[\Omega_2]$, $g = Re$ *and* λg, $\frac{1}{2}\lambda^2 g^2 \in \operatorname{dom} \Omega^{-1}$ *for a* $\lambda \in \mathbb{F}$, $a_1, ..., a_n \in \mathbb{C}$ $(n \in \mathbb{N})$, *then*

$$\sum_{j=1}^{n} a_j (E_1 j\lambda x) = \sum_{j=1}^{n} a_j (E_2 j\lambda x) E_2 [j^2 \lambda^2 R^2 (Dx)^2] \quad (n \in \mathbb{N}). \tag{3.13}$$

Since X is a Leibniz algebra, we have $E_1(jx) = (E_1x)^j$ $(j = 1, ..., n)$. Thus we obtain

Corollary 3.2 *(see Przeworska-Rolewicz, 2004). Let* $X \in \mathsf{L}(D)$*. If* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$*,* $(L_1, E_1) \in G_{R,1}[\Omega]$*,* $(L_2, E_2) \in G_{R,2}[\Omega_2]$*,* $(u, x) \in \text{graph } \Omega_2$*,* $a_1, ..., a_n \in \mathbb{C}$ *(*$n \in \mathbb{N}$*), then* $u = E_1x$ *and*

$$\sum_{j=1}^{n} a_j(E_1x)^j = \sum_{j=1}^{n} a_j(E_2jx)E_2[j^2R^2(Dx)^2] \quad (n \in \mathbb{N}). \tag{3.14}$$

Corollary 3.3 *(see Przeworska-Rolewicz, 2004). Let* $X \in \mathsf{L}(D)$*. If* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$*,* $(L_1, E_1) \in G_{R,1}[\Omega]$*,* $(L_2, E_2) \in G_{R,2}[\Omega_2]$*,* $g = Re$ *and* λg*,* $\frac{1}{2}\lambda^2 g^2 \in \text{dom } \Omega^{-1}$ *for a* $\lambda \in \mathbb{F}$*,* $a_1, ..., a_n \in \mathbb{C}$ *(*$n \in \mathbb{N}$*), then*

$$\sum_{j=1}^{n} a_j(E_1\lambda x)^j = \sum_{j=1}^{n} a_j(E_2j\lambda x)E_2[j^2\lambda^2R^2(Dx)^2] \quad (n \in \mathbb{N}). \tag{3.15}$$

4. Harmonic logarithms

We shall use the so-called *Roman factorial* defined as

$$[n]! = \begin{cases} n! & \text{if } n \geq 0; \\ \frac{(-1)^{n+1}}{(-n-1)!} & \text{if } n < 0 \end{cases} \qquad (n \in \mathbb{N}) \tag{4.1}$$

and *Roman coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \qquad (n, k \in \mathbb{Z}) \tag{4.2}$$

(see Roman and Rota, 1978). In particular, we have $\begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -k \end{bmatrix} = \frac{(-1)^{k+1}}{k!}$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 4.1 *(see Przeworska-Rolewicz, 2000) Suppose that* $X \in \mathbf{Lg}(D)$ *(*$\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$*),* F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *such that there is* $(L, E) \in G_{R,1}[\Omega]$*. We admit the following convention:* $R^{-n}L = D^nL$ *(*$n \in \mathbb{N}$*) for* $FL = 0$*. Harmonic logarithms of order* $p \in \mathbb{N}_0$ *are elements*

$$\lambda_n^{(p)}(u) = [n]!R^n(Lu)^p \quad \textit{for } u \in I(X) \cap \text{dom } \Omega, \; n \in \mathbb{Z}, \; p \in \mathbb{N}_0. \tag{4.3}$$

For instance, if $g = Re \in I(X) \cap \text{dom } \Omega$, then

$$\lambda_0^{(p)}(g) = (Lg)^p \quad (p \in \mathbb{N}_0) \quad ;$$

$$\lambda_n^{(1)}(g) = \begin{cases} g^n[Lg - (1 + \frac{1}{2} + ... + \frac{1}{n})e] & \text{if } n \in \mathbb{N}_0; \\ g^{-n} & \text{if } -n \in \mathbb{N}. \end{cases}$$

Note that harmonic logarithms are not logarithms in the sense of Definition 2.2, although are constructed with the use of these logarithms.

