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A C^1 function which is nowhere strongly paraconvex and nowhere semiconcave

by

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Abstract: X be an infinite-dimensional Banach space which admits an equivalent Fréchet smooth norm (or, more generally, a C^1 Lipschitz bump function). Then there exists a C^1 Lipschitz function f on X such that, on any open ball, f is neither strongly paraconvex nor semiconcave. In particular, f is approximately convex in the sense of Ngai, Luc and Théra, but it is not strongly paraconvex on any ball.

Keywords: approximately convex function, strongly $\alpha(\cdot)$-paraconvex function, semiconcave function, C^1 function.

1. Introduction

A number of notions of generalized convex functions (surfaces, sets) was considered in the literature. Functions f defined on $\mathbf{R}^n$ (or corresponding surfaces in $\mathbf{R}^{n+1}$), which are locally representable in the form $f(x) = g(x) - c\|x\|^2$, where g is convex and $c > 0$, were treated in the literature many times under different names and different (equivalent) definitions (see, e.g., Reshetnyak, 1956; Rockafellar, 1982; Cannarsa, and Sinestrari, 2004). Most frequently, these functions are called (locally) semiconvex or lower-C^2 functions.

The first step to further generalization of convex functions in this direction was made in 1979 by Rolewicz (1979), who defined the notion of (γ)-paraconvex functions (where $1 < \gamma \leq 2$) in general Banach spaces.

This notion coincides with the notion of a (general) semiconvex function (Alberti, Ambrosio, and Cannarsa, 1992; Albano, Cannarsa, 1999) with the modulus $\omega(t) = t^{\gamma-1}$, and with the notion of strongly $\alpha(\cdot)$-paraconvex function (Rolewicz, 2000) with $\alpha(t) = t^{\gamma}$.

A further generalization was made in Ngai, Luc, Théra (2000), where the notion of approximately convex functions (more general than the notions of strongly paraconvex and semiconvex functions) on a Banach space was considered. Approximately convex functions in $\mathbf{R}^n$ coincide (Daniilidis, Georgiev, 2004) with lower-C^1 functions of Spingarn (1981). Moreover, in Zajíček (2007, Remark 2.6), the following observation was made:

A continuous function f on $\mathbf{R}^n$ is approximately convex if and only if f is locally strongly paraconvex (equivalently: locally uniformly approximately convex).

Further, an example from Zajíček (2007, Remark 2.6) shows that the above statement fails in ℓ^2. The aim of the present note is to generalize and improve this example.

For our construction, we suppose that X is an infinite-dimensional Banach space which admits an equivalent Fréchet smooth norm (or, more generally, a C^1 Lipschitz bump function). Then we construct a Lipschitz function f on X which is C^1 smooth (and therefore also approximately convex) but is not strongly paraconvex on any ball. (We even show that this property applies also to $-f$; i.e., f is not semiconcave on any ball.)

2. Preliminaries

In the following, X will be always a (real) Banach space and $B(x, r)$ will denote an open ball with center x and radius r.

In the following definitions, f is a real function defined on an open subset G of X.

Recall that $A \in X^*$ is called a (Fréchet; uniform) strict derivative of f at $a \in X$ if

$$\lim_{x,y \to a, x \neq y} \frac{f(y) - f(x) - \langle A, y - x \rangle}{\|y - x\|} = 0$$

(where we allow that $x = a$ or $y = a$).

Denote by $\mathcal{M}$ the system of all functions $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0.

The following definition is taken from Rolewicz (2006); the definitions in Rolewicz (2000) and Rolewicz (2005) are slightly different, but essentially equivalent.

Definition 1 *Let* $\alpha \in \mathcal{M}$ *be such that* $\lim_{t \to 0+} \alpha(t)/t = 0$ *and* $\Omega \subset X$ *be a convex open set. A continuous function* $f : \Omega \to \mathbf{R}$ *is called* strongly $\alpha(\cdot)$-paraconvex *if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) + \min(\lambda, 1 - \lambda)\alpha(\|x - y\|), \quad (1)$$

whenever $\lambda \in [0, 1]$ *and* $x, y \in \Omega$. *We will say that* f *is* strongly paraconvex *if it is strongly* $\alpha(\cdot)$*-paraconvex for some* $\alpha \in \mathcal{M}$ *with* $\lim_{t \to 0+} \alpha(t)/t = 0$.

