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Hemivariational inequalities governed by the p-Laplacian - Neumann problem

by

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Abstract: A hemivariational inequality involving p-Laplacian is studied under the hypothesis that the nonlinear part fulfills the unilateral growth condition. The existence of solutions for problems with Neumann boundary conditions is established by making use of Chang's version of the critical point theory for nonsmooth locally Lipschitz functionals, combined with the Galerkin method. The approach is based on the recession technique introduced previously by the author.

Keywords: Neumann problem, noncoercive hemivariational inequality, unilateral growth condition, critical point theory, locally Lipschitz functional.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The problem under consideration is as follows: Find $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p u(x) & \in & -\partial j(x, u(x)) \text{ a.e. on } \Omega \\ \frac{\partial u}{\partial n}|_{\partial\Omega} & = & 0, \quad 2 \le p < \infty. \end{cases} \tag{1}$$

where $-\Delta_p u := -\operatorname{div}\big(|Du|^{p-2}Du\big)$ stands for the p-Laplacian operator. By $\partial j(x,u)$ we denote the generalized gradient of Clarke (Clarke, 1983) of a locally Lipschitz $\mathbb{R} \ni \xi \mapsto j(x,\xi)$ (for a.e. $x \in \Omega$). For the right hand side of (1) we suppose only that it satisfies the unilateral growth condition (Naniewicz, 1994)

$$j^0(x;\xi,-\xi) \le \kappa(1+|\xi|^q), \quad \forall\, \xi \in \mathbb{R}, \text{ for a.e. } x \in \Omega, \; q < p^\star, \; p^\star = \frac{N_p}{N-p}.$$

Thus, the problem to be studied involves nonlinear, nonconvex function $j(\cdot, u)$ which is not summable for every $u \in W_0^{1,p}(\Omega)$ and consequently, the corresponding energy functional $\mathcal{R}(u) = \frac{1}{p}\|Du\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega j(x, u(x))\, dx$ has no longer the whole space $W^{1,p}(\Omega)$ as its effective domain. The direct use of the critical point theory developed for locally Lipschitz functionals (Chang, 1981) is therefore not available. We use the Galerkin method and solve the discretized problems in finite dimensional subspaces of $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ by making use of the recession technique for semicoercive problems introduced in Naniewicz (2003) and then pass to the limit to get a solution.

The class of hemivariational inequalities considered in the paper can be referred to as variational problems with discontinuities, widely studied recently. For the Neumann problem involving p-Laplacian we refer to Papalini (2002) where under the classical growth condition the existence and multiplicity of solutions have been established. Dirichlet problems driven by the p-Laplacian can be found in Gasiński & Papageorgiou (2001a, b), Papageorgiou & Papalini (2000), Halidias & Naniewicz (2004), Naniewicz (2004) and the references therein. See also Arcoya & Orsina (1997), Bouchala & Drabek (2000), Anane & Gossez (1990) for such problems involving smooth potentials.

The notion of hemivariational inequalities has been first introduced in the early eighties with the works of P.D. Panagiotopoulos (Panagiotopoulos 1981, 1983). The main reason for its birth was the need for description of important problems in physics and engineering, where nonmonotone, multivalued boundary or interface conditions occur, or where some nonmonotone, multivalued relations between stress and strain, or reaction and displacement have to be taken into account. The theory of hemivariational inequalities (as the generalization of variational inequalities, see Duvaut & Lions, 1972) has been proved to be very useful in understanding of many problems of mechanics and engineering involving nonconvex, nonsmooth energy functionals. For the general study of hemivariational inequalities in both scalar and vector-valued function spaces the reader is referred to Motreanu & Naniewicz (1996, 2001, 2002, 2003), Motreanu & Panagiotopoulos (1995, 1996, 1999), Goeleven, Motreanu & Panagiotopoulos (1997), Naniewicz (1995, 1997), Naniewicz & Panagiotopoulos (1995), Panagiotopoulos (1985, 1993), Radulescu (1993), Gasiński & Papageorgiou (2005) and the references quoted there.

2. Mathematical background

Let us recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the generalized gradient of Clarke (Clarke, 1983).

Let Y be a subset of a Banach space X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$|f(y) - f(x)| \le K\|y - x\|_X$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^0(x;v)$, is defined as follows:

$$f^0(x;v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+tv) - f(y)}{t}$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^0(x;v)$ is finite, positively homogeneous, subadditive and satisfies the conditions $|f^0(x;v)| \leq K\|v\|_X$ and $f^0(x;-v) = (-f)^0(x;v)$. Now we are ready to introduce the generalized gradient $\partial f(x)$ defined by Clarke (1983):

$$\partial f(x) = \{w \in X^* : f^0(x;v) \geq \langle w, v \rangle_X \text{ for all } v \in X\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are as follows:

(a) $\partial f(x)$ is a nonempty, convex, weakly-star compact subset of $X^\star$ and $\|w\|_{X^\star} \leq K$ for every w in $\partial f(x)$.

(b) For every v in X, one has

$$f^0(x;v) = \max\{\langle w, v \rangle : w \in \partial f(x)\}.$$

(c) If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (P.S.)-condition introduced by Chang (Chang, 1981):

Definition 1 *A locally Lipschitz function f is said to satisfy the Palais - Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and*

$$\lambda(x_n) = \min_{w \in \partial f(x_n)} \|w\|_{X^\star} \to 0$$

possesses a convergent subsequence.

Let us mention some facts about the first nonzero eigenvalue of the p-Laplacian.

Consider the first nonzero eigenvalue λ_1 of $(-\Delta_p, W^{1,p}(\Omega))$ for the p-Laplacian with homogeneous Neumann boundary condition. It is well known (see Papalini, 2002) that $\lambda_1 > 0$ and it is characterized by (Rayleigh quotient):

$$\lambda_1 := \inf\left\{\frac{\|Dw\|^p_{L^p(\Omega;\mathbb{R}^N)}}{\|w\|^p_{L^p(\Omega)}} : w \in \mathcal{W},\ w \neq 0\right\},$$

where $\mathcal{W} := \{v \in W^{1,p}(\Omega) \colon \int_\Omega v(x)\,dx = 0\}$. Thus, for any $v \in \mathcal{W}$ we have

$$\|Dv\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1 \|v\|^p_{L^p(\Omega)},$$

which means that the norms $\|D(\cdot)\|_{L^p(\Omega;\mathbb{R}^N)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ are equivalent on $\mathcal{W}$. Moreover, each eigenfunction $w \in \mathcal{W}$ corresponding to λ_1 has the properties that $\|Dw\|^p_{L^p(\Omega;\mathbb{R}^N)} = \lambda_1 \|w\|^p_{L^p(\Omega)}$ and it is a solution of the problem

$$\begin{cases} -\Delta_p w &= \lambda_1 |w|^{p-2} w \ \text{ a.e. on } \Omega \\ \frac{\partial w}{\partial n}|_{\partial\Omega} &= 0, \quad 2 \leq p < \infty. \end{cases} \tag{2}$$

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on a Banach space. A point $x \in X$ is said to be a critical point if $0 \in \partial f(x)$ and $c = f(x)$ is called a critical value of f.

The results below characterize conditions under which the existence of critical points follows. They are due to Chang (Chang, 1981) and extend to a nonsmooth setting the well known theorems of Ambrosetti and Rabinowitz. They will be used to obtain the main results of the paper.

THEOREM 1 *If a locally Lipschitz function $f : X \to \mathbb{R}$ on the reflexive Banach space X satisfies the (PS)-condition and there exist a positive constant $\rho > 0$ and $e \in X$ with $\|e\| > \rho$ such that*

$$\max\{f(0), f(e)\} < \inf_{\|x\|=\rho} \{f(x)\},$$

then f has a critical point $u \in X$ with its critical value $c = f(u)$ characterized by

$$c = \inf_{g\in G} \max_{t\in[0,1]} f(g(t))$$

where

$$G = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}.$$

THEOREM 2 *Suppose that a reflexive Banach space X can be represented as $X = X_1 \oplus X_2$ with a finite dimensional X_1. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the (PS)-condition and there exist positive constants $b_1 < b_2$ and a neighborhood N of 0 in X_1 such that*

$$\inf_{x\in X_2} \{f(x)\} \geq b_2,$$
$$\inf_{x\in\partial N} \{f(x)\} \leq b_1,$$

∂N being the boundary of N. Then f has a critical point.

3. Auxiliary results

It is well known that not vanishing constant functions on Ω are eigenfunctions corresponding to the first eigenvalue $\lambda_0 = 0$ of the p-Laplacian in $W^{1,p}(\Omega)$. Let us denote by $V_0 = \{s\theta\}_{s\in\mathbb{R}}$ the one-dimensional subspace of $W^{1,p}(\Omega)$ spanned by a constant function $\theta \in W^{1,p}(\Omega)$ normalized by $\theta > 0$ and $\|\theta\|_{W^{1,p}(\Omega)} = 1$ ($\theta(x) = \theta_0 := 1/|\Omega|^{\frac{1}{p}}$ for a.e. $x \in \Omega$). Concerning the first nonzero eigenvalue we know that (see Papalini, 2002)

$$\lambda_1 := \inf\left\{\frac{\|Dw\|^p_{L^p(\Omega;\mathbb{R}^N)}}{\|w\|^p_{L^p(\Omega)}} : w \in \mathcal{W},\ w \neq 0\right\},$$

is positive, i.e. $\lambda_1 > 0$, where $\mathcal{W} := \left\{w \in W^{1,p}(\Omega) \colon \int_\Omega w(x)\,dx = 0\right\}$, and if $w \in \mathcal{W}$ has the properties that $\|w\|^p_{L^p(\Omega)} = 1$ and $\|Dw\|^p_{L^p(\Omega;\mathbb{R}^N)} = \lambda_1$ then w is the normalized eigenfunction of the problem

$$\begin{cases} -\Delta_p w = \lambda_1 |w|^{p-2}w \ \text{ a.e. in } \Omega \\ \frac{\partial w}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

Thus, for any $u \in W^{1,p}(\Omega)$ we have the decomposition $u = e\overline{\theta} + \widehat{u}$ with $e = \left|\int_\Omega u(x)\,dx\right| \geq 0$, $\widehat{u} \in \mathcal{W}$ and $\overline{\theta} \in \{\pm\theta\} \subset V_0$, for which

$$\|D\widehat{u}\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1 \|\widehat{u}\|^p_{L^p(\Omega)}. \tag{4}$$

Hence the equivalence of the norms $\|D(\cdot)\|_{L^p(\Omega;\mathbb{R}^N)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ on $\mathcal{W}$ results.