THEOREM 4.1 *(see Przeworska-Rolewicz, 2000) Suppose that X is a complete linear metric locally convex space ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ is closed,* $\ker D \neq \{0\}$ *and F is a continuous initial operator for D corresponding to a right inverse R almost quasinilpotent on* $\ker D$. *Suppose, moreover, that $X \in \mathbf{Lg}(D)$, there are $(L, E) \in G_{R,1}[\Omega]$, $g = Re \in I(X) \cap \mathrm{dom}\ D$, $g^{-1} \in A_R(D)$ and $\{S_h\}_{h \in A(\mathbb{R})}$ is a family of multiplicative true shifts. Then*

$$\lambda_n^{(p)}(g + he) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(g) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.4}$$

Theorem 4.1 is a generalization of the well-known binomial theorem with harmonic logarithms appearing in Umbral Calculus (see Roman and Rota, 1978; Loeb and Rota, 1989) for harmonic logarithms induced by a right invertible operator $D \in L(X)$ and $(L, E) \in G[\Omega]$.

Let $X \in \mathbf{Lg}(D)$, $(L, E) \in G_{R,1}[\Omega]$ $(R \in \mathcal{R}_D)$ and let $g = Re \in I(X) \cap \mathrm{dom}\ \Omega$. Consider the algebra

$$\mathcal{X}(L; g) = \mathrm{lin}\ \{g^n (Lg)^p : n \in \mathbb{Z},\ p \in \mathbb{N}_0\}.$$

Clearly, $\mathcal{X}(L; g)$ is a Leibniz algebra whenever X is a Leibniz algebra.

THEOREM 4.2 *(see Przeworska-Rolewicz, 2000) Suppose that $X \in \mathbf{Lg}(D)$ is a Leibniz algebra, $(L, E) \in G_{R,1}[\Omega]$ for an $R \in \mathcal{R}_D$ and $g = Re \in I(X) \cap \mathrm{dom}\ \Omega$. Then*

$$\mathcal{X}(L; g) = \mathrm{lin}\ \{\lambda_n^{(p)}(g) : n \in \mathbb{Z},\ p \in \mathbb{N}_0\}. \tag{4.5}$$

Now, there is a possibility to extend results obtained for algebras considered in Umbral Calculus to algebras $\mathcal{X}(L; g)$ induced by a right invertible operator $D \in L(X)$ and $(L, E) \in G[\Omega]$.

Concerning linear equations with scalar coefficients and with the right-hand side belonging to $\mathcal{X}(L; g)$, we have the following

THEOREM 4.3 *(see Przeworska-Rolewicz, 2000) Suppose that X is a complete linear metric space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and a commutative Leibniz algebra with unit e, $D \in R(X)$,* $\ker D \neq \{0\}$, *F is a multiplicative initial operator for D corresponding to an $R \in \mathcal{R}_D \cap AQN(\ker D)$, $X \in \mathbf{Lg}(D)$, $(L, E) \in G[\Omega]$ and $g \in I(X) \cap \mathrm{dom}\ \Omega$. Then every equation*

$$P(D)x = y, \qquad y \in \mathcal{X}(L; g) \quad \big(P(t) \in \mathbb{F}[t]\big) \tag{4.6}$$

has all solutions belonging again to $\mathcal{X}(L; g)$. If, in addition, $g^{-1} \in A_R(D)$ then $\mathcal{X}(L; g) \subset A_R(D)$.

Note that in the proof of Theorem 4.3 we have applied in an essential way properties of the so-called *D-R hulls* (see von Trotha, 1981 and also Przeworska-Rolewicz, 1996).

An analogue of Theorem 4.1 for $u \neq g = Re$ is

THEOREM 4.4 *Suppose that* X *is a complete linear metric locally convex space* ($\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ *is closed,* $\ker D \neq \{0\}$ *and* F *is a continuous initial operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$. *Suppose, moreover, that* $X \in \mathbf{Lg}(D)$, *there are* $(L, E) \in G_{R,m}[\Omega]$ ($m \in \mathbb{N}$), $u \in I(X) \cap \operatorname{dom} D^m$, $u^{-1} \in A_R(D)$ *and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of multiplicative true shifts. Then*

$$\lambda_n^{(p)}(S_h u) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.7}$$

COROLLARY 4.1 *Suppose that all assumptions of Theorem 4.4 are satisfied. Then*

$$\lambda_n^{(p)}(u) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(S_{-h} u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.8}$$

COROLLARY 4.2 *Suppose that all assumptions of Theorem 4.4 are satisfied. Then*

$$(LS_h u)^p = \frac{1}{n!} D^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} h^k \lambda_{n-k}^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.9}$$

Denote by $I_n(Y)$ the set of all elements from $Y \subset X$ having *n-th roots*:

$$I_n(Y) = \{x \in Y : \underset{y \in I(Y)}{\exists}\ y^n = x\} \quad (n \in \mathbb{N}).$$

If $x \in I_n(Y)$ and $y^n = x$ then we write $y = x^{1/n}$, $(n \in \mathbb{N})$.

