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Stability of higher-level energy norms of strong solutions to a wave equation with localized nonlinear damping and a nonlinear source term

by

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Abstract: We derive global in time a priori bounds on *higher-level energy norms* of strong solutions to a semilinear wave equation: in particular, we prove that despite the influence of a nonlinear source, the evolution of a smooth initial state is *globally bounded in the strong topology* $\sim H^2 \times H^1$. And the bound is *uniform* with respect to the corresponding norm of the initial data.

It is known that an m-accretive semigroup generator monotonically propagates smoothness of the initial condition; however, this result does not hold in general for Lipschitz perturbations of monotone systems where higher order Sobolev norms of the solution may blow-up asymptotically as $t \to \infty$. Due to nonlinearity of the system, the only a priori global-in-time bound that follows from classical methods is that on finite energy: $\sim H^1 \times L^2$.

We show that under some correlation between growth rates of the damping and the source, the norms of topological order *above the finite energy level* remain globally bounded. Moreover, we also establish this result when damping exhibits *sublinear or superlinear growth at the origin, or at infinity*, which has immediate applications to asymptotic estimates on the decay rates of the finite energy.

The approach presented in the paper is not specific to the wave equation, and can be extended to other hyperbolic systems: e.g. plate, Maxwell, and Schrödinger equations.

Keywords: wave equation, localized nonlinear damping, nonlinear source, stability, uniform boundedness, higher energy, strong solutions.

1. Introduction

Stability and energy decay rates for dissipative wave equations, as well as other hyperbolic-like structures (plates, shells), have attracted considerable attention in the past years, see books by Haraux (1981, 1987), Lagnese (1983), Komornik (1994), Lasiecka (2002) and references therein. The problem of interest in this paper is a semilinear wave equation driven by a nonlinear source $f \in C^1(\mathbb{R})$ and monotone nonlinear dissipation $g \in C(\mathbb{R})$:

$$w_{tt}(x,t) - \Delta w(x,t) + \chi(x)g(w_t(x,t)) = f(w(x,t)), \quad \{x,t\} \in \Omega \times [0,T[\tag{1}$$

on a smooth domain $\Omega \subset \mathbb{R}^n$. The growth of the source feedback map $f(s)$ can reach the critical Sobolev's exponent, while the dissipation $\chi g(w_t)$ is geometrically restricted to the support of the cutoff function $\chi(x)$. Equation (1) will be equipped with appropriate homogenous boundary conditions (Neumann, Dirichlet, Robin, or an appropriate combination thereof) which we will specify later.

Let us first discuss the above problem within the framework of *finite energy space*, which corresponds to the usual topology: $H^1(\Omega)$ for the acoustic pressure (or displacement) component w, and $L_2(\Omega)$ for the time rate of change w_t. It is by now a standard result that *finite energy solutions* exist locally in time. In order to avoid a potential finite-time blowup, one needs to put some stability-dissipativity restrictions on f, such as, for instance, condition (5) below. This assumption can be relaxed if one is interested only in global (in time) existence of solutions. However, since the main focus of this paper is on the decay rates of finite energy to 0, such a stability assertion is necessary. Among other implications, this assumption gives us global Hadamard wellposedness for the problem.

When discussing long-term behavior of equation (1), the first question is:

Question 1. *Under what conditions on the dissipation do the solutions decay uniformly (in finite energy norm) to zero?*

This is a classical uniform stabilization problem that has been considered extensively in the literature. It is known that while "strong stabilization" does hold without any additional conditions imposed on the dissipation (due to Holmgren's uniqueness theorem), the *uniform* stabilization does require appropriate equipartition of potential and kinetic energy, which imposes certain growth restriction on the damping. In the case of geometrically constrained dissipation, not only must the support of the cutoff $\chi(x)$ occupy a sufficiently "large" subset of a boundary collar, but also the growth bounds imposed on g at infinity must be of linear type (from above and below). In this case the rate of decay depends solely on the growth of the damping near the origin (Lasiecka, Tataru, 1993). A comprehensive treatment of decay rates for equation (1) was presented in Lasiecka and Toundykov (2006) and Toundykov (2007), which extends the results obtained earlier in Lasiecka, Tataru (1993) for the case of boundary damping.

One of the results in Lasiecka and Toundykov (2006) shows that linearly-bounded growth of the dissipative feedback map g (possibly restricted in space by cutoff $\chi(x)$) yields exponential decay of the finite energy norms. Any deviation of g from linear-like behavior near *the origin* weakens the decay rates to algebraic or logarithmic (or iterated logarithmic: ($\log\log$) etc). This result settles the issue completely in the case of dissipation that is linearly bounded at infinity. The next natural question to address is:

Question 2. *What can be said about the decay rates of the energy when the damping is either sublinear or superlinear at infinity?*

It is the lack of equipartition between the potential and kinetic energy components that destroys uniformity (in the same finite energy topology) of the decay rates: either we have too much of kinetic damping (superlinear case) or too little (sublinear case).