Lemma 1 *Assume that*

$(H1)$ $j(\cdot,0) \in L^1(\Omega)$ *and* $j(x,\cdot)$ *is Lipschitz continuous on the bounded subsets of* $\mathbb{R}$ *uniformly with respect to* $x \in \Omega$, *i.e.,* $\forall\, r > 0\ \exists\, K_r > 0$ *such that* $\forall\, |y_1|, |y_2| \leq r$,

$$|j(x,y_1) - j(x,y_2| \leq K_r|y_1 - y_2|, \quad \textit{for a.e. } x \in \Omega;$$

$(H2)$ *One of the two conditions below holds (the Ambrosetti-Rabinowitz type conditions):*

(i) There exist $\mu > p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ *and a constant* $k \geq 0$ *such that*

$$\mu j(x,\xi) - j^0(x,\xi;\xi) \geq -a(x) - k|\xi|^\sigma, \ \ \forall\, \xi \in \mathbb{R} \textit{ and for a.e. } x \in \Omega;$$

(ii) There exist $0 < \nu < p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ *and a constant* $k \geq 0$ *such that*

$$-\nu j(x,\xi) - j^0(x,\xi;-\xi) \geq -a(x) - k|\xi|^\sigma, \ \forall\, \xi \in \mathbb{R} \textit{ and for a.e. } x \in \Omega;$$

$(H3)$ *Suppose that* $J^\infty(\overline{\theta}) > 0$ *for each* $\overline{\theta} \in \{\pm\theta\}$, *where*

$$J^\infty(\overline{\theta}) := \liminf_{\substack{t\to+\infty \\ \eta \xrightarrow[L^p(\Omega)]{} \overline{\theta}}} \int_\Omega -j^0\big(x, t\eta(x); -\overline{\theta}(x)\big)\,dx, \quad \overline{\theta} \in \{\pm\theta\},$$

is the recession function of nonconvex, nonsmooth $J(\cdot) = \int_\Omega j(x,\cdot)\,dx$ *as introduced in Naniewicz (2003) to study semicoercive problems (see also Goeleven & Théra, 1995; Baiocchi, Buttazzo, Gastaldi & Tomarelli, 1988). Moreover, suppose that for a sequence* $\{u_n\} \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *there exists* $\varepsilon_n \searrow 0$ *such that the conditions below are fulfilled:*

$$\int_\Omega |Du_n(x)|^{p-2}\big\langle Du_n(x), Dv(x) - Du_n(x)\big\rangle_{\mathbb{R}^N}\,dx + \int_\Omega j^0\big(x, u_n(x); v(x) - u_n(x)\big)\,dx \geq -\varepsilon_n\|v - u_n\|_{W^{1,p}(\Omega)}, \quad \forall\, v \in \mathrm{Lin}(\{u_n,\theta\}), \tag{5}$$

and

$$\left|\frac{1}{p}\int_\Omega |Du_n(x)|^p\,dx + \int_\Omega j\big(x, u_n(x)\big)\,dx\right| \leq C, \quad C > 0, \tag{6}$$

where $\mathrm{Lin}(\{u_n,\theta\})$ *is the linear subspace of* $W^{1,p}(\Omega)$ *spanned by* $\{\theta, u_n\}$. *Then the sequence* $\{u_n\}$ *is bounded in* $W^{1,p}(\Omega)$, *i.e. there exists* $M > 0$ *such that*

$$\|u_n\|_{W^{1,p}(\Omega)} \leq M. \tag{7}$$

Proof. Suppose, on the contrary, that the claim is not true, i.e. there exists a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$, for which (5) and (6) hold. Under $(H2)_{(i)}$, combining (6) multiplied by $\mu > p$ with (5) (with $v = 2u_n$ substituted) we get

$$\mu C + \varepsilon_n\|u_n\|_{W^{1,p}(\Omega)} \geq \tfrac{\mu-p}{p}\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega \Big(\mu j(u_n) - j^0(u_n; u_n)\Big)\,dx. \tag{8}$$

From (4) the decomposition results: $u_n = e_n\theta_n + \widehat{u}_n$, where $e_n = \left|\int_\Omega u_n\,dx\right|$, $\widehat{u}_n \in \mathcal{W}$, $\theta_n \in \{\pm\theta\}$, $\|\theta\|_{W^{1,p}(\Omega)} = 1$, such that

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1\,\|\widehat{u}_n\|^p_{L^p(\Omega)}. \tag{9}$$

Since $Du_n = D\widehat{u}_n$, by $(H2)_{(i)}$ we have

$$\mu C + \varepsilon_n\|u_n\|_{W^{1,p}(\Omega)} \geq \tfrac{\mu-p}{p}\|Du_n\|^p_{W^{1,p}(\Omega)} - c_1\|u_n\|^\sigma_{L^p(\Omega)} - \|a\|_{L^1(\Omega)} \tag{10}$$

$$\geq c\tfrac{\mu-p}{p}\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)} - c_2\|u_n\|^\sigma_{W^{1,p}(\Omega)} - \|a\|_{L^1(\Omega)}. \tag{11}$$

Hence

$$\mu C+\varepsilon_n(\|\widehat{u}_n\|_{W^{1,p}(\Omega)}+e_n)\geq c\frac{\mu-p}{p}\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)}-c_3\|\widehat{u}_n\|^\sigma_{W^{1,p}(\Omega)}-c_4e_n^\sigma-\|a\|_{L^1(\Omega)}. \tag{12}$$

Thus, it follows that $e_n \to \infty$ because, otherwise, we would get the boundedness of $\{\widehat{u}_n\}$ and consequently, the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$, contrary to our supposition. Dividing (12) by e_n we obtain the estimate

$$\begin{aligned}\tfrac{\mu C+\|a\|_{L^1(\Omega)}}{e_n}+\varepsilon_n(\|\tfrac{\widehat{u}_n}{e_n}\|_{W^{1,p}(\Omega)}+1)\geq e_n^{p-1}c\tfrac{\mu-p}{p}\|\tfrac{\widehat{u}_n}{e_n}\|^p_{W^{1,p}(\Omega)}\\ -c_3e_n^{\sigma-1}\|\tfrac{\widehat{u}_n}{e_n}\|^\sigma_{W^{1,p}(\Omega)}-c_4e_n^{\sigma-1}\end{aligned} \tag{13}$$

which, in view of $e_n \to \infty$ and $\sigma < p$, allows for the conclusion that

$$\|\tfrac{\widehat{u}_n}{e_n}\|_{W^{1,p}(\Omega)} \to 0. \tag{14}$$

Now, let us turn back to (5). By passing to a subsequence one can suppose also that $\theta_n = \theta$ (or $\theta_n = -\theta$). Thus, substituting $v = \widehat{u}_n$ into (5) yields

$$\begin{aligned}e_n^p\int_\Omega |D(\tfrac{\widehat{u}_n}{e_n})+D\theta)|^{p-2}\big\langle D(\tfrac{\widehat{u}_n}{e_n})+D\theta),-D\theta\big\rangle_{\mathbb{R}^N}\,dx\\ +e_n\int_\Omega j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n}+\theta);-\theta\big)\,dx\geq -\varepsilon_n e_n.\end{aligned}$$

In view of $D\theta = 0$ this gives

$$\int_\Omega j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n}+\theta);-\theta\big)\,dx \geq -\varepsilon_n \tag{15}$$

and consequently

$$J^\infty(\theta)\leq \limsup_{n\to\infty}\int_\Omega -j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n}+\theta);-\theta\big)\,dx\leq 0, \tag{16}$$

the contradiction with $(H3)$.

Under $(H2)_{(ii)}$, combining (6) multiplied by $\nu < p$ with (5) (with $v = 0$ substituted), we arrive at

$$\nu C+\varepsilon_n\|u_n\|_{W^{1,p}(\Omega)}\geq \tfrac{p-\nu}{p}\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)}+\int_\Omega\big(-\nu j(u_n)-j^0(u_n;-u_n)\big)\,dx.$$

Now we can proceed as previously to establish the result. The proof of Lemma 1 is complete. ■

Lemma 2 *Assume that* $(H1)$ *and the hypotheses below hold:*
$(H4)$ *The unilateral growth condition (Naniewicz, 1994):*

There exist $1 \le q < p^\star$, $p^\star = \frac{Np}{N-p}$, *and a constant* $\kappa \ge 0$ *such that*

$$j^0(x,\xi;-\xi) \le \kappa(1+|\xi|^q), \quad \forall\, \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

$(H5)$ *Uniformly for a.e.* $x \in \Omega$,

$$\liminf_{\xi \to 0} \frac{pj(x,\xi)}{|\xi|^p} \ge \varphi(x) \ge 0,$$

with $\varphi(x) \in L^\infty(\Omega)$, $\varphi(x) > 0$ *on a set of positive measure of* Ω *and* $\varphi(x) < \lambda_1$ *for a.e.* $x \in \Omega$.

Then there exists $\rho > 0$ *such that*

$$\mathcal{R}(u) := \tfrac{1}{p}\|Du\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega j(u)\,dx \ge \eta, \quad \eta = \text{const} > 0, \tag{17}$$

is valid for any $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *with* $\|u\|_{W^{1,p}(\Omega)} = \rho$.

