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Fixed points of mappings in Klee admissible spaces

by

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Abstract: In this paper we generalize the Lefschetz fixed point theorem from the case of metric ANR-s to the case of acceptable subsets of Klee admissible spaces. The results presented in this paper were announced in an earlier publication of the authors.

Keywords: Lefschetz number, fixed points, topological vector spaces, Klee admissible spaces.

1. Introduction

The famous Lefschetz fixed point theorem proved by S. Lefschetz in 1923 is still studied by many authors. The metric case of this theorem is quite well developed (see: Fournier, 1975; Górniewicz, 2005; Górniewicz, Ślosarski, 2007; Granas, 1967; Granas and Dugundji, 2003; Kryszewski, 1987). Nonmetric case was considered in Andres and Górniewicz (2003), Fournier (1975), Fournier and Granas (1973), Gróniewicz, Rozpłoch (1996). In this paper we shall prove some further generalizations.

2. Klee admissible spaces

In what follows we shall consider linear vector spaces over the field of real numbers R (compare Andres and Górniewicz, 2003; Rolewicz, 1972). All mappings are assumed to be continuous.

Definition 2.1 *Let E be a topological vector space. We shall say that E is a Klee admissible space provided for any compact subset $K \subset E$ and for any open neighbourhood V of $0 \in E$ there exists a map:*

$$\pi_V : K \to E$$

such that the following two conditions are satisfied:
(2.1.1) $\pi_V(x) \in (x + V)$, *for any* $x \in K$,
(2.1.2) there exists a natural number $n = n_K$ *such that* $\pi_V(K) \subset E^n$, *where* E^n *is an* n*-dimensional subspace of* E.

Roughly speaking, a space E is Klee admissible, if compact mappings into E can be approximated by compact finite dimensional mappings. Firstly, from the Schauder approximation theorem it follows that any normed space is Klee admissible (compare Fournier, 1973; or Andres and Górniewicz, 2003). It is known (compare Andres and Górniewicz, 2003; Fournier and Granas, 1973; or Górniewicz and Rozpłoch, 1996) that any locally convex topological vector space is Klee admissible. The authors do not know any example of a topological vector space which is not Klee admissible but the following problem is still open:

OPEN PROBLEM 1 *Is it true that any topological vector space is Klee admissible?*

We need the following definition:

DEFINITION 2.2 (Borsuk, 1966) *Let* X, Y *be two topological Hausdorff spaces. We shall say that* X *is* r*-dominated by* Y *provided there are two maps* $r : Y \to X$ *and* $s : X \to Y$ *such that* $r \circ s = Id_X$, *where* $Id_X : X \to X$ *is a mapping defined by* $Id_X(x) = x$ *for every* $x \in X$.

Now, we shall formulate the main notion of this section.

DEFINITION 2.3 *A topological Hausdorff space* X *is called neighbourhoodly acceptable (written* $X \in NA_C$*) provided there exists a Klee admissible space* E *and an open subset* $U \subset E$ *such that* X *is* r*-dominated by* U*; we shall say that* X *is* A_C*-space if* X *is* r*-dominated by a Klee admissible space* E.

Let us remark that NA_C-spaces (A_C-spaces) play the same role in the case of nonmetric topological spaces as ANR-s (AR-s) in the case of metric spaces, i.e., Klee admissible spaces play the role of normed spaces considered in the case of metric spaces (compare Borsuk, 1966; Fournier, 1975; Fournier and Granas, 1973; Górniewicz and Rozpłoch, 1996; Górniewicz and Ślosarski, 2007; Granas, 1967; Granas and Dugundji, 2003).

REMARK 2.1 *Observe that if* $X \in NA_C$ *and* U *is an open subset of* X, *then* $U \in NA_C$, *too.*

3. Abstract version of the Lefschetz fixed point theorem

Let Top_2 be the category of pairs of Hausdorff topological spaces and continuous mappings of such pairs. By a pair (X, A) in Top_2 we understand a Hausdorff topological space X and its subset, a pair $(X, \emptyset)$ for short we shall denote by X. By a map $f : (X, A) \to (Y, B)$ we shall understand a continuous map from X to Y such that $f(A) \subset B$. In what follows we shall use the following notations: if $f : (X, A) \to (Y, B)$ is a map of pairs, then by $f_X : X \to Y$ and $f_A : A \to B$ we shall denote the respective induced mappings. Let us denote by $Vect_G$ the category of graded vector spaces over the field of rational numbers Q and linear maps of degree zero between such spaces. By $H : Top_2 \to Vect_G$ we shall denote the singular homology functor (see Rolewicz, 1972; or Granas and Dugundji, 2003) with the coefficients in Q. Thus, for any pair (X, A) we have

$$H(X, A) = \{H_q(X, A)\}_{q \geq 0}$$

a graded vector space in $Vect_G$ and for any map $f : (X, A) \to (Y, B)$ we have the induced linear map of degree zero:

$$f_* = \{f_{*q}\} : H(X, A) \to H(Y, B),$$

where $f_{*q} : H_q(X, A) \to H_q(Y, B)$ is a linear map of q-dimensional homology of (Y, B).