In this situation, asymptotic behavior of the finite energy strongly depends on topological properties of the flow; to claim decay rates at the phase-space level, one must establish *global-in-time* bounds on solutions in topology strictly *above* the order of the finite energy itself. These higher-order norms are what we shall henceforth refer to as the "*higher energy*" which can be associated with $H^2(\Omega) \times H^1(\Omega)$ topology (later on we will make the definition of "high energy" more precise).

This dependence on higher norms is necessary even in some 1-dimensional settings (Vancostenoble and Martinez, 2000) with localized dissipation. Lasiecka and Toundykov (2006) present a detailed account on how additional regularity of "high energy" solutions reflects on stability when the dissipative feedback behaves sub- or super-linearly at infinity. However, this raises yet another, and fundamental, question:

Question 3. *When are higher-order norms globally bounded in time? Furthermore, can we expect such a regularity estimate to be uniform with respect to the high-energy norm of the initial state?*

We know that given sufficiently smooth initial data, higher energy of solutions remains bounded on every finite interval (see Proposition a-3 in the Appendix). However, it is far from being clear whether the bound remains unform for all times. This problem, classical within the realm of dynamical systems, has a simple solution when the system is contractive (no source present in the model). Indeed, nonlinear semigroup theory (Barbu, 1993) provides an affirmative answer. However, in a non-contractive case there is no natural mechanism to ensure stability of higher energies, since *dissipativity of the source is guaranteed only at the level of finite energy space.*

At the level of higher-order norms the nonlinear source term actively "pumps" energy into the system. In general, the best one can claim is that the higher energies obey an exponential bound that blows up as $t \to \infty$. Thus, even though it is relatively easy to prove that the finite energy is globally bounded (and uniformly with respect to norm of the initial state), the norms in finer topologies do not need to obey any global estimates.

These considerations lead to yet another question:

Question 4. *What additional conditions can one impose on the damping and the source, to guarantee that the higher energies of solutions stay globally (uniformly) bounded?*

In general, information on global behavior of solutions to PDE in higher-order spaces has, of course, its own merits. However, in our case, an additional motivation is to use this uniform regularity to measure the decay rates of finite energy solutions, under super- or sub-linear damping at infinity. The expected answer must be conditional and should depend on the interaction between the source and the damping, in particular it relies on

- how strong the source is at the higher energy level
- how fast the solutions decay to zero at the lower ("finite") energy level.

It is the balance between these two factors that provides an answer to Question 4: we set up an optimization problem between rapid decays and high regularity needed to compensate for the effect of the source. One can schematically illustrate this situation by the following relational loop

$$\begin{array}{c}\text{Regularity of}\\ \text{High Energy}\end{array} \Longrightarrow \begin{array}{c}\text{Decay Rates of}\\ \text{Finite Energy}\end{array} \Longrightarrow \begin{array}{c}\text{Regularity of}\\ \text{High Energy}\end{array}$$

which, when expressed quantitatively, naturally leads to a fixed point-type argument that involves optimization of regularity and decay estimates. In this paper we show how to resolve this optimization problem and state what conditions can be imposed on the interaction between the source and damping, in order to affirmatively answer Question 4 (see Theorem 2 below).

1.1. The model

Throughout the paper we shall mostly focus on the $n = 3$-dimensional set-up. It captures all the technical difficulties that arise in higher dimensions, and straightforwardly simplifies to lower-dimensional models. We will, however derive some of the results for 2-dimensional domains as well.

Let $\Omega \subset \mathbb{R}^{n=3}$ be a smooth bounded connected domain with connected boundary Γ. Let $Q_T := \Omega\times]0,T[$ and $\Sigma_T := \Gamma\times]0,T[$; the form $\|\cdot\|$ without a subscript will denote the norm in $L^2(\Omega)$. This paper aims at investigating long-term behavior of solutions to the following system:

$$\begin{aligned} w_{tt} - \Delta w + \chi(x) g(w_t) = f(w) \quad &\text{in} \quad Q_T \\ w(0) = w_0, \quad w_t(0) = w_1 \quad &\text{in} \quad \Omega \end{aligned} \tag{2}$$

with boundary conditions of Dirichlet or Robin type:

$$\left(\alpha \frac{\partial w}{\partial \nu} + \beta w\right)\bigg|_{\Sigma_T} = 0, \qquad \alpha \geq 0, \quad \beta > 0. \tag{3}$$

Nonlinear functions g and f represent Nemytski operators associated with scalar continuous real-valued functions $g(s)$ and $f(s)$, respectively.

Map g is continuous, monotone increasing, zero at the origin, and represents interior dissipation, that is localized to a subset of the domain by the cutoff map $\chi(x)$. For the system to be well-posed, the scalar function f must correspond to a locally Lipschitz Nemytski operator $H^1(\Omega) \to L^2(\Omega)$; in particular, we shall assume

$$|f'(s)| \le C(1 + |s|^k) \tag{4}$$

where $0 \le k \le 2/(2-n)$ if $n \ge 3$, or $0 \le k < \infty$ if $n = 2$.