Proof. Suppose the assertion is not true. Thus, there exist sequences $\{u_n\} \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\rho_n \searrow 0$ such that $\|u_n\|_{W^{1,p}(\Omega)} = \rho_n$ and $\mathcal{R}(u_n) \le \rho_n^{p+1}$. So we have

$$\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega pj(u_n)\,dx \le p\rho_n^{p+1}. \tag{18}$$

Further, from $(H5)$ it follows that for any $\varepsilon > 0$ one can find $\delta > 0$ such that

$$pj(x,\xi) \ge \varphi(x)|\xi|^p - \varepsilon|\xi|^p, \quad |\xi| \le \delta, \text{ uniformly for all } x \in \Omega.$$

Moreover, $(H4)$ allows to conclude that (see Lemma 2.1, pp. 119-120, Naniewicz, 1997):

$$j(x,\xi) \ge -\kappa_1(1+|\xi|^q), \quad \forall\, \xi \in \mathbb{R},\ x \in \Omega; \quad \kappa_1 = \text{const} > 0.$$

Thus, it is easy to see that

$$pj(x,\xi) \ge (\varphi(x)-\varepsilon)|\xi|^p - \gamma|\xi|^q, \quad \forall\, \xi \in \mathbb{R},\ x \in \Omega,$$

for some positive constant $\gamma = \gamma(\delta) > 0$. Then, by (18) it follows that

$$\begin{aligned}
&\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega (\varphi(x)-\varepsilon)|u_n(x)|^p\,dx \\
&\le p\rho_n^{p+1} + \gamma\int_\Omega |u_n(x)|^q\,dx \\
&\le p\rho_n^{p+1} + \gamma_1\|u_n\|^q_{W^{1,p}(\Omega)}.
\end{aligned}$$

Taking into account that $u_n = \widehat{u}_n + e_n\theta_n$, $\widehat{u}_n \in W$, $e_n = \left|\int_\Omega u_n\,dx\right|$, $\theta_n \in \{\pm\theta\}$ and $\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1\|\widehat{u}_n\|^p_{L^p(\Omega)}$ we obtain

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|\widehat{u}_n\|^p_{L^p(\Omega)} + \int_\Omega (\varphi(x)-\varepsilon)|\widehat{u}_n(x)+e_n\theta_n|^p\,dx + \int_\Omega \lambda_1|\widehat{u}_n(x)|^p\,dx \leq p\rho_n^{p+1} + \gamma_1\rho_n^q.$$

Hence

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|\widehat{u}_n\|^p_{L^p(\Omega)} + \int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|\widehat{u}_n(x)+e_n\theta_n|^p\,dx$$
$$+\int_\Omega \lambda_1|\widehat{u}_n(x)|^p\,dx + \int_{\{\varphi>\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|\widehat{u}_n(x)+e_n\theta_n|^p\,dx \leq p\rho_n^{p+1} + \gamma_1\rho_n^q.$$

By the inequality $|a\pm b|^p \geq \frac{1}{2^{p-1}}|a|^p - |b|^p$, $a,b\in\mathbb{R}$, it follows that

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|\widehat{u}_n\|^p_{L^p(\Omega)} + \int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|\widehat{u}_n(x)+e_n\theta_n|^p\,dx$$
$$+\int_{\{\varphi>\varepsilon\}} \Big(\big(\varphi(x)-\varepsilon\big)\big(\tfrac{e_n^p}{2^{p-1}}|\theta|^p - |\widehat{u}_n|^p\big) + \lambda_1|\widehat{u}_n|^p\Big)\,dx \leq p\rho_n^{p+1} + \gamma_1\rho_n^q.$$

This can be written as

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|\widehat{u}_n\|^p_{L^p(\Omega)} + \int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|\widehat{u}_n(x)+e_n\theta_n|^p\,dx$$
$$+\frac{e_n^p}{2^{p-1}|\Omega|}\int_{\{\varphi>\varepsilon\}} \big(\varphi(x)-\varepsilon\big)\,dx + \int_{\{\varphi>\varepsilon\}} \big(\lambda_1-\varphi(x)+\varepsilon\big)|\widehat{u}_n|^p\,dx \leq p\rho_n^{p+1} + \gamma_1\rho_n^q.$$

Let us set $y_n = \frac{1}{\widehat{\rho}_n}\widehat{u}_n$, where $\widehat{\rho}_n := \|D\widehat{u}_n\|_{L^p(\Omega;\mathbb{R}^N)}$. Dividing this inequality by $\widehat{\rho}_n^p$ yields

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + \int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|y_n(x)+\tfrac{e_n}{\widehat{\rho}_n}\theta_n|^p\,dx$$
$$+\big(\tfrac{e_n}{\widehat{\rho}_n}\big)^p\frac{1}{2^{p-1}|\Omega|}\int_{\{\varphi>\varepsilon\}} \big(\varphi(x)-\varepsilon\big)\,dx + \int_{\{\varphi>\varepsilon\}} \big(\lambda_1-\varphi(x)+\varepsilon\big)|y_n|^p\,dx$$
$$\leq \big(\tfrac{\rho_n}{\widehat{\rho}_n}\big)^p\big(p\rho_n+\gamma_1\rho_n^{q-p}\big).$$

By making use of the inequality $|a_1+a_2|^p \leq 2^{p-1}(|a_1|^p+|a_2|^p)$, $a_1,a_2\in\mathbb{R}$, we obtain

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + 2^{p-1}\int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)|y_n(x)|^p\,dx$$
$$+\big(\tfrac{e_n}{\widehat{\rho}_n}\big)^p\frac{2^{p-1}}{|\Omega|}\int_{\{\varphi<\varepsilon\}} \big(\varphi(x)-\varepsilon\big)\,dx + \big(\tfrac{e_n}{\widehat{\rho}_n}\big)^p\frac{1}{2^{p-1}|\Omega|}\int_{\{\varphi>\varepsilon\}} \big(\varphi(x)-\varepsilon\big)\,dx$$
$$+\int_{\{\varphi>\varepsilon\}} \big(\lambda_1-\varphi(x)+\varepsilon\big)|y_n|^p\,dx \leq \big(\tfrac{\rho_n}{\widehat{\rho}_n}\big)^p\big(p\rho_n+\gamma_1\rho_n^{q-p}\big).$$

Further, notice that

$$\Big(\frac{\rho_n}{\widehat{\rho}_n}\Big)^p = 1 + \Big(\frac{\|u_n\|_{L^p(\Omega)}}{\widehat{\rho}_n}\Big)^p \le 1 + \Big(\frac{\|\widehat{u}_n\|_{L^p(\Omega)}}{\|D\widehat{u}_n\|_{L^p(\Omega;\mathbb{R}^N)}}\Big)^p + \Big(\frac{e_n}{\widehat{\rho}_n}\Big)^p \|\theta_n\|^p_{L^p(\Omega)}$$
$$\le 1 + \frac{2^{p-1}}{\lambda_1} + 2^{p-1}\Big(\frac{e_n}{\widehat{\rho}_n}\Big)^p.$$

Thus, we arrive at the estimate

$$\begin{aligned}
&\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + 2^{p-1}\int_{\{\varphi<\varepsilon\}}\big(\varphi(x)-\varepsilon\big)|y_n(x)|^p\,dx\\
&+\big(\tfrac{e_n}{\widehat{\rho}_n}\big)^p\Bigg(\tfrac{2^{p-1}}{|\Omega|}\int_{\{\varphi<\varepsilon\}}\big(\varphi(x)-\varepsilon\big)\,dx + \tfrac{1}{2^{p-1}|\Omega|}\int_{\{\varphi>\varepsilon\}}\big(\varphi(x)-\varepsilon\big)\,dx\\
&-2^{p-1}(p\rho_n+\gamma_1\rho_n^{q-p})\Bigg) + \int_{\{\varphi>\varepsilon\}}\big(\lambda_1-\varphi(x)+\varepsilon\big)|y_n|^p\,dx\\
&\le (1+\tfrac{2^{p-1}}{\lambda_1})\big(p\rho_n+\gamma_1\rho_n^{q-p}\big).
\end{aligned}$$

By $(H5)$ and $\rho_n \to 0$ it follows that for sufficiently large n and small $\varepsilon > 0$,

$$\tfrac{2^{p-1}}{|\Omega|}\int_{\{\varphi<\varepsilon\}}\big(\varphi(x)-\varepsilon\big)\,dx + \tfrac{1}{2^{p-1}|\Omega|}\int_{\{\varphi>\varepsilon\}}\big(\varphi(x)-\varepsilon\big)\,dx - 2^{p-1}(p\rho_n - \gamma_1\rho_n^{q-p}) \ge \gamma_0, \qquad \gamma_0 > 0,$$

which leads to

$$\begin{aligned}
&\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} + 2^{p-1}\int_{\{\varphi<\varepsilon\}}\big(\varphi(x)-\varepsilon\big)|y_n(x)|^p\,dx\\
&+\big(\tfrac{e_n}{\widehat{\rho}_n}\big)^p\gamma_0 + \int_{\{\varphi>\varepsilon\}}\big(\lambda_1-\varphi(x)+\varepsilon\big)|y_n|^p\,dx \le (1+\tfrac{1}{\lambda_1})\big(p\rho_n+\gamma_1\rho_n^{q-p}\big).
\end{aligned} \tag{19}$$

Thus, $\{\frac{e_n}{\widehat{\rho}_n}\}$ is bounded. Further, since the norm $\|D(\cdot)\|_{L^p(\Omega;\mathbb{R}^N)}$ is equivalent to the usual norm $\|\cdot\|_{W^{1,p}(\Omega)}$ on $\mathcal{W} = \big\{v \in W^{1,p}(\Omega)\colon \int_\Omega v\,dx = 0\big\}$, $y_n \in \mathcal{W}$ and $\|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)} = 1$, we get the boundedness of $\{y_n\}$ in $W^{1,p}(\Omega)$. This, together with the boundedness of $\{\frac{e_n}{\widehat{\rho}_n}\}$, allows to conclude that for some $y \in W^{1,p}(\Omega)$, $e_0 \in \mathbb{R}_+$ and $\theta_0 \in \{\pm\theta\}$ a subsequence can be extracted (again denoted by the same symbol) such that $y_n \rightharpoonup y$ weakly in $W^{1,p}(\Omega)$, $y_n \to y$ strongly in $L^p(\Omega)$ (the Rellich theorem), $\frac{e_n}{\widehat{\rho}_n} \to e_0$ and $\theta_n = \theta_0$. Passing to the limit with $n \to \infty$ in (19) and taking into account the weak lower semicontinuity of the norm leads to the inequality