THEOREM 4.5 *Suppose that* X *is a complete linear metric locally convex space* ($\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$), $D \in R(X)$ *is closed,* $\ker D \neq \{0\}$ *and* F *is a continuous initial operator for* D *corresponding to a right inverse* R *almost quasinilpotent on* $\ker D$. *Suppose, moreover, that* $X \in \mathbf{Lg}(D)$, *there are* $(L, E) \in G_{R,m}[\Omega]$ ($m \in \mathbb{N}$), $u \in I_p(X) \cap \operatorname{dom} D^m$, $u^{-1} \in A_R(D)$ *and* $\{S_h\}_{h\in A(\mathbb{R})}$ *is a family of multiplicative true shifts. Then*

$$S_h u = \big(\sum_{k=0}^{\infty} [k]! h^k D^k u^p\big)^{1/p} \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.10}$$

THEOREM 4.6 *Suppose that all assumptions of Theorem 4.4 are satisfied. Then*

$$\lambda_n^{(p)}(S_h u) = S_h \lambda_n^{(p)}(u) \qquad for\ n \in \mathbb{Z},\ p \in \mathbb{N}_0. \tag{4.11}$$

5. Generic elements and related summation formulae

Suppose that all assumptions of Theorem 3.3 are satisfied with that $\mathbb{F} = \mathbb{R}$. It means that $X \in L(D)$ with unit $e \in \mathrm{dom}\ \Omega^{-1}$ is a complete m-pseudoconvex algebra and $(L, E) \in G[\Omega]$, D is closed, $g = Re$, $\lambda g \in \mathrm{dom}\ \Omega^{-1}$ for an $R \in \mathcal{R}_D$ and a $\lambda \in \mathbb{F}$ and the initial operator F corresponding to R is multiplicative. If the series

$$\sum_{n=0}^{\infty} g^{kn} \qquad (k \in \mathbb{N}) \tag{5.1}$$

is convergent then, by Theorem 3.4, $e - g^k \in I(X)$ and

$$(e - g^k)^{-1} = \sum_{n=0}^{\infty} g^{kn}. \tag{5.2}$$

Write

$$\tilde{\Omega} = \big\{\{a_m(r)\}_{m\in\mathbb{Z}} \subset \mathrm{dom}\ \Omega : \tag{5.3}$$
$$\sum_{-\infty}^{\infty} a_m(r)g^m \ \textit{is convergent for all}\ r \in \mathbb{R},\ a_m(r) \in \mathrm{dom}\ \Omega\big\}$$

and

$$a(r) = \sum_{-\infty}^{\infty} a_m(r)g^m \quad \textit{for} \quad \sum_{-\infty}^{\infty} a_m(r)g^m \in \tilde{\Omega},, \tag{5.4}$$

i.e. by definition, $a(r) \in \tilde{\Omega}$.

Theorem 5.1 *Suppose that all assumptions of Theorem 3.3 are satisfied,* $\mathbb{F} = \mathbb{R}$, $(L, E) \in G[\Omega]$, $a \in \tilde{\Omega}$, $g = Re \in I(X)$ *and* $g^2 - e \in I(X)$. *Then there is an* $a(r)$ *such that for all* $r \in \mathbb{R}$

$$E[a(r)(g - g^{-1})] = \sum_{m=-\infty}^{\infty} a_m(r)g^m \in \tilde{\Omega}, \tag{5.5}$$

namely,

$$a(r) = g(g^2 - e)^{-1}L \sum_{m=-\infty}^{\infty} a_m(r)g^m. \tag{5.6}$$

Moreover,

$$E\{[a(r_1) + a_(r_2)](g - g^{-1})\} = \tag{5.7}$$
$$= \sum_{m=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} a_k(r_1)a_{m-k}(r_2)\Big]g^m \quad \textit{for all}\ r_1, r_2 \in \mathbb{R}.$$

Proof. By our assumptions, $g \in I(X)$, $g^2 - e \in I(X)$. This, and Corollary 3.1 together imply that $g - g^{-1} = g^{-1}(g^2 - e) \in I(X)$. Hence

$$\sum_{m=-\infty}^{\infty} a_m(r)g^m = EL \sum_{m=-\infty}^{\infty} a_m(r)g^m =$$
$$= E\{[g(g^2-e)^{-1}]^{-1}a(r)\} = E[a(r)(g^2-1)g^{-1}] = E[a(r)(g-g^{-1})].$$