Note that the polynomial bound on f may include the critical Sobolev exponent for the embedding $H^1 \to L^2$, that represents the threshold above which finite-energy norms may blow up in finite time (e.g. see Georgiev and Todorova, 1994; Serrin, Todorova and Vitillaro, 2003).

In addition, we need to impose further stability conditions to ensure global existence and filter out non-trivial steady states:

$$\sup_{s \ne 0} f(s)/s < \lambda_1, \quad \text{and} \quad f(0) = 0 \tag{5}$$

where λ_1 is the smallest eigenvalue of the operator $A = -\Delta$ defined on a subset of $L^2(\Omega)$ functions that have two distribution derivatives and satisfy boundary conditions of the form (3).

1.1.1. On boundary conditions

The analysis of decay rates in Lasiecka and Toundykov (2006) also addresses models with mixed Dirichlet/Neumann/Robin boundary conditions. In this case the solutions may develop singularities near the junction where different types of boundary dynamics meet each other (e.g. Grisvard, 1985, 1989).

Present discussion can be generalized to the mixed case as well, provided one takes into account that the domain of the corresponding Laplacian is not necessarily a subset of $H^2(\Omega)$. We would have to restrict the use of Sobolev embeddings throughout the proofs, in order to be consistent with the available elliptic regularity.

To keep the presentation focused we will not address the mixed setting, however, the approach presented below can be generalized to accommodate more complex boundary dynamics (see Lasiecka and Toundykov, 2006).

1.1.2. Geometry of the domain

The cutoff function $\chi \in L^\infty\big(\Omega; [0, \infty[\,\big)$, potentially restricts the action of the velocity feedback $g(w_t)$. We are primarily interested in the case when χ is supported only on a sub-collar of Ω: i.e. there exists a nonempty segment of the boundary Γ_C, and some fixed $\gamma > 0$ such that

$$\Omega_\chi := \big\{x \in \Omega \;:\; \operatorname{dist}(x, \Gamma_C) \le \gamma\big\} \subseteq \operatorname{supp}\chi. \tag{6}$$

Also assume that χ has a positive a.e. uniform bound from below on Ω_χ: $\operatorname{ess\,inf}_{\Omega_\chi}(\chi) > 0$. The remaining portion of the boundary (possibly overlapping with Γ_C) is denoted Γ_U:

$$\Gamma = \overline{\Gamma}_C \cup \Gamma_U. \tag{7}$$

The part Γ_C is assumed to have positive measure, be connected, and relatively open. If $\overline{\Gamma}_C \neq \Gamma$ then the same assertions apply to Γ_U, otherwise take $\Gamma_U = \emptyset$. When nonempty, we assume that Γ_U satisfies the necessary geometrical assumptions for the damping to be effective:

ASSUMPTION 1 (Geometry of the unobserved segment) *If the unobserved portion Γ_U of the boundary is nonempty then*

(a) Assert that for some $x_0 \in \mathbb{R}^n$, $(x - x_0) \cdot \nu(x) \leq 0$ on Γ_U with ν being the outward normal field.

(b) If a Neumann-type boundary condition holds (3) with $\alpha > 0$, then we need Ω to be strictly convex near Γ_U. More specifically:

$$\Gamma_U = \{x \in \mathbb{R}^n : \ell(x) = 0\}, \quad \nabla \ell \neq 0 \quad \text{on} \quad \Gamma_U$$

with the surface $z = \ell(x)$ having a convex epigraph. See Lasiecka and Toundykov (2006) as well as Lasiecka, Triggiani and Zhang (2000), p. 302, for more details.

1.2. Known results: uniform decay of finite energy and local stability of higher norms

System (2), (3), with at most critical source, and stability assumption (5), generates a nonlinear semigroup flow on the phase space

$$\mathcal{H} = D(A^{1/2}) \times L^2(\Omega) \subset H^1(\Omega) \times L^2(\Omega)$$

(e.g. see Chueshov, Eller and Lasiecka, 2002, Theorem 7.2). We begin with the following regularity result, which provides some improvement over the classical theory when g is *superlinear* at infinity, we state and prove it for the 3-dimensional case.

PROPOSITION 1 *Let g be monotone continuous, $g(0) = 0$. Assume that $f \in C^1(\mathbb{R})$ satisfies $|f'(s)| \leq C_f(1+|s|^2)$. Suppose that $\{w_0, w_1\} \in D(A) \times D(A^{1/2})$. Then*

$$\{w, w_t\} \in L^\infty\left(0, T; D(A) \times D(A^{1/2})\right) \qquad \forall T < \infty \tag{8}$$

The proof can be found in the Appendix. ■

REMARK 1 (Notation) *In some instances, for example in Chueshov, Eller and Lasiecka (2002), the (local) bound of type (8) is denoted, with a slight abuse of notation, by $L^\infty\big([0,\infty); X\big)$. However, let us point out that in either case the estimate is not global and may blowup as $T \to \infty$.*

The ultimate goal of the present discussion will be to establish conditions under which the estimate in (8) can be made *independent of* T.