The following definition is taken from Cannarsa and Sinestrari (2004); the definitions in Alberti, Ambrosio, Cannarsa (1992) and Albano, Cannarsa (1999) are slightly different but essentially equivalent.

Definition 2 *A continuous real function f on an open convex set $\Omega \subset X$ is called semiconcave with modulus $\omega \in \mathcal{M}$ if*

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\| \quad (2)$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called semiconcave on Ω if it is semiconcave on Ω with some modulus $\omega \in \mathcal{M}$. A function f is called semiconvex if $-f$ is semiconcave.

Now we define approximately (or approximate) convex functions in the sense of Ngai, Luc and Théra (2000) (*let us note that the term "approximately convex functions" is used for a long time for another type of functions, namely for ε-convex functions in the sense of Hyers and Ulam*).

Definition 3 (Ngai, Luc, Théra, 2000) *A continuous function f on an open set $\Omega \subset X$ is called approximately convex at $x_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \varepsilon\lambda(1-\lambda)\|x-y\| \quad (3)$$

whenever $\lambda \in [0,1]$ and $x, y \in B(x_0, \delta)$. We say that f is approximately convex on Ω if it is approximately convex at each $x_0 \in \Omega$.

We say that f is uniformly approximately convex on an open convex set $\Omega \subset X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that (3) *holds whenever $\lambda \in [0,1]$, $x, y \in \Omega$, and $\|x-y\| < \delta$.*

Remark 1

(i) It is easy to show (see Zajíček, 2007) that f is semiconvex on an open convex $\Omega \subset X$ if and only if f is strongly paraconvex on Ω.

(ii) It is easy to see that each strongly paraconvex (or semiconvex) function on an open convex $\Omega \subset X$ is uniformly approximately convex on Ω. (For a converse of this fact, see Rolewicz, 2001).

(iii) If f is a C^1 smooth function on an open convex $\Omega \subset X$, then f is approximately convex on Ω, see Ngai, Luc, Théra (2000, Proposition 3.1)

The following lemma is an immediate consequence of Veselý, Zajíček (1989, Proposition 3.7) (see also Zajíček, 1991, Theorem A).

Lemma 1 *Let f be a real function defined on an open subset G of a Banach space X. Then f is strictly differentiable at a point $a \in G$ if and only if f is continuous at a and for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left| \frac{f(y+kv) - f(y)}{k} - \frac{f(y) - f(y-hv)}{h} \right| < \varepsilon, \quad (4)$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in B(a, \delta)$, $y + kv \in B(a, \delta)$.

The following lemma is an easy (geometrically more understandable) reformulation of the definition of uniform approximate convexity.

LEMMA 2 *Let X be a Banach space and f be a real function defined on an open convex set $\Omega \subset X$. Then, f is uniformly approximately convex on Ω if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h} > -\varepsilon, \tag{5}$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in \Omega$, $y + kv \in \Omega$, and $k + h < \delta$.

Proof. Consider arbitrary $\|v\| = 1$, $k > 0$ and $h > 0$, for which $y - hv \in \Omega$, $y + kv \in \Omega$. Denote $x_1 := y + kv$, $x_2 := y - hv$, $t := h/(h + k)$. Then $1 - t = k/(k + h)$, $y = tx_1 + (1 - t)x_2$ and an easy computation shows that

$$\frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h}$$

$$= \frac{1}{t(1-t)\|x_1-x_2\|}\left(tf(x_1)+(1-t)f(x_2)-f(tx_1+(1-t)x_2)\right).$$

The rest of the proof is clear. ■

3. Construction

We will say that a real function f is c-Lipschitz on a subset of a Banach space X, if f is Lipschitz with Lipschitz constant c (not necessarily minimal). If f is defined on the whole X, we put $\|f\| = \sup_{x\in X} |f(x)|$.