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y\|^p_{L^p(\Omega)} + 2^{p-1}\int_{\{\varphi<\varepsilon\}}\big(\varphi(x)-\varepsilon\big)|y(x)|^p\,dx$$
$$+e_0^p\gamma_0 + \int_{\{\varphi>\varepsilon\}}\big(\lambda_1-\varphi(x)+\varepsilon\big)|y|^p\,dx \le 0,$$

which is valid for an arbitrary $\varepsilon > 0$. Therefore we get

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y\|^p_{L^p(\Omega)} + e_0^p\gamma_0 + \int_\Omega(\lambda_1-\varphi)|y|^p\,dx \le 0. \tag{20}$$

Using the quotient characterization of λ_1 and (H5) we arrive at $e_0 = 0$ and

$$\|Dy\|^p_{L^p(\Omega;\mathbb{R}^N)} = \lambda_1\|y\|^p_{L^p(\Omega)}, \tag{21}$$

$$\int_\Omega\big(\lambda_1-\varphi(x)\big)|y(x)|^p\,dx = 0. \tag{22}$$

Now we show that $y \neq 0$. Indeed, from the results obtained it follows that

$$\|Dy_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \lambda_1\|y_n\|^p_{L^p(\Omega)} \to 0$$

and by the compactness of the imbedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ we get

$$\|y_n\|_{L^p(\Omega)} \to \|y\|_{L^p(\Omega)}.$$

Since $\|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)} = 1$, we arrive at $\|y\|^p_{L^p(\Omega)} = \frac{1}{\lambda_1}$ which establishes the assertion. Finally, in view of $y \neq 0$ the contradiction between (22) and $(H5)$ is clearly seen. The proof of Lemma 2 is complete. ■

Lemma 3 *Assume the hypotheses* $(H1), (H3)$ *and*
$(H4)_1$ *There exists* $1 \le s < p$ *and a constant* $\kappa \ge 0$ *such that*

$$j^0(x,\xi;-\xi) \le \kappa(1+|\xi|^s), \quad \forall\,\xi\in\mathbb{R} \text{ and for a.e. } x\in\Omega.$$

Moreover, suppose that for a sequence $\{u_n\} \subset W^{1,p}(\Omega)\cap L^\infty(\Omega)$ *there exists* $\varepsilon_n \searrow 0$ *such that the condition below is fulfilled:*

$$\int_\Omega |Du_n(x)|^{p-2}\big\langle Du_n(x), Dv(x)-Du_n(x)\big\rangle_{\mathbb{R}^N}\,dx$$
$$+\int_\Omega j^0\big(x,u_n(x);v(x)-u_n(x)\big)\,dx \ge -\varepsilon_n\|v-u_n\|_{W^{1,p}(\Omega)},$$
$$\forall\, v\in \operatorname{Lin}(\{u_n,\theta\}). \tag{23}$$

where $\operatorname{Lin}(\{u_n,\theta\})$ *is the linear subspace of* $W^{1,p}(\Omega)$ *spanned by* $\{\theta, u_n\}$. *Then the sequence* $\{u_n\}$ *is bounded in* $W^{1,p}(\Omega)$, *i.e. there exists* $M > 0$ *such that*

$$\|u_n\|_{W^{1,p}(\Omega)} \le M. \tag{24}$$

Proof. Suppose, on the contrary, that the claim is not true, i.e. there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ with $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$ for which (23) holds. By substituting $v = 0$ into (23) we obtain

$$\varepsilon_n \|u_n\|_{W^{1,p}(\Omega)} \geq \|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \int_\Omega j^0(u_n; -u_n)\, dx. \tag{25}$$

Taking into account the decomposition: $u_n = e_n\theta_n + \widehat{u}_n$, where $e_n = \left|\int_\Omega u_n\, dx\right|$, $\widehat{u}_n \in \mathcal{W}$, $\theta_n \in \{\pm\theta\}$, $\|\theta\|_{W^{1,p}(\Omega)} = 1$, in view of $Du_n = D\widehat{u}_n$ and $(H4)_1$ we have

$$\begin{aligned}\varepsilon_n \|u_n\|_{W^{1,p}(\Omega)} &\geq \|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} - \kappa\|u_n\|^s_{L^s(\Omega)} - \kappa|\Omega| \\ &\geq c\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)} - c_1\|u_n\|^s_{W^{1,p}(\Omega)} - \kappa|\Omega|.\end{aligned} \tag{26}$$

Hence

$$\varepsilon_n(\|\widehat{u}_n\|_{W^{1,p}(\Omega)} + e_n) \geq c\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)} - c_2\|\widehat{u}_n\|^s_{W^{1,p}(\Omega)} - c_3 e_n^s - \kappa|\Omega|. \tag{27}$$

Thus, it follows that $e_n \to \infty$ because, otherwise, we would get the boundedness of $\{\widehat{u}_n\}$ and consequently, the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$, contrary to our supposition. Dividing (27) by e_n leads to the estimate

$$\begin{aligned}\varepsilon_n(\|\tfrac{\widehat{u}_n}{e_n}\|_{W^{1,p}(\Omega)} + 1) \geq e_n^{p-1} c\|\tfrac{\widehat{u}_n}{e_n}\|^p_{W^{1,p}(\Omega)} - c_2 e_n^{s-1}\|\tfrac{\widehat{u}_n}{e_n}\|^s_{W^{1,p}(\Omega)} \\ - c_3 e_n^{s-1} - \tfrac{\kappa|\Omega|}{e_n},\end{aligned} \tag{28}$$

which, in view of $e_n \to \infty$ and $s < p$, allows for the conclusion that

$$\|\tfrac{\widehat{u}_n}{e_n}\|_{W^{1,p}(\Omega)} \to 0. \tag{29}$$

Then, we proceed like in the proof of Lemma 1. ■

Lemma 4 *Assume that* $(H1)$-$(H2)$ *hold. Moreover, let*
$(H6)$ $\int_\Omega j(x,0)\, dx \leq 0$ *and either*

$$\liminf_{s\to+\infty} \int_\Omega j(x,s)\, dx < 0, \tag{30}$$

or

$$\liminf_{s\to-\infty} \int_\Omega j(x,s)\, dx < 0, \tag{31}$$

or there exists $v_0 \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ *such that (Motreanu & Panagiotopoulos, 1999):*

$$\liminf_{s\to+\infty} s^{-\sigma} \int_\Omega j\big(x, sv_0(x)\big)\, dx < \frac{k}{\sigma - \mu}\|v_0\|^\sigma_{L^\sigma(\Omega)} \tag{32}$$

with the positive constants k, μ, σ *entering* $(H2)$.

Then there exists $e \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, $e \neq 0$, *such that*

$$\mathcal{R}(se) \leq 0, \quad \forall\, s \geq 1.$$

Proof. If (30) or (31) is fulfilled then the assertion holds for $e = s_0\theta$, or $e = s_0(-\theta)$ with sufficiently large $s_0 > 0$, respectively.

For the case (32) we follow the lines of Motreanu & Panagiotopoulos (1999). For all $\tau \neq 0$, $\xi \in \mathbb{R}$, the formula below of the generalized gradient (with respect to τ) holds

$$\partial_\tau(\tau^{-\mu} j(x,\tau\xi)) = \tau^{-\mu-1}[-\mu j(x,\tau\xi) + \partial_\xi j(x,\tau\xi)(\tau\xi)],$$

for the constant $\mu > p$ fulfilling (H2). Since the function $\tau \mapsto \tau^{-\mu} j(x,\tau\xi)$ is differentiable a.e. on $\mathbb{R}$, the equality above and a classical property of Clarke's generalized directional derivative imply that

$$t^{-\mu} j(x,t\xi) - j(x,\xi) = \int_1^t \frac{d}{d\tau}(\tau^{-\mu} j(x,\tau\xi))d\tau$$
$$\leq \int_1^t \tau^{-\mu-1}[-\mu j(x,\tau\xi) + j^0(x,\tau\xi;\tau\xi)]d\tau, \quad \forall\, t > 1, \text{ a.e. } x \in \Gamma, \ \xi \in \mathbb{R}.$$

In view of assumption (H2) we infer that

$$t^{-\mu} j(x,t\xi) - j(x,\xi) \leq \int_1^t \tau^{-\mu-1}[a(x) + k\tau^{\sigma}|\xi|^{\sigma}]\, d\tau$$
$$= \left[a(x)\big(-\frac{1}{\mu}t^{-\mu} + \frac{1}{\mu}\big) + k|\xi|^{\sigma}\big(\frac{1}{\sigma-\mu}t^{\sigma-\mu} - \frac{1}{\sigma-\mu}\big)\right]$$
$$\leq \mu^{-1}a(x) + (\mu-\sigma)^{-1}k|\xi|^{\sigma}, \quad \forall\, t > 1, \text{ a.e. } x \in \Gamma, \ \xi \in \mathbb{R}. \tag{33}$$

Set $\xi = sv_0(x)$ with $x \in \Gamma$ and $s > 0$. We find from (33) the estimate

$$j(x,tsv_0(x)) \leq t^{\mu}[j(x,sv_0(x)) + \mu^{-1}a(x)$$
$$+(\mu-\sigma)^{-1}ks^{\sigma}|v_0(x)|^{\sigma}], \quad \forall\, t > 1, \ s > 0, \text{ a.e. } x \in \Gamma. \tag{34}$$

Combining (34) with (32) yields

$$\mathcal{R}(tsv_0) \leq \tfrac{1}{p}t^p s^p \|Dv_0\|^p_{L^p(\Omega;\mathbb{R}^N)}$$
$$+t^{\mu}s^{\sigma}\Big[s^{-\sigma}\int_\Omega j(x,sv_0(x))dx + k(\mu-\sigma)^{-1}\|v_0\|^{\sigma}_{L^{\sigma}(\Omega)} + s^{-\sigma}\mu^{-1}\|a\|_{L^1(\Omega)}\Big],$$
$$\forall\, t > 1, \ s > 0. \tag{35}$$

Assumption (32) allows for fixing some number $s_0 > 0$ such that

$$s^{-\sigma}\int_\Omega j(x,sv_0(x))dx + k(\mu-\sigma)^{-1}\Big(\|v_0\|^{\sigma}_{L^{\sigma}(\Omega)}\Big) + s^{-\sigma}\mu^{-1}\big(\|a\|_{L^1(\Omega)}\big) < 0.$$

With such an $s_0 > 0$ we can pass to the limit as $t \to +\infty$ in (35) and obtain (in view of $\mu > p$) that $\mathcal{R}(ts_0v_0) \to -\infty$ as $t \to +\infty$. Consequently, setting $e = t_0s_0v_0$ with sufficiently large $t_0 > 0$ we establish the assertion. This completes the proof of Lemma 4. ∎

4. Finite dimensional approximation

Let Λ be the family of all finite dimensional subspaces F of $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ including θ and an element e as constructed in the proof of Lemma 4.