A non-empty space X is called acyclic provided:

(i) $H_q(X) = 0$ for all $q \geq 1$,

(ii) $H_0(X) \approx Q$.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \ for\ some\ n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $dim\widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of graded vector spaces $E = \{E_q\}$. We call u a Leray endomorphism if

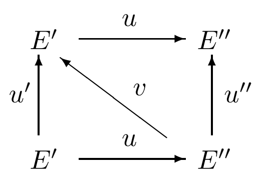
(i) all u_q are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial. For such u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (compare Górniewicz and Rozpłoch 1996; or Granas, 1967). The following important property of the Leray endomorphism is a consequence of the well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 3.1 *Assume that, in the category of graded vector spaces, the following diagram commutes*



Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case, $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u : E \to E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly nilpotent endomorphism $u : E \to E$, we have $N(u) = E$, we get:

PROPOSITION 3.2 *If $u : E \to E$ is a weakly-nilpotent endomorphism, then $\Lambda(u) = 0$.*

Let $f : (X, X_0) \to (X, X_0)$ be a map, $f_* : H(X, X_0) \to H(X, X_0)$ be a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic $f \sim g$, then $\Lambda(f)$ is well defined iff $\Lambda(g)$ is well defined; and, in this case, $\Lambda(f) = \Lambda(g)$. Let us observe that if X is an acyclic space or, in particular, contractible, then for every $f : X \to X$ the endomorphism $f_* : H(X) \to H(X)$ is a Leray endomorphism and $\Lambda(f_*) = 1$. Consequently, if $X \in AR$ or, in particular, X is a convex subset in a normed space, then for every continuous map $f : X \to X$ the Lefschetz number $\Lambda(f) = \Lambda(f_*) = 1$. We have the following lemma (see: Bowszyc, 1968/1969; Fournier, 1975; Fournier and Granas, 1973; Granas and Dugundji, 2003).

LEMMA 3.1 *Let $f : (X, X_0) \to (X, X_0)$ be a map of pairs. If two of the endomorphisms $f_* : H(X, X_0) \to H(X, X_0)$, $(f_X)_* : H(X) \to H(X)$, $(f_{X_0})_* : H(X_0) \to H(X_0)$ are Leray endomorphisms, so is the third; in that case:*

$$\Lambda(f_*) = \Lambda((f_X)_*) - \Lambda((f_{X_0})_*)$$

or equivalently

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_{X_0}).$$

We shall also use the following proposition:

PROPOSITION 3.3 *Assume that for a mapping $f : X \to X$ the Lefschetz number $\Lambda(f)$ is well defined and let p be a prime number, then $\Lambda(f^p)$ of f^p is well defined and $\Lambda(f) \equiv \Lambda(f^p)$ mod p.*

For the proof see Peitgen (1976) or Granas (1967).

We shall use the following notion:

DEFINITION 3.1 *A map $f : X \to X$ is called a Lefschetz map provided $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that $Fix(f) = \{x \in X; f(x) = x\} \neq \emptyset$.*

Finally, we shall prove the following abstract version of the Lefschetz fixed point theorem (for the metric case see 2.7 in Górniewicz and Ślosarski, 2007).

THEOREM 3.1 *(Abstract version of the Lefschetz fixed point theorem). Let* (X, A) *be a pair in* Top_2 *and let* $f : (X, A) \to (X, A)$ *be a map such that:*
(3.1.1) $f_* : H(X, A) \to H(X, A)$ *is weakly nilpotent,*
(3.1.2) $f_A : A \to A$ *is a Lefschetz map.*
Then $f_X : X \to X$ *is a Lefschetz map.*

Proof. First, in view of Proposition 3.2, we have $\Lambda(f) = 0$. Consequently, by Lemma 3.1, $\Lambda(f_X)$ is well defined and $\Lambda(f_X) = \Lambda(f_{X_0})$. Hence, $\Lambda(f_X) \neq 0$ implies $\Lambda(f_{X_0}) \neq 0$ and, by assumption (3.1.2), $Fix(f_{X_0}) \neq \emptyset$. Finally, since $Fix(f_{X_0}) \subset Fix(f_X)$, our theorem is proved. ∎

4. Consequences of Theorem 3.1

First, we recall the following result proved by G. Fournier and A. Granas (1973).