We will work with an infinite-dimensional Banach space X which admits a Lipschitz C^1 bump function F (i.e., F is not identically zero and $\operatorname{supp} F := \{x \in X : F(x) \neq 0\}$ is bounded). We can clearly suppose that $F(0) > 0$, F is 1-Lipschitz, and $\operatorname{supp} F \subset B(0, 1)$. Recall (see, e.g., Deville, Godefroy and Zizler, 1993, Fact I.2.1), that if X admits an equivalent Fréchet smooth norm, then X admits such F.

Setting In the following, we will suppose that X is an infinite dimensional Banach space and F is a 1-Lipschitz C^1 function such that $\operatorname{supp} F \subset B(0, 1)$ and $0 < \gamma := F(0) \leq 1$.

LEMMA 3 *Let X, F and γ be as in the Setting. Let $y \in X$ and $1 > \tau > 0$. Set*

$$F_{y,\tau}(x) := \tau F((x-y)/\tau), \qquad x \in X.$$

Then $F_{y,\tau}$ is a 1-Lipschitz C^1 function such that $\operatorname{supp} F_{y,\tau} \subset B(y, \tau)$. Further, there exist a unit vector $v \in X$, and $0 < h < \tau$, $0 < k < \tau$ such that

$$\frac{F_{y,\tau}(y+kv)-F_{y,\tau}(y)}{k} - \frac{F_{y,\tau}(y)-F_{y,\tau}(y-hv)}{h} < -2\gamma. \tag{6}$$

Proof. Choose an arbitrary unit vector $v \in X$. Since, clearly, $\operatorname{supp} F_{y,\tau} \subset B(y,\tau)$, we can choose $0 < h < \tau$, $0 < k < \tau$ such that $F_{y,\tau}(y+kv) = F_{y,\tau}(y-hv) = 0$. Since $F_{y,\tau}(y) = \tau\gamma$, we easily obtain (6). All other assertions of the lemma are obvious. ∎

LEMMA 4 *Let X, F and γ be as in the Setting. Let p be a C^1 function on X and $c > 0$, $1 > \delta > 0$ be given. Then there exists a function $q = q(p,c,\delta)$ such that*

(i) *q is C^1, $\|q\| \le c$, and q is c-Lipschitz;*

(ii) *for each $x \in X$ and $\eta > 0$ there exist points y_1, y_2 in $B(x,\delta)$, unit vectors v_1, v_2 in X, and positive real numbers h_1, k_1, h_2, k_2, such that $y_i - h_i v_i \in B(x,\delta)$, $y_i + k_i v_i \in B(x,\delta)$, $h_i + k_i < \eta$ ($i = 1,2$), and, denoting $s := p + q$,*

$$\frac{s(y_1+k_1v_1)-s(y_1)}{k_1} - \frac{s(y_1)-s(y_1-h_1v_1)}{h_1} > c\gamma$$

$$\frac{s(y_2+k_2v_2)-s(y_2)}{k_2} - \frac{s(y_2)-s(y_2-h_2v_2)}{h_2} < -c\gamma. \quad (7)$$

Proof. Using Zorn's lemma, we find a maximal set $Z \subset X$ which is $\delta/2$-discrete (i.e., $\|z_1 - z_2\| \ge \delta/2$ for every distinct $z_1, z_2 \in Z$). Since $B(0,\delta/9)$ is not relatively compact, we can choose $\xi \in (0,\delta/9)$ and a sequence $w_1, w_2, \dots$ in $B(0,\delta/9)$ such that $\|w_k - w_l\| \ge 3\xi$, whenever $k \ne l$. For each $z \in Z$ and $k \in \mathbf{N}$, set $y_{z,k} := z + w_k$. Since p is C^1, it is easy to prove (see Nijenhuis, 1974, (2), p. 970) that p is strictly differentiable at all points. So, for each $z \in Z$ and $k \in \mathbf{N}$, we can find by Lemma 1 a number $0 < \delta_{z,k} < \min(\xi, 1/k)$ such that