For every subspace $F \in \Lambda$ we introduce the functional $\mathcal{R}_F : F \to \mathbb{R}$ which is the restriction of $\mathcal{R}$ to F, i.e.

$$\mathcal{R}_F(v) = \tfrac{1}{p}\|Dv\|^p_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega j\big(x, v(x)\big)\, dx, \quad \forall\, v \in F. \tag{36}$$

It is obvious that the functional $\mathcal{R}_F$ is locally Lipschitz and its generalized gradient is expressed by

$$\partial\mathcal{R}_F(v) \subset i_F^\star A i_F v + \overline{i}_F^\star \partial\mathcal{J}(v), \qquad \forall\, v \in F, \tag{37}$$

where $i_F : F \to W^{1,p}(\Omega)$, $\overline{i}_F : F \to L^\infty(\Omega)$ are the inclusion maps with their dual projections $i_F^\star : (W^{1,p}(\Omega))^\star \to F^\star$ and $\overline{i}_F^\star : L^1(\Omega) \to F^\star$, respectively, while $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\star$ is defined by

$$\big\langle Au, v\big\rangle_{W^{1,p}(\Omega)} = \int_\Omega |Du|^{p-2}\big\langle Du, Dv\big\rangle_{\mathbb{R}^N}\, dx. \tag{38}$$

By $\partial\mathcal{F}(\cdot)$ the generalized Clarke gradient of $\mathcal{J} : L^\infty(\Omega) \to \mathbb{R}$ given by

$$\mathcal{J}(v) = \int_\Omega j(x, v(x))\, dx, \quad \forall\, v \in L^\infty(\Omega)$$

have been denoted. Notice that in view of (H1), the functional $\mathcal{J}$ is locally Lipschitz on $L^\infty(\Omega)$, so the generalized gradient $\partial\mathcal{J}(\cdot)$ is well defined. The pairing over $F^\star \times F$ will be denoted by $\big\langle \cdot, \cdot \big\rangle_F$.

PROPOSITION 1 *Assume the hypotheses (H1)-(H6). Then, for each $F \in \Lambda$ there exists $u_F \in F$ such as to satisfy the hemivariational inequality*

$$\int_\Omega |Du_F|^{p-2}\big\langle Du_F, Dv - Du_F\big\rangle_{\mathbb{R}^N}\, dx + \int_\Omega j^0(u_F; v - u_F)\, dx \geq 0, \quad \forall\, v \in F. \tag{39}$$

Moreover, there exist constants $M > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ not depending on $F \in \Lambda$ such that

$$\|u_F\|_{W^{1,p}(\Omega)} \leq M, \quad \forall\, F \in \Lambda \tag{40}$$

$$\gamma_1 \leq \mathcal{R}(u_F) \leq \gamma_2, \quad \forall\, F \in \Lambda. \tag{41}$$

Proof. First we show that the functional $\mathcal{R}_F : F \to \mathbb{R}$ satisfies the Palais-Smale condition in the sense of Chang (Chang, 1981). Let $\{u_n\} \subset F$ and $\{w_n\} \subset F^\star$ be sequences such that $|\mathcal{R}_F(u_n)| \leq c$, for all $n \geq 1$, with a constant $c > 0$, and $w_n \in \partial\mathcal{R}_F(u_n)$, $\|w_n\|_{F^\star} = \varepsilon_n \to 0$ as $n \to \infty$. Since F is finite dimensional, it remains to show that $\{u_n\}$ is bounded in F. According to (37) we see that w_n can be expressed as follows

$$w_n = i_F^\star A u_n + \overline{i}_F^\star \chi_n, \quad \text{with } \chi_n \in \partial\mathcal{J}(u_n). \tag{42}$$

Let us notice that the hypothesis of Theorem 2.7.3 in Clarke (Clarke, 1983), p. 80, is verified. Therefore we obtain

$$\partial\mathcal{J}(v) \subset \int_\Omega \partial j(x, v(x))\, dx, \quad \forall\, v \in L^\infty(\Omega).$$

Thus

$$\begin{aligned}\big\langle Au_n, v - u_n\big\rangle_{W^{1,p}(\Omega)} + \int_\Omega j^0(u_n; v - u_n)\, dx &\geq \big\langle w_n, v - u_n\big\rangle_F \geq -\varepsilon_n \|v - u_n\|_F \\ &\geq -c\varepsilon_n \|v - u_n\|_{W^{1,p}(\Omega)}, \quad \forall\, v \in F, \quad c = \text{const} > 0,\end{aligned}$$

because the norms $\|\cdot\|_F$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ are equivalent in F (F is finite dimensional). Since $\mathrm{Lin}(\theta, u_n) \subset F$, the hypotheses of Lemma 1 are verified. Consequently $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, which means that

$$\|u_F\|_{W^{1,p}(\Omega)} \leq M_F \tag{43}$$

for some $M_F > 0$.

Following the lines of the proof of Lemma 2 (with $W^{1,p}(\Omega)$ replaced by F) we conclude the existence of positive constants $\rho_F > 0$ and $\eta_F > 0$ such that

$$\mathcal{R}_F(v) \geq \eta_F, \quad \forall\, v \in \{w \in F \colon \|w\|_F = \rho_F\}. \tag{44}$$

By Lemma 4 we know that $\mathcal{R}(te) \leq 0$ for any $t \geq 1$, therefore $\rho_F < \|e\|_F$. Thus, taking into account that $\mathcal{R}_F(0) \leq 0$ and $\mathcal{R}_F(e) \leq 0$ we are allowed to apply Theorem 1 to deduce the existence of a critical point $u_F \in F$ of $\mathcal{R}_F$. This leads to the finite dimensional hemivariational inequality (39) (see Motreanu & Panagiotopoulos, 1999).

Let us recall that the critical value $\mathcal{R}_F(u_F)$ is characterized by (see Motreanu & Panagiotopoulos, 1999)

$$\mathcal{R}_F(u_F) = \inf_{\gamma \in C_F} \max_{t \in [0,1]} \mathcal{R}_F(\gamma(t)), \tag{45}$$

where

$$C_F = \{\gamma \in C([0,1], F) \gamma(0) = 0,\ \gamma(1) = e\},$$

is the the family of all continuous curves in F joining points 0 and e in F i.e. $\gamma(0) = 0$ and $\gamma(1) = e$, $\gamma(t) \subset F$. Further, from Lemma 2 it follows that for a certain positive $\rho > 0$ one can find $\eta > 0$ with

$$\mathcal{R}(v) \geq \eta, \quad \forall\, v \in S_\rho \cap L^\infty(\Omega), \tag{46}$$

where $S_\rho := \{v \in W^{1,p}(\Omega) \colon \|v\|_{W^{1,p}(\Omega)} = \rho\}$, while Lemma 4 ensures the existence of $e \in W^{1,p}(\Omega)$, $e \neq 0$, such that

$$\mathcal{R}(te) \leq 0, \quad \forall\, t \geq 1. \tag{47}$$

Therefore, for any $F \in \Lambda$, if $\gamma \in C_F([0,1]; F)$ then γ meets points of S_ρ which means that

$$\max_{t\in[0,1]} \mathcal{R}_F\big(\gamma(t)\big) \geq \eta. \tag{48}$$

Hence

$$\eta \leq \mathcal{R}(u_F) = \inf_{\gamma\in C_F} \max_{t\in[0,1]} \mathcal{R}_F(\gamma(t)) \leq \max_{t\in[0,1]} \mathcal{R}(te), \quad \forall\, F \in \Lambda \tag{49}$$

and (41) results.

Now we are ready to show that $M_F > 0$ in (43) is independent of $F \in \Lambda$. For this purpose suppose that a sequence $\{u_{F_n}\}_{F_n\in\Lambda}$ of solutions of (P_{F_n}) has the property that $\|u_{F_n}\|_{W^{1,p}(\Omega)} \to \infty$. Taking into account (39) and (49) it is easy to check that the hypotheses (6) and (5) of Lemma 1 hold (with F replaced by F_n and $\varepsilon_n = 0$). Following the lines of the proof of Lemma 1 we arrive at the contradiction, which establishes the assertion. The proof of Proposition 1 is complete. ■

Proposition 2 *Assume the hypotheses* $(H1), (H3), (H4)_1$. *Moreover, let*

$(H6)_1$
$$\liminf_{t\to\pm\infty} \int_\Omega j(x,t)\,dx = -\infty.$$

Then, for each $F \in \Lambda$ *there exists* $u_F \in F$ *such as to satisfy the hemivariational inequality* (39). *Moreover, condition* (40) *is fulfilled.*

Proof. Since $W^{1,p}(\Omega) = \mathbb{R} \oplus \mathcal{W}$, where $\mathcal{W} = \{v \in W^{1,p}(\Omega) \colon \int_\Omega v(x)\,dx = 0\}$,

$$\|v\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1 \|v\|^p_{L^p(\Omega)}, \quad \forall\, v \in \mathcal{W},$$

and by $(H4)_1$ there follows (see Lemma 2.1, pp. 119-120, Naniewicz, 1997):

$$j(\xi) \geq -c(1 + |\xi|^s), \quad \forall\, \xi \in \mathbb{R},$$

so we easily deduce that

$$\inf\{\mathcal{R}_F(v) \colon v \in \mathcal{W}\} = \alpha > -\infty,$$

with $\alpha \in \mathbb{R}$ independent of the choice of $F \in \Lambda$. By making use of $(H6)_1$ we infer the existence of $\rho > 0$ with the property that

$$|\xi| = \rho \quad \Rightarrow \quad \mathcal{R}_F(\xi) < \alpha.$$

By Theorem 2 this allows the conclusion that $\mathcal{R}_F$ has a critical point $u_F \in F$ which means that (39) holds (see Motreanu & Panagiotopoulos, 1999).