We define the *finite-energy* of the state at time t to be a topological equivalent of its phase-space norm:

$$E(t) = E(w(t), w_t(t)) = \frac{1}{2}\|w(t), w_t(t)\|^2_{\mathcal{H}} = \frac{1}{2}\|A^{1/2}w(t)\|^2 + \frac{1}{2}\|w_t(t)\|^2$$

where A is the corresponding Laplace operator on Ω. Note that due to a Poincaré-type estimate, the potential energy is equivalent to the gradient

$$c_\Omega\|\nabla w\| \le \|A^{1/2}w\| \le C_\Omega\|\nabla w\|. \tag{9}$$

We briefly summarize known results on the decay of finite-energy solutions:

Theorem 1 (Lasiecka and Toundykov, 2006) *Let g be continuous monotone increasing, $g(0) = 0$ and $f \in C^1(\mathbb{R})$ satisfy (5). Also assume that the geometrical observability condition (Assumption 1) holds. Then the energy of the system (2), (3) is non-negative and obeys the following relations:*

a) For a sufficiently large (fixed) $T > 0$ one can find a concave function h and a constant $C_{\text{dec}} > 0$ such that

$$E(T) + (h + \text{Id})^{-1}\left(\frac{1}{C_{\text{dec}}}E(T)\right) \le E(0)\,.$$

b) Asymptotic decay of the finite energy is given by

$$E(t) \le S\left(\frac{t}{T} - 1\right) \qquad \forall t \ge T$$

where $S(t)$ solves the following (monotone) ODE

$$\dot{S} + h^{-1}\left(\frac{S}{C_{\text{dec}}}\right) = 0, \qquad S(0) = E(0)\,.$$

Map h is a concave function (defined in Lasiecka, Toundykov, 2006, Section 3.2) that depends on the damping. Parameter $C_{\text{dec}} > 0$ satisfies:

- *If $g(s)$ is linearly bounded at infinity then parameter C_{dec} depends only on initial energy $E(0)$ and constant T (which, in turn, relates to the diameter of Ω). Thus, decay is uniform with respect to the norm of the initial state.*
- *If $g(s)$ is either superlinear or sublinear at infinity then C_{dec} depends on a* ***global bound on the higher energy****:*

$$\|w_t\|_{L^\infty\left(0,\infty;\, L^p(\Omega)\right)} \quad \text{and/or} \quad \|\nabla w\|_{L^\infty\left(0,\infty;\, L^p(\Omega)\right)} \tag{10}$$

for a certain $p > 2$, which may depend on damping g. So, a priori (when $f \neq 0$) we may only claim that decay is uniform with respect to the global bound on the entire trajectory in higher topology.

As was pointed out earlier, the estimate (10) on higher norms for superlinear and sublinear dissipation is known to be necessary in some models of this type (e.g. see Vancostenoble and Martinez, 2000). The underlying physical reason is that in an over-damped or under-damped system, there respectively appears an *excess* of kinetic or potential energy, which must be contained by means other than the dissipation. Naturally, one would like to know when such additional regularity is available, to begin with.

1.3. Main result: Global uniform bounds on higher energies

This section presents the main result of the paper. In some sense it states that due to the structure of the dynamics, higher-order norms of the solution cannot blowup asymptotically if the finite energy decays fast enough.

The results can be extended to general nonlinear damping; however, to make the statement clearer let us concentrate on a few specific instances of linear, superlinear and sublinear growths. Each of the three growth types will be considered in two cases: either near the origin ($g(s)$ for $|s| < 1$) or at infinity (for $|s| \geq 1$). Let $m, M > 0$ be fixed, then

(Linearly bounded) $$m|s| \leq |g(s)| \leq M|s|. \tag{11}$$

(Sublinear) $$|g(s)| = |s|^\theta, \qquad \text{some } 0 \leq \theta < 1. \tag{12}$$

(Superlinear) $$|g(s)| = |s|^r, \qquad \text{some } r > 1. \tag{13}$$

Remark 2 *There is no need for g to have a polynomial structure, rather, lower and upper polynomial bounds would suffice. The proofs, however, will become more cumbersome. To keep the presentation concise we shall focus only on the aforementioned polynomial representations of non-linearly-bounded behaviors.*

Theorem 2 (Main theorem: global bounds on higher-level energy) *Assume:*

1. *$g \in C(\mathbb{R})$ is monotone increasing with $g(0) = 0$.*
2. *$f \in C^1(\mathbb{R})$ satisfies (5) and $|f'(s)| \leq C|s|^k$. Here $0 \leq k \leq \frac{2}{2-n}$ when dimension $n \geq 3$, and $0 \leq k < \infty$ when $n = 2$. (If the bound on the derivative necessarily includes a constant term: $|f'(s)| \leq C(1+|s|^{k_1})$, then conditions of the theorem for* both *$k = 0$ and $k = k_1$ must be satisfied).*
3. *The unobserved segment Γ_U (when non-empty) satisfies the geometric Assumption 1, and:*
4. *Take smooth initial condition $\{w_0, w_1\} \in D(A) \times D(A^{1/2})$.*