$$\left| \frac{p(y+kv)-p(y)}{k} - \frac{p(y)-p(y-hv)}{h} \right| < c\gamma, \quad (8)$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in B(y_{z,k}, \delta_{z,k})$, $y + kv \in B(y_{z,k}, \delta_{z,k})$. Now set

$$q := \sum_{z \in Z, k \in \mathbf{N}} f_{z,k}, \qquad \text{where} \qquad f_{z,k} := c(-1)^k F_{y_{z,k},\delta_{z,k}},$$

and $F_{y_{z,k},\delta_{z,k}}$ is defined as in Lemma 3. Obviously,

$$\operatorname{supp} f_{z,k} \subset B(y_{z,k}, \delta_{z,k}) \subset B(y_{z,k}, \xi) \subset B(z, 2\delta/9).$$

Consequently,

$$\operatorname{dist}(\operatorname{supp} f_{z,k}, \operatorname{supp} f_{z,l}) > \xi, \quad \text{whenever} \quad z \in Z,\ k \ne l, \text{ and} \quad (9)$$

$$\operatorname{dist}(\operatorname{supp} f_{z,k}, \operatorname{supp} f_{\tilde z,l}) > \frac{\delta}{2} - \frac{4\delta}{9} = \frac{\delta}{18}, \quad \text{if} \quad z, \tilde z \in Z, z \ne \tilde z,\ k, l \in \mathbf{N}. \quad (10)$$

Lemma 3 immediately implies that each function $f_{z,k}$ is C^1 and c-Lipschitz. Since $\operatorname{diam}(\operatorname{supp} f_{z,k}) \leq 1$, clearly $\|f_{z,k}\| \leq c$. So, using (9) and (10), we easily obtain (i). (For the proof that q is c-Lipschitz, we can use, e.g., Federer, 1969, 2.2.7).

To prove (ii), suppose that $x \in X$ and $\eta > 0$ are given. By maximality of Z, we can choose $z \in Z$ such that $\|x - z\| < \delta/2$. Find an odd $k \in \mathbf{N}$ with $2/k < \eta$ and set $y_1 := y_{z,k}$ and $y_2 := y_{z,k+1}$. By Lemma 3, we can choose a unit vector $v_1 \in X$, $0 < h_1 < \delta_{z,k}$, $0 < k_1 < \delta_{z,k}$ such that (6) holds with $\tau := \delta_{z,k}$, $y := y_1$, $v := v_1$, $h := h_1$, and $k := k_1$. Since $q = f_{z,k} = -cF_{y_{z,k},\delta_{z,k}}$ on $B(y_{z,k}, \delta_{z,k})$, we obtain

$$\frac{q(y_1 + k_1 v_1) - q(y_1)}{k_1} - \frac{q(y_1) - q(y_1 - h_1 v_1)}{h_1} > 2c\gamma\,. \tag{11}$$

Using (11) and (8), we easily obtain

$$\frac{s(y_1 + k_1 v_1) - s(y_1)}{k_1} - \frac{s(y_1) - s(y_1 - h_1 v_1)}{h_1} > 2c\gamma - c\gamma = c\gamma.$$

Since the points $y_1 + h_1 v_1$, $y_1 + k_1 v_1$ belong to $B(z, 2\delta/9) \subset B(x, \delta)$, and $h_1 + k_1 < 2\delta_{z,k} < 2/k < \eta$, we see that y_1, v_1, h_1 and k_1 have all the desired properties. The existence of the desired v_2, h_2, and k_2 follows quite analogously. ∎

THEOREM 1 *Let X be an infinite-dimensional Banach space which admits an equivalent Fréchet smooth norm (or, more generally, a C^1 Lipschitz bump function). Then there exists a C^1 Lipschitz function f on X such that, on any open ball, f is neither strongly paraconvex nor semiconcave.*

Proof. Let F and γ be as in the Setting. Using Lemma 4, it is easy to construct a sequence $(f_n)_{n \in \mathbf{N}}$ of C^1 functions on X such that:

(i) $\|f_k\| \leq 4^{-k}\gamma^k$ and f_k is Lipschitz with constant $4^{-k}\gamma^k$ for each $k \geq 1$.