In order to show (40) suppose, on the contrary, that the claim is not true, i.e. there exist a family $\{F_n\}_{n=1}^{\infty} \subset \Lambda$ and the corresponding sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega)$ of solutions of (39) (with F replaced by F_n) such that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. By substituting $v = 0$ into (39) we have

$$\int_\Omega j^0(u_n; -u_n)\, dx \geq \|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)}. \tag{50}$$

From (4) the decomposition results: $u_n = e_n\theta_n + \widehat{u}_n$, where $e_n = \left|\int_\Omega u_n\, dx\right|$, $\widehat{u}_n \in \mathcal{W}$, $\theta_n \in \{\pm\theta\}$, $\|\theta\|_{W^{1,p}(\Omega)} = 1$, such that

$$\|D\widehat{u}_n\|^p_{L^p(\Omega;\mathbb{R}^N)} \geq \lambda_1 \|\widehat{u}_n\|^p_{L^p(\Omega)}. \tag{51}$$

Since $Du_n = D\widehat{u}_n$, by $(H4)_1$ we have

$$k + k\|u_n\|^s_{W^{1,p}(\Omega)} \geq c\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)}. \tag{52}$$

Hence

$$k + k_1\|\widehat{u}_n\|^s_{W^{1,p}(\Omega)} + k_1 e_n^s \geq c\|\widehat{u}_n\|^p_{W^{1,p}(\Omega)}. \tag{53}$$

Thus, it follows that $e_n \to \infty$ because, otherwise, we would get the boundedness of $\{\widehat{u}_n\}$ and consequently, the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$, contrary to our supposition. Dividing (53) by e_n we get the estimate

$$\tfrac{k}{e_n} + k_1 e_n^{s-1}\|\tfrac{\widehat{u}_n}{e_n}\|^s_{W^{1,p}(\Omega)} + k_1 e_n^{s-1} \geq e_n^{p-1} c\|\tfrac{\widehat{u}_n}{e_n}\|^p_{W^{1,p}(\Omega)}$$

which, in view of $e_n \to \infty$ and $s < p$, allows for the conclusion that

$$\|\tfrac{\widehat{u}_n}{e_n}\|_{W^{1,p}(\Omega)} \to 0. \tag{54}$$

Now let us turn back to (39). By passing to a subsequence one can suppose also that $\theta_n = \theta$ (or $\theta_n = -\theta$). Thus, substituting $v = \widehat{u}_n$ into (39) yields

$$\begin{aligned} &e_n^p \int_\Omega |D(\tfrac{\widehat{u}_n}{e_n}) + D\theta)|^{p-2}\big\langle D(\tfrac{\widehat{u}_n}{e_n}) + D\theta), -D\theta\big\rangle_{\mathbb{R}^N}\, dx \\ &+ e_n \int_\Omega j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n} + \theta); -\theta\big)\, dx \geq -\varepsilon_n e_n. \end{aligned}$$

In view of $D\theta = 0$ this yields

$$\int_\Omega j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n}+\theta);-\theta\big)\,dx \ge -\varepsilon_n \tag{55}$$

and consequently, by (54),

$$J^\infty(\theta) \le \limsup_{n\to\infty}\int_\Omega -j^0\big(e_n(\tfrac{\widehat{u}_n}{e_n}+\theta);-\theta\big)\,dx \le 0,$$

the contradiction with $(H3)$. The proof of Proposition 2 is complete. ■

For the restriction $\mathcal{J}_F := \mathcal{J}|_F : F \to \mathbb{R}$ we have $\partial\mathcal{J}_F(u_F) \subset \overline{i}_F^\star \partial\mathcal{J}(u_F)$. Therefore, Propositions 1 and 2 can be reformulated as follows:

Corollary 1 *Assume that the hypotheses* $\{(H1)$-$(H6)\}$ *or* $\{(H1),(H3),(H4)_1,(H6)_1\}$ *hold. Then for each* $F \in \Lambda$
Problem (P_F)*: Find* $u_F \in F$ *and* $\chi_F \in L^1(\Omega)$ *such that*

$$\int_\Omega |Du_F|^{p-2}\big\langle Du_F, Dv - Du_F\big\rangle_{\mathbb{R}^N}\,dx + \int_\Omega \chi_F(v-u_F)\,dx = 0, \quad \forall\, v \in F, \tag{56}$$

$$\chi_F \in \partial j(u_F)\ \text{a.e. in } \Omega, \tag{57}$$

has at least one solution.

According to the results obtained we know that to any $F \in \Lambda$ a pair $(u_F, \chi_F) \in F \times L^1(\Omega))$ can be assigned, which is a solution of the problem (P_F). Moreover, the family $\{u_F\}_{F\in\Lambda}$ is uniformly bounded in $W^{1,p}(\Omega)$, i.e. (40) holds. The question arises concerning the behavior of $\{\chi_F\}_{F\in\Lambda}$.

Proposition 3 *Assume that a pair* $(u_F, \chi_F) \in F \times L^1(\Omega))$ *satisfies* (56) *and* (57)*. Then,* $\{\chi_F\}_{F\in\Lambda}$ *is weakly precompact in* $L^1(\Omega)$*.*

Proof. See the proof of Proposition 3.3, pp. 198-199, Naniewicz, 2004. ■

5. Main result

Now we are ready to formulate our main result.

Theorem 3 *Assume the hypotheses* $(H1)$-$(H6)$ *and also:*
$(H7)$ *For any sequence* $\{v_k\} \subset L^\infty(\Omega)$*,* $v_k \to 0$ *strongly in* $L^p(\Omega)$*, the condition*

$$\int_\Omega \min\big\{\psi(x)v_k(x) \colon \psi(x) \in \partial j\big(x, v_k(x)\big)\big\}\,dx \le 0,$$

implies

$$\limsup_{k\to\infty}\int_\Omega j\big(x, v_k(x)\big)\,dx \le 0.$$

Then there exists $u \in W^{1,p}(\Omega)$ *with* $u \neq 0$ *and* $j(u) \in L^1(\Omega)$*, such as to satisfy the hemivariational inequality*

$$\int_\Omega |Du|^{p-2} \langle Du, Dv - Du \rangle_{\mathbb{R}^N} \, dx + \int_\Omega j^0(u; v-u) \, dx \geq 0, \quad \forall\, v \in W^{1,p}(\Omega). \tag{58}$$

Moreover, there exists $\chi \in L^1(\Omega)$ *with the property that*

$$\int_\Omega |Du|^{p-2} \langle Du, Dv - Du \rangle_{\mathbb{R}^N} \, dx + \int_\Omega \chi(v - u) \, dx = 0, \quad \forall\, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{59}$$

$$\chi u \in L^1(\Omega) \quad \textit{and} \quad \chi \in \partial j(u) \;\; \textit{a.e. in } \Omega. \tag{60}$$

Proof. The proof is carried out in a sequence of steps.

Step 1. For every $F \in \Lambda$ we introduce

$$U_F = \big\{ u_F \in W^{1,p}(\Omega) \colon \text{for some } \chi_F \in L^1(\Omega), \; (u_F, \chi_F) \text{ is a solution of } (P_F) \big\}$$

and

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} U_{F'}.$$

By Proposition 1, W_F is nonempty (even U_F is nonempty) and contained in the ball $B_M = \{ v \in W^{1,p}(\Omega) \colon \|v\|_{W^{1,p}(\Omega)} \leq M \}$. We denote by weakcl$(W_F)$ the closure of W_F in the weak topology of $W^{1,p}(\Omega)$. Proposition 1 ensures that weakcl(W_F) is weakly compact in $W^{1,p}(\Omega)$. We claim that the family $\{\text{weakcl}(W_F)\}_{F \in \Lambda}$ has the finite intersection property. Indeed, if $F_1, \ldots, F_k \in \Lambda$ then $W_{F_1} \cap \ldots \cap W_{F_k} \supset W_F$, with $F = F_1 + \ldots + F_k$ and the assertion follows. So, we are allowed to conclude that there exists an element $u \in W^{1,p}(\Omega)$ with

$$u \in \bigcap_{F \in \Lambda} \text{weakcl}(W_F).$$

Let us choose $G \in \Lambda$ arbitrarily. Since $W^{1,p}(\Omega)$ is reflexive, one can extract an increasing sequence of subspaces $\{G_n\} \subset \Lambda$, each containing G, and for each n an element $u_n \in U_{G_n}$ such that $u_n \to u$ weakly in $W^{1,p}(\Omega)$ as $n \to \infty$ (Proposition 11, p. 274, Browder & Hess, 1972). Let us denote by $\{\chi_n\} \subset L^1(\Omega)$ the corresponding sequences with the property that for each n a pair (u_n, χ_n) is a solution of (P_{G_n}). By Proposition 3 we can suppose without loss of generality that $\chi_n \to \chi^G$ weakly in $L^1(\Omega)$ for some $\chi^G \in L^1(\Omega)$. Thus, we have asserted that

$$u_n \to u \quad \text{weakly in } W^{1,p}(\Omega) \tag{61}$$

$$\chi_n \to \chi^G \quad \text{weakly in } L^1(\Omega) \tag{62}$$

and that (56), with F replaced by G_n, reads

$$\big\langle Au_n, v-u_n\big\rangle_{W^{1,p}(\Omega)} + \int_\Omega \chi_n(v-u_n)\,dx = 0, \quad \forall\, v\in G_n, \tag{63}$$

where $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\star$ is defined by (38).

Step 2. Now we claim that $\chi^G \in \partial j(u)$ a.e. in Ω. (See the proof of Theorem 4.1, Step 2, p. 201, Naniewicz, 2004).

Step 3. Now it will be shown that

$$\limsup_{n\to\infty} \int_\Omega j^0(u_n; v-u_n)\,dx \le \int_\Omega j^0(u; v-u)\,dx, \tag{64}$$

holds for any $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. (See the proof of Theorem 4.1, Step 3, pp. 201-202, Naniewicz, 2004).