Let $r_1, r_2 \in \mathbb{R}$ be arbitrarily fixed. Since X is a Leibniz algebra, we find

$$E\{[a(r_1) + a_(r_2)](g-g^{-1})\} = E[a(r_1)(g-g^{-1})]E[a(r_2)(g-g^{-1})] =$$
$$= \Big[\sum_{k=-\infty}^{\infty} a_k(r_1)g^k\Big]\Big[\sum_{j=-\infty}^{\infty} a_j(r_2)g^j\Big] = \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_k(r_1)a_j(r_2)g^{j+k} =$$
$$= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k(r_1)a_{n-k}(r_2)g^n = \sum_{m=-\infty}^{\infty}\Big[\sum_{k=-\infty}^{\infty} a_k(r_1)a_{m-k}(r_2)\Big]g^m.$$

■

The element $E[a(r)(g-g^{-1})]$ is said to be *generic* for an element $a \in \tilde{\Omega}$.

Example 5.1 *Let $a(r) = \frac{1}{2}ir$ for $r \in \mathbb{R}$. Let $J_m(r)$ $(m \in \mathbb{Z})$ be the classic Bessel functions. Then the generic function is*

$$\mathrm{e}^{\frac{1}{2}ir(z-z^{-1})} = \sum_{m=-\infty}^{\infty} J_m(r)z^m.$$

Moreover, for all $r_1, r_2 \in \mathbb{R}$, we have

$$J_m(r_1+r_2) = \sum_{k=-\infty}^{\infty} J_{m-k}(r_1)J_k(r_2) \qquad (m \in \mathbb{Z})$$

(see Wawrzyńczyk, 1984; Vich, 1987).

Putting in Formula (5.7) $r_1 = r_2 = r$, we obtain

Corollary 5.1 *Suppose that all assumptions of Theorem 5.1 are satisfied. Then*

$$a_m(r) = \frac{1}{2}\sum_{k=-\infty}^{\infty} a_{m-k}(r)a_k(r) \quad for\ r \in \mathbb{R}\ (m \in \mathbb{Z}). \tag{5.8}$$

Since

$$\sum_{m=-\infty}^{\infty} a_m(0)g^m = E(0) = e \quad whenever\ a(0) = 0,$$

we get

Corollary 5.2 *Suppose that all assumptions of Theorem 5.1 are satisfied and* $a(0) = 0$. *Then* $a_m(0) = \delta_{m0}e$ *for all* $m \in \mathbb{Z}$ *(where* δ_{m0} *is the Kronecker symbol, i.e.* $\delta_{00} = 1$ *and* $\delta_{m0} = 0$ *for* $m \neq 0$*).*

An important property of generic elements is given in the following

Theorem 5.2 *Suppose that all assumptions of Theorem 3.3 are satisfied,* $\mathbb{F} = \mathbb{R}$, $(L, E) \in G[\Omega]$, $\mathbf{a}, \mathbf{b} \in \tilde{\Omega}$, $g = Re \in I(X)$ *and* $g^2 - e \in I(X)$. *Write*

$$E_a(r) = E[a(r)(g - g^{-1}],\ E_b(r) = E[b(r)(g - g^{-1}], \qquad (5.9)$$
$$E_{a+b}(r) = E\{[a(r) + b(r)](g - g^{-1})\} \quad (r \in \mathbb{R}),$$

where

$$a(r) = \sum_{m=-\infty}^{\infty} a_m(r)g^m, \quad b(r) = \sum_{m=-\infty}^{\infty} b_m(r)g^m \quad (r \in \mathbb{R}). \qquad (5.10)$$

Then

$$E_{a+b} = E_aE_b = E_{a*b}, \qquad (5.11)$$

where

$$E_{a*b}(r) = \sum_{n=-\infty}^{\infty} \Big[\sum_{j=-\infty}^{\infty} a_{n-j}(r)b_j(r)\Big]g^n.$$

Proof. By our assumptions, for all $r \in \mathbb{R}$

$$E_{a+b}(r) = E\{[a(r) + b(r)](g - g^{-1}\} = E[a(r)(g - g^{-1}]E[b(r)(g - g^{-1}] =$$
$$= \Big[\sum_{m=-\infty}^{\infty} a_m(r)g^m\Big]\Big[\sum_{m=-\infty}^{\infty} b_m(r)g^m\Big] =$$
$$= \sum_{m=-\infty}^{\infty} \Big[\sum_{j=-\infty}^{\infty} a_m(r)b_j(r)g^{j+m}\Big] = \sum_{n=-\infty}^{\infty} \Big[\sum_{j=-\infty}^{\infty} a_{n-j}(r)b_j(r)\Big]g^n.$$

■

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