*THEN: (i) the higher order energy is **globally bounded**: $\|\nabla w_t\| + \|w_{tt}\| < C$ for all $t \geq 0$; (ii) this estimate is **uniform** with respect to the higher energy of the initial state $C = C(\|\Delta w_0\| + \|\nabla w_1\|)$*

PROVIDED one of the following cases applies:

a) ***Linearly bounded damping*** *((11) $\forall s \in \mathbb{R}$).*

b) ***Sublinear damping at the origin*** *((12) $\forall |s| \leq 1$) with sub-linearity exponent $\theta \in]0,1[$; and linearly bounded damping at infinity ((11) $\forall |s| > 1$). In addition assume*

- *In 3 dimensions $2/(k+4) < \theta$, with $k \leq 2$*
- *In 2 dimensions $1/(k+2) < \theta$, with $k < \infty$.*

c) ***Superlinear damping at the origin*** *((13) $\forall |s| \leq 1$) with super-linearity exponent $r > 1$; and linearly bounded damping at infinity ((11) $\forall |s| > 1$). In addition*

- *In 3 dimensions $r < (4+k)/2$, with $k \leq 2$*
- *In 2 dimensions $r < 2+k$, with $k < \infty$.*

d) ***Sublinear damping at infinity*** *((12) $\forall |s| > 1$) and linearly bounded at the origin ((11) $\forall |s| \leq 1$). In addition, we require*

- *In 3 dimensions $(k+1)/3 < \theta$, with $k \leq 2$*
- *In 2 dimensions $0 < \theta$, with $k < \infty$.*

e) ***Superlinear damping at infinity*** *((13) $\forall |s| > 1$) and linearly bounded at the origin ((11) $\forall |s| \leq 1$). In addition*

- *In $n = 3$ dimensions assume $r < (11 + 2\sqrt{10})/9 \approx 1.925$, and the following restrictions apply:*

$$\begin{array}{llll} \textit{if} & 1 < r \leq 5/3 & \textit{then} & k \leq (14-6r)/(3r+1) \\ \textit{if} & 5/3 < r < 17/9 & \textit{then} & k < (20/3) - (5/3r) - 3r \\ \textit{if} & 17/9 \leq r < (11+2\sqrt{10})/9 & \textit{then} & k < 2 - [9(r-1)^2/(2r)]. \end{array}$$

- *In $n = 2$ dimensions assume $r \leq (3+\sqrt{5})/2 \approx 2.618$, and the following restrictions apply:*

$$\begin{array}{llll} \textit{if} & 1 < r < 2 & \textit{then} & k < (3-r)/(r-1) \\ \textit{if} & r = 2 & \textit{then} & k \leq 1 \quad \textit{(closed range)} \\ \textit{if} & 2 < r < 1+\sqrt{2} & \textit{then} & k < 5 - 2r \\ \textit{if} & 1+\sqrt{2} \leq r < (3+\sqrt{5})/2 & \textit{then} & k < 1 - (r-1)^2/r. \end{array}$$

REMARK 3 *The condition $(k+1)/3 < \theta$ for sublinear damping at infinity in Theorem 2 agrees with the special case $k = 0$ derived in Lasiecka and Toundykov (2006), Theorem 3.*

Also note that results of Theorem 2 for sublinear and superlinear damping near the origin *yield a larger range of admissible exponents when the source term is stronger. The reason for such unexpected behavior is that asymptotic decay of $\|w\|_{H^1(\Omega)}$ and higher powers of f contribute to $\|f(w)\|$ being small, which* *in some sense diminishes the influence of the source. However, this effect is relatively weak and helps only when the damping is linearly bounded at infinity, i.e. when the decay of finite energy does not involve higher-order norms.*

1.4. Examples

When studying stability, the most interesting cases of equation (2) are where the damping is *not* linearly bounded at infinity. Then, the asymptotic behavior of the finite energy $E(t)$ directly relates to the smoothness of the flow. Below we provide some examples of sub- and super-linear dissipation at infinity, and show how Theorem 2 helps us determine energy decay rates.

1.4.1. Sublinear damping at infinity

Suppose $|g(s)| \sim |s|^\theta$, some $0 < \theta < 1$, at infinity (with linear bounds near zero) and $|f(s)| \leq C|s|^{k+1}$. Then (see Theorem 2) whenever

- $0 \leq k < 3\theta - 1 \quad$ in $n = 3$ dimensions
- or any $k \geq 0 \quad$ in $n = 2$ dimensions

we can claim global bounds on the higher-level norms. Hence, finite energy will decay, Lasiecka and Toundykov (2006) at the rate $1/(q-1)$, where

$$q = (p - \theta - 1)/(p - 2)$$

provided $\sup_{t\geq 0} \|w_t\|_{L^p(\Omega)} < \infty$. Since this higher norm is globally bounded we can set $p = 6$ (in 3 dimensions), and conclude

$$E(t) \sim Ct^{-4/(1-\theta)} \quad \text{as} \quad t \to \infty \qquad (n = 3).$$

Alternatively, in $n = 2$ dimensions, due to Sobolev embedding $H^1 \to L^p$, $2 \leq p < \infty$, we can chose p to be arbitrarily large, thus obtaining arbitrarily fast polynomial decay (however, necessarily sub-exponential).