Step 4. Now we show that

$$\chi^G u \in L^1(\Omega). \tag{65}$$

$$\liminf_{n\to\infty} \int_\Omega \chi_n u_n\,dx \ge \int_\Omega \chi^G u\,dx. \tag{66}$$

For this purpose let $\{\epsilon_k\} \subset L^\infty(\Omega)$ be such that (Hedberg, 1978):

$$\begin{aligned} \{(1-\epsilon_k)u\} \subset W^{1,p}(\Omega)\cap L^\infty(\Omega), \quad 0\le \epsilon_k \le 1 \\ \widetilde{u}_k := (1-\epsilon_k)u \to u \text{ strongly in } W^{1,p}(\Omega) \text{ as } k\to\infty. \end{aligned} \tag{67}$$

Without loss of generality it can be assumed that $\widetilde{u}_k \to u$ a.e. in Ω. Since it is already known that $\chi^G \in \partial j(u)$, one can apply (H4) to obtain $\chi^G(-u) \le j^0(u;-u) \le \kappa(1+|u|^q)$. Hence

$$\chi^G \widetilde{u}_k = (1-\epsilon_k)\chi^G u \ge -\kappa(1+|u|^q). \tag{68}$$

This implies that the sequence $\{\chi^G\widetilde{u}_k\}$ is bounded from below by an integrable function and $\chi^G\widetilde{u}_k \to \chi^G u$ a.e. in Ω. On the other hand, one gets

$$\int_\Omega \chi_n(\widetilde{u}_k - u_n)\,dx \le \int_\Omega j^0(u_n; \widetilde{u}_k - u_n)\,dx.$$

Thus, passing to the limit with $n\to\infty$ yields

$$\int_\Omega \chi^G \widetilde{u}_k\,dx - \liminf_{n\to\infty}\int_\Omega \chi_n u_n\,dx \le \limsup_{n\to\infty}\int_\Omega j^0(u_n;\widetilde{u}_k - u_n)\,dx,$$

and due to (64) we are led to the estimate

$$\begin{aligned}\int_\Omega \chi^G \widetilde{u}_k\,dx &\le \liminf_{n\to\infty}\int_\Omega \chi_n u_n\,dx + \int_\Omega j^0(u;\widetilde{u}_k - u)\,dx\\ &\le \liminf_{n\to\infty}\int_\Omega \chi_n u_n\,dx + \int_\Omega j^0(u;-\epsilon_k u)\,dx\\ &\le \liminf_{n\to\infty}\int_\Omega \chi_n u_n\,dx + \int_\Omega \epsilon_k\kappa(1+|u|^q)\,dx \le C, \quad C=\text{const}\,.\end{aligned}$$

Thus, by Fatou's lemma we are allowed to conclude that $\chi^G u\in L^1(\Omega)$, i.e. (65) holds. Taking into account that $\epsilon_k\to 0$ a.e. in Ω as $k\to\infty$ (passing to a subsequence if necessary) we establish (66), as required.

Step 5. It will be shown that

$$\begin{aligned}\langle Au, v-u\rangle_{W^{1,p}(\Omega)} + \int_\Omega \chi^G\,(v-u)\,dx = 0, \quad &\forall\, v\in\bigcup_{n=1}^\infty G_n\supset G\\ &\chi^G\in\partial j(u).\end{aligned} \qquad (Q^G)$$

Since A is bounded and $\{u_F\}_{F\in\Lambda}\subset\{v\in W^{1,p}(\Omega)\colon \|v\|_{W^{1,p}(\Omega)}\le M\}$, there exists $K>0$ such that $\{Au_F\}_{F\in\Lambda}\subset\{l\in W^{1,p}(\Omega)^\star\colon \|l\|_{W^{1,p}(\Omega)^\star}\le K\}$. From (63) it follows that for any fixed $G\in\Lambda$ we get

$$\left|\int_\Omega \chi^G v\,dx\right| \le K\|v\|_{W^{1,p}(\Omega)}, \quad \forall\, v\in\bigcup_{n=1}^\infty G_n,\ \ \chi^G\in\partial j(u), \tag{69}$$

because $\{G_n\}$ is an increasing sequence. Further, by making use of (65) and (66) we have $\chi^G u\in L^1(\Omega)$ and

$$\limsup_{n\to\infty}\langle Au_n, u_n - v\rangle_{W^{1,p}(\Omega)} \le \int_\Omega \chi^G(v-u)\,dx, \quad \forall\, v\in\bigcup_{n=1}^\infty G_n. \tag{70}$$

Since $u_n\in G_n$ and $u_n\to u$ weakly in $W^{1,p}(\Omega)$, the closure of $\bigcup_{n=1}^\infty G_n$ in the strong topology of $W^{1,p}(\Omega)$, $\overline{\bigcup_{n=1}^\infty G_n}$, must contain u. Thus there exists a sequence $\{w_i\}\subset\bigcup_{n=1}^\infty G_n$ converging strongly to u in $W^{1,p}(\Omega)$ as $i\to\infty$. We claim that for such a sequence,

$$\int_\Omega \chi^G\, w_i\,dx \to \int_\Omega \chi^G\, u\,dx \quad \text{as } i\to\infty. \tag{71}$$

Indeed, let $\{\widetilde{u}_k\}_{k=1}^\infty$ be given by (67). From (68) it follows that

$$-\kappa(1+|u|^q)\le \chi^G\,\widetilde{u}_k \le |\chi^G u|, \quad k=1,2\dots, \tag{72}$$

with the bounds $-\kappa(1+|u|^q)$ and $|\chi^G u|$ being integrable in Ω. Thus, there exists a constant $C>0$ such that

$$\left|\int_\Omega \chi^G\, \widetilde{u}_k\, dx\right| \le C\|\widetilde{u}_k\|_{W^{1,p}(\Omega)}, \quad k=1,2,\ldots. \tag{73}$$

Denote by $\mathcal{A}$ a linear subspace spanned by $\{\widetilde{u}_k\}_{k=1}^\infty$ and define a linear functional $\widehat{l}_{\chi^G} : \big(\bigcup_{n=1}^\infty G_n + \mathcal{A}\big) \to \mathbb{R}$ by the formula

$$\widehat{l}_{\chi^G}(v) := \int_\Omega \chi^G\, v\, dx, \quad v \in \bigcup_{n=1}^\infty G_n + \mathcal{A}.$$

Taking into account (69) and (73), from the Hahn-Banach theorem it follows that $\widehat{l}_{\chi^G}$ admits its linear continuous extension onto $W^{1,p}(\Omega)$, $l_{\chi^G} \in W^{1,p}(\Omega)^\star$. By the dominated convergence,

$$\int_\Omega \chi^G\, \widetilde{u}_k\, dx \to \int_\Omega \chi^G\, u\, dx, \quad \text{as } k\to\infty,$$

so we get $l_{\chi^G}(u) = \int_\Omega \chi^G\, u\, dx$ which, in particular, implies (71), as claimed.

Taking into account (70) and (71) we conclude that

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{W^{1,p}(\Omega)} \le 0, \tag{74}$$

which, by the pseudomonotonicity of A, implies

$$Au_n \to Au \ \text{ weakly in } W^{1,p}(\Omega) \tag{75}$$

$$\langle Au_n, u_n\rangle_{W^{1,p}(\Omega)} \to \langle Au, u\rangle_{W^{1,p}(\Omega)}. \tag{76}$$

Hence from (63) we are led to (Q^G), as desired. Notice that (75) and (76) imply the strong convergence $u_n \to u$ in $W^{1,p}(\Omega)$.

Step 6. It remains to show that there exists $\chi \in \partial j(u)$ with the associated linear functional defined by

$$\widehat{l}_\chi(v) := \int_\Omega \chi\, v\, dx, \quad \forall\, v \in W^{1,p}(\Omega)\cap L^\infty(\Omega),$$

admitting a continuous extension $l_\chi \in W^{1,p}(\Omega)^\star$ such that

$$Au + l_\chi = 0, \quad \langle l_\chi, u\rangle_{W^{1,p}(\Omega)} = \int_\Omega \chi\, u\, dx. \tag{77}$$

For every $G \in \Lambda$ let us introduce

$$V^{(G)} = \big\{\chi^G \in L^1(\Omega) \colon (Q^G) \text{ holds}\big\}$$

and

$$Z^{(G)} = \bigcup_{\substack{G' \in \Lambda \\ G' \supset G}} V^{(G')}.$$

As in the proof of Proposition 3 we show that the family $\{\chi^G\}_{G\in\Lambda}$ is weakly precompact in $L^1(\Omega)$. Denoting by $\mathrm{weakcl}(Z^{(G)})$ the closure of $Z^{(G)}$ in the weak topology of $L^1(\Omega)$ we prove, analogously, that the family $\{\mathrm{weakcl}(Z^{(G)})\}_{G\in\Lambda}$ has the finite intersection property. Thus, there exists an element $\chi \in \partial j(u)$ such that for any $G \in \Lambda$ there is

$$\langle Au, v\rangle_{W^{1,p}(\Omega)} + \int_\Omega \chi\, v\, dx = 0, \quad \forall\, v \in G.$$

Since $G \in \Lambda$ has been chosen arbitrarily and Λ is dense in $W^{1,p}(\Omega)$, (77) results, as desired.

Step 7. It remains to demonstrate (58). For that we refer the reader to the proof of Theorem 4.1, Step 7, pp. 205-206, Naniewicz, 2004.

Step 8. In order to show that $j(u) \in L^1(\Omega)$ we use (40) and (41) to get

$$\int_\Omega j(u_n)\, dx \le \gamma_2 - \tfrac{1}{p}\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} \le \gamma_2.$$

Next, in view of the conditions

$$j(u_n) \ge -\kappa_0(1 + |u_n|^q),$$

$j(u_n) \to j(u)$ a.e. in Ω as $n \to \infty$ and $u_n \to u$ strongly in $L^q(\Omega)$, we are allowed to apply Fatou's lemma which yields the assertion.