THEOREM 4.1 *If* X *is a* NA_C*-space and* $f : X \to X$ *is a compact map, then* f *is a Lefschetz map.*

Our first application of Theorem 3.1 concerns compact absorbing contractions.

DEFINITION 4.1 *(compare Fournier, 1973; Fournier and Granas, 1973; Górniewicz, 2005; Górniewicz, Rozpłoch, 1996; Górniewicz, Ślosarski, 2007; Granas and Dugundji, 2003) A map* $f : X \to X$ *is called a compact absorbing contraction (written* $f \in CAC(X)$*) provided there exists an open set* $U \subset X$ *such that:*
(4.1.1) $f(U) \subset U$ *and the map* $f_U : U \to U$ $f_U(x) = f(x)$ *for every* $x \in U$ *is compact,*
(4.1.2) for every $x \in X$ *there exists* $n = n_x$ *such that* $f^n(x) \in U$.

We let

$$K(X) = \{f : X \to X;\ f \text{ is compact}\}.$$

Evidently we have $[K(X) \subset CAC(X)$. We prove:

THEOREM 4.2 *If* $X \in NA_C$ *and* $f \in CAC(X)$*, then* f *is a Lefschetz map.*

Proof. Let U be chosen according to Definition 4.1. In view of Remark 2.1 $U \in NA_C$. Let $\overline{f} : (X, U) \to (X, U)$ be defined by $\overline{f}(x) = f(x)$ for every $x \in X$ and let $f_U : U \to U$ be induced by $\overline{f}$. In view of Theorem 4.1 the map f_U is a Lefschetz map. Consequently, if we prove that $\overline{f_*}$ is weakly nilpotent, then our claim follows from Theorem 3.1. Let K be a compact subset of X. Since U is open in X, then Definition 4.1 implies that there exists $n = n_K$ such that $\overline{f}^n(K) \subset U$. Finally, from the fact that the singular homology theory is a functor with compact carriers we deduce that $\overline{f_*}$ is a weakly nilpotent linear map of degree zero and the proof is complete. ∎

Observe that Theorem 4.2 can be formulated in the following slightly more general form:

Theorem 4.3 *Let (X, d) be a metric space and let $f : X \to X$ be a CAC-mapping. Assume, further, that there exists an NA_C-space $A \subset X$ such that $\overline{f(U)} \subset A$, where U is chosen according to Definition 4.1, then f is a Lefschetz map.*

Now we shall generalize the class of CAC- mappings.

Definition 4.2 *Let $f, h : (X, A) \to (X, A)$ be two mappings. We shall say that $f_X : X \to X$ is a generalized compact absorbing contraction (written $f_X \in GCAC(X)$) provided the following conditions are satisfied:*
(4.2.1) $f_A : A \to A$ is a Lefschetz map,
(4.2.2) for every compact $K \subset X$ there exists $n = n_K$ such that $f^n(h(K)) \subset A$ (or $h(f^n(K)) \subset A$ and $f(h^{-1}(A)) \subset h^{-1}(A)$),
(4.2.3) $h_* : H(X, A) \to H(X, A)$ is an epimorphism ($h_* : H(X, A) \to H(X, A)$ is a monomorphism).

Remark 4.1 *Observe that if $X \in NA_C$ A is an open subset of X and $h = Id_{(X,A)}$ then the class $GCAC$ reduces to CAC- mappings.*

For more information on $GCAC$ mappings see Górniewicz, Ślosarski (2007). Now we shall formulate the following generalization of the Lefschetz fixed point theorem.

Theorem 4.4 *If $f_X \in GCAC(X)$, then f_X is a Lefschetz map.*

Proof. We have a map $f : (X, A) \to (X, A)$ such that f_A is a Lefschetz map and $f_X \in GCAC(X)$. In view of the abstract version of the Lefschetz fixed point theorem for the proof it is sufficient to show that f_* is weakly nilpotent. Let $h : (X, A) \to (X, A)$ be chosen according to the definition of $GCAC$- mappings. Firstly, let h_* be a monomorphism and let $z \in H(X, A)$. We have to prove that there exists $n = n_z$ such that $(f_*)^n(z) = 0$. Observe that $(f_*)^n = (f^n)_*$. Since we consider the homology functor H with compact carriers we can assume that $supp(z) \subset K$, where K is a compact subset of X. By assumption, there exists $n = n_K$ such that $h(f^n(K)) \subset A$. It implies that $(h \circ f^n)_*(z) = 0$. On the other hand we have

$$[0 = (h \circ f^n)_*(z) = (h_* \circ (f^n)_*)(z) = h_*((f_*)^n(z))] \Rightarrow [(f_*)^n(z) = 0].$$