(ii) If $n \geq 2$, then, denoting $s_n := \sum_{k=1}^{n} f_k$, for each $x \in X$ and $\eta > 0$ there exist points y_1, y_2 in $B(x, 1/n)$, unit vectors v_1, v_2 in X, and positive real numbers h_1, k_1, h_2, k_2, such that $y_i - h_i v_i \in B(x, 1/n)$, $y_i + k_i v_i \in B(x, 1/n)$, $h_i + k_i < \eta$ $(i = 1, 2)$, and

$$\begin{gathered}\frac{s_n(y_1 + k_1 v_1) - s_n(y_1)}{k_1} - \frac{s_n(y_1) - s_n(y_1 - h_1 v_1)}{h_1} > 4^{-n}\gamma^{n+1} \quad \text{and} \\ \frac{s_n(y_2 + k_2 v_2) - s_n(y_2)}{k_2} - \frac{s_n(y_2) - s_n(y_2 - h_2 v_2)}{h_2} < -4^{-n}\gamma^{n+1}.\end{gathered} \tag{12}$$

Indeed, it is sufficient to put $f_1 := 0$ and, having constructed $f_1, \dots, f_{n-1}$ $(n \geq 2)$, then set $f_n := q(\sum_{k=1}^{n-1} f_k, 4^{-n}\gamma^n, 1/n)$, where q is as in Lemma 4.

Now set $f := \sum_{k=1}^{\infty} f_k$. Since $\gamma \leq 1$, the condition (i) easily implies that f is Lipschitz and C^1 on X. Now suppose that f is strongly paraconvex on a ball $B(x,\omega)$. Choose $n \in \mathbf{N}$ with $1/n < \omega$. By Remark 1(ii), f is uniformly approximately convex on $B(x,\omega)$. So, by Lemma 2, we can choose $\eta > 0$ such that

$$\frac{f(y+kv)-f(y)}{k} - \frac{f(y)-f(y-hv)}{h} > -(1/3)4^{-n}\gamma^{n+1}, \tag{13}$$

whenever $\|v\| = 1$, $k > 0$, $h > 0$, $y - hv \in B(x,\omega)$, $y + kv \in B(x,\omega)$, and $h+k < \eta$. By (ii), we find $y_2 \in B(x,1/n)$, a unit vector v_2, and positive numbers h_2, k_2, such that $y_2 - h_2v_2 \in B(x,1/n)$, $y_2 + k_2v_2 \in B(x,1/n)$, $h_2 + k_2 < \eta$, and

$$\frac{s_n(y_2+k_2v_2)-s_n(y_2)}{k_2} - \frac{s_n(y_2)-s_n(y_2-h_2v_2)}{h_2} < -4^{-n}\gamma^{n+1}.$$

Set $r_n := \sum_{k=n+1}^{\infty} f_k$. By (i), we obtain that r_n is Lipschitz with constant

$$\sum_{k=n+1}^{\infty} 4^{-k}\gamma^k = 4^{-(n+1)}\gamma^{n+1}\frac{1}{1-\gamma/4} \le \frac{1}{3}4^{-n}\gamma^{n+1}.$$

Consequently,

$$\left|\frac{r_n(y_2+k_2v_2)-r_n(y_2)}{k_2} - \frac{r_n(y_2)-r_n(y_2-h_2v_2)}{h_2}\right| \le \frac{2}{3}4^{-n}\gamma^{n+1}.$$

So, since $f = s_n + r_n$, we obtain

$$\frac{f(y_2+k_2v_2)-f(y_2)}{k_2} - \frac{f(y_2)-f(y_2-h_2v_2)}{h_2} < -(1/3)4^{-n}\gamma^{n+1},$$

which is a contradiction with (13).

Quite analogously (using now the first inequality of (12)) it is possible to prove that $-f$ is not uniformly approximately convex on $B(x,\omega)$. So, by Remark 1, f is not semiconcave on $B(x,\omega)$. ■

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