Step 9. The existence of a nontrivial solution $u \ne 0$ follows from $(H7)$. The supposition $u = 0$ leads to the contradiction. Indeed, since $\{u_n\} \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $u_n \to 0$ strongly in $W^{1,p}(\Omega)$ as shown previously, by making use of (63) with $v = 2u_n$ we get $\int_\Omega \min\{\psi u_n \colon \psi \in \partial j(u_n)\}\, dx \le \int_\Omega \chi_n u_n\, dx = -\|Du_n\|^p_{L^p(\Omega;\mathbb{R}^N)} \le 0$. Hence, $\limsup_{n\to\infty} \int_\Omega j(u_n)\, dx \le 0$, and consequently, $\limsup_{n\to\infty} \mathcal{R}(u_n) \le 0$, which contradicts (41). This contradiction yields the assertion. The proof of Theorem 3 is complete. ■

Analogously we prove the following:

THEOREM 4 *Assume the hypotheses* $(H1)$, $(H3)$, $(H4)_1$, $(H6)_1$. *Then there exists* $u \in W^{1,p}(\Omega)$ *with* $j(u) \in L^1(\Omega)$, *such as to satisfy the hemivariational inequality* (58). *Moreover, there exists* $\chi \in L^1(\Omega)$ *such that* (59) *and* (60) *hold.*

From (59) and (60) we obtain now

COROLLARY 2 *Assume that the hypotheses* $\{(H1)\text{-}(H7)\}$ *or* $\{(H1),(H3),(H4)_1,(H6)_1\}$ *are fulfilled. Then the problem: Find* $u \in W^{1,p}(\Omega)$, $\chi \in L^1(\Omega)$ *such that*

$$(P) \qquad \begin{cases} -\Delta_p u &= -\chi \quad \text{in } \Omega \text{ (in the distributional sense)} \\ \chi &\in \partial j(u) \quad \text{a.e. in } \Omega \\ \chi u &\in L^1(\Omega) \\ j(u) &\in L^1(\Omega) \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \text{ (in the distributional sense).} \end{cases}$$

has at least one nontrivial solution.

6. Remarks and comments

As it is well known, see Naniewicz & Panagiotopoulos (1995), in case of the classical growth condition of the form

$$|\partial j(\xi)| \leq c(1 + |\xi|^{p-1}), \quad \xi \in \mathbb{R}^s, \tag{78}$$

the problem described by hemivariational inequality (58) admits a solution $u \in V$ and, moreover, there exist $\chi \in L^q(\Omega; \mathbb{R}^s)$, $1/p + 1/q = 1$, and $\psi \in V^\star$ such that

$$\begin{aligned} &\chi \in \partial j(u) \text{ a.e. in } \Omega \quad \text{and} \quad \psi \in \partial\Phi(u), \\ &g = Au + \psi + l_\chi, \end{aligned}$$

where $l_\chi \in V^\star$ is a linear continuous functional defined by

$$\langle l_\chi, v\rangle := \int_\Omega \chi \cdot v \, d\Omega, \quad v \in V. \tag{79}$$

Recall that the subdifferential $\partial\Phi(u) \subset V^\star$ in the sense of Convex Analysis (Ekeland & Temam, 1976) is defined for $u \in \operatorname{Dom}\Phi$ by means of the formula

$$\Phi(v) - \Phi(u) \geq \langle \psi, v - u\rangle, \quad \forall\, v \in V.$$

Thus, in case of (78) a statement that $u \in V$ is a solution of hemivariational inequality (58) is equivalent to

$$g - Au - l_\chi \in \partial\Phi(u). \tag{80}$$

The situation changes essentially when (78) is replaced by the unilateral growth condition $(H4)$. In such a case we have only ensured the $L^1(\Omega)$-regularity of χ and consequently, the corresponding functional l_χ (given by the formula (79)) is linear on its domain $\operatorname{Dom} l_\chi \supset L^\infty(\Omega; \mathbb{R}^s) \cap V$, but not necessarily continuous. It may happen that l_χ does not have the continuous extension onto the whole space V (l_χ is discontinuous). If it is the case, l_χ can be extended onto the whole space V as a function from V into $\mathbb{R} \cup \{+\infty, -\infty\}$ by setting

$$l_\chi(v) := \begin{cases} \int_\Omega \chi \cdot v \, d\Omega & \text{if } \chi \cdot v \in L^1(\Omega) \\ +\infty & \text{if } \int_\Omega [\chi \cdot v]^+ \, d\Omega = +\infty \\ -\infty & \text{if } \int_\Omega [\chi \cdot v]^+ \, d\Omega < +\infty \text{ and } \int_\Omega [\chi \cdot v]^- \, d\Omega = +\infty, \end{cases} \tag{81}$$

for each $v \in V$. Thus, we deal with a functional $l_\chi : V \to \mathbb{R} \cup \{+\infty, -\infty\}$ which is discontinuous whenever $l_\chi(v) = +\infty$ or $l_\chi(v) = -\infty$ for at least one point of V. Notice that l_χ can be expressed as a difference of two convex lower semicontinuous proper functions $l_\chi^+(v) := \int_\Omega [\chi \cdot v]^+ \, d\Omega$ and $l_\chi^-(v) := \int_\Omega [\chi \cdot v]^- \, d\Omega$, $v \in V$, i.e.

$$l_\chi(v) = l_\chi^+(v) - l_\chi^-(v), \quad \forall \, v \in V. \tag{82}$$

Denote by $\mathcal{L}(V)$ the class of all linear densely defined functions $l : V \to \mathbb{R} \cup \{+\infty, -\infty\}$ which can be represented by a difference of two convex lower semicontinuous proper functions $l^+ : V \to \mathbb{R} \cup \{+\infty\}$ and $l^- : V \to \mathbb{R} \cup \{+\infty\}$, i.e. $l = l^+ - l^-$, with the convention that

$$l(v) := \begin{cases} l^+(v) - l^-(v) & \text{if } v \in \operatorname{Dom} l^+ \cap \operatorname{Dom} l^- \\ +\infty & \text{if } v \notin \operatorname{Dom} l^+ \\ -\infty & \text{if } v \in \operatorname{Dom} l^+ \text{ and } v \notin \operatorname{Dom} l^-. \end{cases} \tag{83}$$

For a convex, lower semicontinuous, proper function $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ we introduce $\widehat{\partial}\varphi(u) \subset \mathcal{L}(V)$ as follows: if $u \notin \operatorname{Dom} \varphi$ then $\widehat{\partial}\varphi(u) = \emptyset$ while if $u \in \operatorname{Dom} \varphi$ then we set

$$l \in \widehat{\partial}\varphi(u) \quad \Leftrightarrow \quad l(u) \in \mathbb{R} \text{ and } \varphi(v) - \varphi(u) \geq l(v - u), \quad \forall \, v \in V. \tag{84}$$

The formal definition of $\widehat{\partial}\varphi(u)$ coincides with that of $\partial\varphi(u) \subset V^\star$ in the sense of convex analysis. However, $\widehat{\partial}\varphi(u)$ apart from containing elements of $\partial\varphi(u)$, may contain also some discontinuous linear functionals which will be called here discontinuous subgradients. Notice that if $u \in \operatorname{Int}(\operatorname{Dom} \Phi)$, where "Int" means "the interior", then by the Banach-Steinhaus theorem it follows that $\widehat{\partial}\varphi(u)$ and $\partial\varphi(u)$ coincide.

In the terminology of Pallaschke & Rolewicz (1997) a function $l \in \widehat{\partial}\varphi(u)$ fulfilling (84) is said to be a $\mathcal{L}(V)$-subgradient of Φ at $u \in V$. We refer the reader to Pallaschke & Rolewicz (1997) for the general abstract subdifferential theory.

As we shall see below, the notion of discontinuous subgradient is specially useful in describing some particular aspects of Theorem 4.

THEOREM 5 *Assume all the hypotheses of Theorem 4. Then there exists $u \in V$ such that*

$$\langle Au - g, v - u \rangle_V + \Phi(v) - \Phi(u) + \int_\Omega j^0(u; v - u)\, d\Omega \geq 0, \quad \forall\, v \in V. \tag{85}$$

Moreover, there exists $\chi \in L^1(\Omega; \mathbb{R}^s)$, $\chi \in \partial j(u)$ a.e. in Ω, such that for l_χ defined by (81) *it follows that*

$$g - Au - l_\chi \in \widehat{\partial}\Phi(u). \tag{86}$$

Proof. Following the lines of the proof of Theorem 4 we can deduce that the inequality

$$\langle Au - g, v - u \rangle_V + \Phi(v) - \Phi(u) + \int_\Omega \chi \cdot (v - u)\, d\Omega \geq 0$$

holds for any $v \in V \cap L^\infty(\Omega; \mathbb{R}^s)$. It can be written equivalently as

$$\Phi(v) - \Phi(u) \geq \langle -Au + g, v - u \rangle_V - l_\chi(v - u),\ \forall\, v \in V \cap L^\infty(\Omega; \mathbb{R}^s), \tag{87}$$

where $l_\chi(v-u) = \int_\Omega \chi \cdot (v - u)\, d\Omega$. It must be shown that this inequality holds for any $v \in V$. If $v \notin \operatorname{Dom}\Phi$ then there is nothing to prove because $\Phi(v) = +\infty$.

Let us consider the case of $v \in \operatorname{Dom}\Phi$. If $\chi \cdot v \in L^1(\Omega)$ then $-\chi \cdot \widetilde{v}_k \geq -|\chi \cdot v|$ which, by Fatou's lemma, yields $\liminf_{k\to\infty} -l_\chi(\widetilde{v}_k) \geq -l_\chi(v)$. Thus, in view of (H) the assertion follows ($\widetilde{v}_k$ has been taken as in the hypothesis (H)). If $\chi \cdot v \notin L^1(\Omega)$ then there is nothing to prove if $l_\chi(v) = +\infty$, while, as it will be shown, the case $l_\chi(v) = -\infty$ cannot happen. Indeed, suppose that $l_\chi(v) = -\infty$, i.e. $\int_\Omega [\chi \cdot v]^+\, d\Omega < +\infty$ and $\int_\Omega [\chi \cdot v]^-\, \Omega = +\infty$. Taking into account (87) we are led to $l_\chi(\widetilde{v}_k) \geq -C$ for a constant C. Hence $D \geq \int_\Omega [\chi \cdot \widetilde{v}_k]^+\, d\Omega \geq \int_\Omega [\chi \cdot \widetilde{v}_k]^-\, d\Omega - C$ for some $D = \text{const}$. But, due to Fatou's lemma this yields $\int_\Omega [\chi \cdot v]^-\, d\Omega \leq C + D$ contrary to $\int_\Omega [\chi \cdot v]^-\, d\Omega = +\infty$. This contradiction completes the proof. ■

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