Let us emphasize again that the result in Lasiecka and Toundykov (2006) confirms such rates only when the entire trajectory is globally bounded in higher topology. However, Theorem 2 refines this statement by saying that the decay is uniform with respect to the higher-order norm of the *initial state* only, because the smoothness of the flow follows automatically.

1.4.2. Superlinear damping at infinity

Suppose damping is linearly bounded at the origin, whereas at infinity it grows as $g(s) = C\sqrt[3]{|s|}s$, $|s| \geq 1$, i.e. has the order $r = 4/3 < 5/3$. Suppose the growth rate k of the source satisfies

$$k \leq \frac{14 - 6(4/3)}{3(4/3) + 1} = \frac{6}{5}, \qquad |f(s)| \sim |s|^{k+1}.$$

For instance, if the source is quadratic: e.g. $|f(s)| = |s|s$, then, according to Theorem 2, even in 3 dimensions the regularity of the initial state will be propagated in a uniform fashion. In particular, a smooth initial condition would imply

$$\sup_{t\geq 0} (\|w_{tt}(t)\| + \|\nabla w_t(t)\|) < \infty\,.$$

From the original equation (2) we have

$$\|\Delta w\| \leq \|w_{tt}\| + C\|w_t^{4/3}\| + \|f(w)\| \leq \|w_{tt}\| + C_{\Omega,E(0)}\|\nabla w_t\| + C(E(0)) < \infty.$$

Consequently, via the Sobolev embedding results, $\|\nabla w\|_{L^6}$ is globally bounded and the finite energy decays as, see Lasiecka and Toundykov (2006, Theorem 2):

$$E(t) \sim Ct^{4r/(r-1)} = Ct^{-16} \qquad \text{when} \quad t \to \infty.$$

Take a stronger damping: $|g(s)| \leq |s|^{1.9}$ (almost at the threshold asserted by Theorem 2 in 3 dimensions). Then we can still estimate the energy decay according to the same law

$$E(t) \sim Ct^{4r/(r-1)} \approx Ct^{-8.45} \qquad \text{when} \quad t \to \infty$$

provided initial data is smooth and $|f(s)| \leq |s|^{1.08}$, e.g. when the source is linear.

Remark 4 *One may observe that in the above examples we get relatively fast algebraic decay rates. The reason for the "rapid" decrease in the energy is that Theorem 2 provides us with conditions which outright yield an extra unit of global smoothness, allowing us to take full advantage of the embedding $H^1 \to L^p$. It might be possible instead to establish weaker conditions that imply global bounds on the flow in spaces $D(A^{1/2+\eta}) \times D(A^\eta)$ (which is still above the finite energy).*

2. Proof of Theorem 2

The result of Theorem 2 follows immediately in the case of $f = 0$. Then the regularity estimate presented in Proposition 1 can be readily extended to a global bound, due to the contractive character of the flow: if g does not grow too rapidly at ∞ (i.e. the domain of the generator coincides with $D(A) \times D(A^{1/2})$) one may appeal to the classical theory of m-accretive operators (Barbu, 1993; Showalter, 1997), alternatively the same outcome follows from the proof of Proposition 1 for any monotone dissipation. Henceforth we will assume the more interesting case of $f \neq 0$.

2.1. Governing inequalities

Let $v \equiv w_t$, and introduce the high-level energy functional:

$$\mathcal{E}(t) = \mathcal{E}(v(t), v_t(t)) \equiv \|v_t\|^2_{L^2(\Omega)} + \|A^{1/2}v\|_{L^2(\Omega)}, \qquad t \geq 0. \tag{14}$$

We begin with the following "fixed point"-type estimate

PROPOSITION 2 (A priori bound on the higher energy) *Under the assumptions of Theorem 2, the following energy relation holds for any $T > 0$*

$$\boxed{\mathcal{E}(T) \leq \mathcal{E}(0) + C\int_0^T E(t)^{\frac{1}{2}+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)}\mathcal{E}(t)^{1/2+\gamma k/\pi}} \tag{15}$$

where

- *In $n = 3$ dimensions $\gamma = 3/2$ and* $\max\{2k, 2\} \leq \pi \leq 6$.
- *In $n = 2$ dimensions $\gamma = 1$ and* $\max\{2k, 2\} \leq \pi < \infty$.