Now, assume that h_* is an epimorphism and let $z \in H(X, A)$. There exists $y \in H(X, A)$ such that $h_*(y) = z$ and again we can assume that $supp(y) \subset K_1$, where K_1 is a compact subset of X. By assumption, there exists $m = m_{K_1}$ such that $f^m(h(K_1)) \subset A$. It implies that $(f^m \circ h)_*(y) = 0$. We have

$$0 = (f^m \circ h)_*(y) = ((f^m)_* \circ h_*)(y) = (f_*)^m(h_*(y)) = (f_*)^m(z).$$

The proof is complete. ■

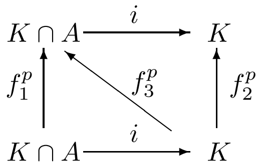
Finally, we would like to prove a generalization of the Schauder fixed point theorem.

Definition 4.3 *Let $f : (X, A) \to (X, A)$ be a given map. We shall say that $f_X : X \to X$ is an acyclically compact absorbing contraction (written $f_X \in ACAC(X)$) provided the following conditions are satisfied,*
(4.3.1) $f_A : A \to A$ is a compact map,
(4.3.2) there exists an acyclic set $K \subset X$ such that $\overline{f_A(A)} \subset K$ and $(K \cap A) \in NA_C$,
(4.3.3) there exists an n such that $f^n(K) \subset A$.

Now we are able to prove:

Theorem 4.5 *If $f_X \in ACAC(X)$, then $Fix(f_X) \neq \emptyset$.*

Proof. By assumption, there exists n such that $(f_X)^n(K) \subset A$ and $(f_X)^{n+k}(K) \subset (f_X)^k(A) \subset K$, for all $k \geq 1$, where A is chosen according to Definition 4.3. We can assume that $n + k = p$ is a prime number for some $k \geq 1$. We have the following commutative diagram:



in which f_1^p, f_2^p, f_3^p are the respective contractions of f_X^p. By assumption, f_1^p is a compact map and $(K \cap A) \in NA_C$, consequently $\Lambda(f_1^p)$ is well defined. By using Proposition 3.1 we deduce that:

$$\Lambda(f_1^p) = \Lambda(f_2^p) = 1.$$

Now, by using Proposition 3.3 we get:

$$\Lambda(f_1^p) \equiv \Lambda(f_1) \; mod \; p,$$

where $f_1 \equiv (f_X)_{/K\cap A}$ is the contraction of f_X. By Definition 4.3 (see 4.3.1 and 4.3.2) a compact map $(f_X)_{/K\cap A}$ is well defined. This implies that $\Lambda((f_X)_{/K\cap A}) \neq 0$. Finally, $Fix((f_X)_{/K\cap A}) \neq \emptyset$ and $Fix((f_X)_{/K\cap A}) \subset Fix(f_X)$, so the proof is complete. ■

By using similar arguments one can prove Theorem 4.5 in a slightly different form. Firstly, we shall say that a set $K \subset X$ is f-admissible, where $f : X \to X$ a continuous map, provided there exists $m \in N$ such that for every $n \geq m$ the following conditions are satisfied:

(a) $f^n : K \to K$,
(b) $f^n_* : H(K) \to H(K)$ is a Leray endomorphism,
(c) $\Lambda(f^n) = \Lambda(f^{n+1}) \neq 0$.

We shall say that a map $f : X \to X$ is an $\widehat{ACAC}$-mapping provided the following conditions are satisfied:

(i) there exists $A \subset X$ such that $f(A) \subset A$ and a map $f_A : A \to A$ given by the formula $f_A(x) = f(x)$ is a compact map,

(ii) there exists an f- admissible subset $K \subset X$ such that $\overline{f_A(A)} \subset K$ and $K \cap A$ is a NA_C-space,

(iii) there exists an n such that $f^n(K) \subset A$.

Now, we are able to formulate the following:

THEOREM 4.6 *If $f : X \to X$ is an $\widehat{ACAC}$-mapping, then $Fix(f) \neq \emptyset$.*

Finally, let us remark that all results presented in this paper can be taken up for admissible multivalued mappings (compare Andres and Górniewicz, 2003; Górniewicz, 2005).

Added in proof: Let us remark that Theorem 4.2 was proved (by using different methods) in the paper by R.P. Agarwal and D. O'Regan (2005) but only for compact mappings.

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