Proof. To carry out the next step we may, without loss of generality, assume that g is differentiable. Alternatively, one may consider (Lipschitz) Yosida approximations of monotone function g, repeat the argument below and pass to the limit in order to arrive at the same result: estimate (17) (see Barbu, 1993, for more details). Differentiation of (2) in time shows that the strong formulation of the original problem weakly satisfies the equation

$$v_{tt} + Av + \chi g'(w_t)v_t = f'(w)v.$$

Now, $L^2(\Omega)$-inner product with v_t yields

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\|A^{1/2}v\|^2 + \int_\Omega \overbrace{\chi g'(w_t)v_t^2}^{\geq 0} = \int_\Omega f'(w)vv_t \leq C\int_\Omega |w|^k\ |v|\ |v_t|. \tag{16}$$

Since g is monotone, the integrand $\chi g'(w_t)v_t^2$ is non-negative. Discard this positive term and integrate over $[0, T]$. Pick a constant $\pi \geq \max\{2k, 2\}$, then apply Holder estimates on the RHS of (16): first with conjugate exponents $\{2,\ 2\}$, then again with $\{\pi/(2k),\ \pi/(\pi - 2k)\}$ (possibly $\{\infty,\ 1\}$ if $k = 0$, or $\{1, \infty\}$ if $\pi = 2k$):

$$\mathcal{E}(T) \leq \mathcal{E}(0) + C\int_0^T \|v_t\|\|w^k v\| \leq \mathcal{E}(0) + C\int_0^T \|v_t\|\|w\|^k_{L^\pi(\Omega)}\|v\|_{L^{(2\pi)/(\pi-2k)}(\Omega)} \tag{17}$$

To derive the next few auxiliary inequalities we will invoke: (i) the definition (14) of the high-level energy; (ii) Sobolev embeddings; (iii) interpolation estimates; as well as (iv) a Poincaré-type inequality. Recall that we have the following embedding $H^{\gamma(1-\frac{2}{\pi})} \hookrightarrow L^\pi$, where in 3 dimensions $\gamma = 3/2$ and $\max\{2k, 2\} \leq \pi \leq 6$, whereas in 2 dimensions $\gamma = 1$ and $\max\{2k, 2\} \leq \pi < \infty$ (for a general reference on Sobolev embeddings see, for instance Adams, 1975):

$$\begin{aligned} \|v_t(t)\| &\leq \mathcal{E}(t)^{1/2} \\ \|w(t)\|^k_{L^\pi} &\leq C_\Omega\|w(t)\|^k_{H^{\gamma(1-2/\pi)}} \leq C_\Omega E(t)^{(\gamma k/2)(1-2/\pi)} \end{aligned}$$

$$\|v(t)\|_{L^{2\pi/(\pi-2k)}} \le C_\Omega \|v(t)\|_{H^{2\gamma k/\pi}} \le C_\Omega \|v(t)\|^{1-2\gamma k/\pi} \|v(t)\|_{H^1}^{2\gamma k/\pi}$$
$$\le C_\Omega E(t)^{\left(\frac{1}{2}-\frac{\gamma k}{\pi}\right)} \mathcal{E}(t)^{\gamma k/\pi}.$$

Apply the last three estimates to (17):

$$\mathcal{E}\big(T\big) \le \mathcal{E}(0) + C\int_0^T E(t)^{1/2+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)} \mathcal{E}(t)^{1/2+\gamma k/\pi}. \tag{18}$$

Which completes the proof of Proposition 2. ■

Corollary 1 *Suppose that*

$$E(t) \le C\left(\sup_{\theta\in[0,t]} \mathcal{E}(\theta)\right)^{\widehat{p}_K} b(t) \tag{19}$$

for some exponent $\widehat{p}_K \ge 0$ *and a continuous function* $b(t)$ *(the strange notation* $\widehat{p}_K$ *is merely a convenience; this choice will be clearer later on). Then the higher energy is globally bounded:* $\sup_{t\in\mathbb{R}_+} \mathcal{E}(t) < \infty$*, provided both of the following conditions hold:*

$$b(t)^{\frac{1}{2}+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)} \in L^1(\mathbb{R}_+), \quad \text{and} \quad \widehat{p}_K\left[\frac{1}{2}+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)\right] + \left(\frac{1}{2}+\frac{\gamma k}{\pi}\right) \le 1$$

where

- *In* $n = 3$ *dimensions* $\gamma = 3/2$ *and* $\max\{2k, 2\} \le \pi \le 6$*;*
- *In* $n = 2$ *dimensions* $\gamma = 1$ *and* $\max\{2k, 2\} \le \pi < \infty$*.*

Proof. The *local* bound $\mathcal{E}(t) \le \text{const}$ for $t \in [0, T]$, any $T > 0$, was established in Proposition 1. We must, however, prove that this bound is global and independent of the length of the observed time-interval. For convenience let

$$\bar{\mathcal{E}}(t) := \sup_{\theta\in[0,t]} \mathcal{E}(\theta). \tag{20}$$

Now start with the estimate (15) and apply the following derivations:

- Since the RHS of (15) is monotone with respect to the upper limit of integration T, we may replace $\mathcal{E}(T)$ on the LHS by $\bar{\mathcal{E}}(T)$.
- Replace $\mathcal{E}(t)$ in the integrand with $\bar{\mathcal{E}}(t)$.
- Note that $\mathcal{E}(0) = \bar{\mathcal{E}}(0)$.
- Use the hypothesis (19) with $T = t$ to rewrite the term $E(t)$ in the integrand on the RHS of (15).

We get

$$\bar{\mathcal{E}}\big(T\big) \le \bar{\mathcal{E}}(0) + C\int_0^T b(t)^{\frac{1}{2}+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)} \bar{\mathcal{E}}(t)^{p_K\left[\frac{1}{2}+\gamma k\left(\frac{1}{2}-\frac{2}{\pi}\right)\right]+(1/2+\gamma k/\pi)} dt.$$

Now a standard Gronwall-type estimate confirms that $\bar{\mathcal{E}}(T)$ is bounded uniformly in T, provided $p_K[1/2 + \gamma k(1/2 - 2/\pi)] + (1/2 + \gamma k/\pi) \leq 1$, and $b(t)^{1/2+\gamma k(1/2-2/\pi)}$ is integrable on $[0, \infty)$. ■

2.2. Decay of finite energy

In order to make use of the Corollary (1) we must determine explicitly function $b(t)$ and the exponent $\widehat{p}_K$ that appear in (19). This information stems from our knowledge of finite-energy decay rates:

LEMMA 1 (Decay of finite energy, Lasiecka and Toundykov, 2006) *Adopt the hypotheses of Theorem 2. Then there exists a (sufficiently large) $T > 0$, so that for any $T_1 \geq T$ the decay of finite energy over the interval $[T, T_1]$ is given by*

$$E(t) \leq S\left(T^{-1}t - 1\right), \qquad T \leq t \leq T_1 \tag{21}$$

where C depends on initial energy $E(0)$, and function $S(t)$ solves a certain ODE of monotone type. When the damping g can be piece-wise bounded above and below by polynomials (including sub-linear exponents), the ODE satisfied by S reduces to:

$$\dot{S} + \left(K^{-1}S\right)^q = 0, \qquad S(0) = E(0) \tag{22}$$

for some $q \geq 1$. Constant K satisfies the estimate

$$K = K(T_1) \leq C\left(\sup_{\theta\in[0,T_1]} \mathcal{E}(\theta)\right)^{p_K}. \tag{23}$$

Moreover,

I. *Suppose g is* ***linearly bounded at infinity****: (11) $\forall |s| \geq 1$, then one can take $p_K = 0$. If, in addition,*
   i) ***Sublinear damping at the origin****: (12) $\forall |s| < 1$ with sub-linearity exponent $0 < \theta < 1$. In this case $q = (1 + \theta)/(2\theta)$.*
   ii) ***Superlinear damping at the origin****: (13) $\forall |s| < 1$ with super-linearity exponent $r > 1$. Then $q = (r + 1)/2$.*

II. *If damping g is* ***sublinear at infinity****: (12) $\forall |s| > 1$ with sub-linearity exponent $0 < \theta < 1$, then*

$$K = K(t) = C\|w_t\|^{2\lambda}_{L^\infty(0,t;L^p(\Omega))} \leq C\mathcal{E}(t)^{p_K}, \qquad p_K = \gamma\lambda\left(1 - 2/p\right) \tag{24}$$

where

- *In $n = 3$ dimensions, we take $p \in]2, 6]$ and $\gamma = 3/2$*
- *In $n = 2$ dimensions, we can take any $p > 2$ and $\gamma = 1$.*

The corresponding optimal (for the energy decay) values for q and λ are

$$q = \frac{p-\theta-1}{p-2}, \qquad \lambda = \frac{p(1-\theta)}{2(p-1-\theta)}. \tag{25}$$

III. *If damping g is* ***superlinear at infinity****: (13) $\forall |s| > 1$ with super-linearity exponent $1 < r$, then*

$$K = K(t) = C\|\nabla w\|^{2(1-\mu)/(2-\mu)}_{L^\infty(0,t;L^p(\Omega))}\,. \tag{26}$$

The optimal values for the parameters μ and q are

$$q = \frac{r(p-1)-1}{r(p-2)}\,, \qquad \mu = \frac{2(p-r-1)}{(r+1)(p-2)}\,. \tag{27}$$

Furthermore

$$K(t) \le C_1 \bar{\mathcal{E}}(t)^{p_K} + C_2 \tag{28}$$

where p_K is given by

$$p_K = \max\left\{\frac{1}{2},\ \frac{\gamma}{2}(r-1)\right\}\cdot\gamma\left(1-\frac{2}{p}\right)\frac{2(1-\mu)}{(2-\mu)} \tag{29}$$

- *In 3 dimensions, $\gamma = 3/2$, $1 < r \le 3$ and $p \in [1+r, 6]$*
- *In 2 dimensions, $\gamma = 1$, $1 < r$ and $p \ge 1+r$.*

Remark 5 *Let us note that in the 3-dimensional case, when dissipation is not linearly bounded at infinity, we impose the upper bound $p \le 6$ only to be consistent with our use of Sobolev embeddings. Finite-energy decay rates themselves only improve as p grows, Lasiecka and Toundykov (2006).*

Proof. The local bound (21) with (22), (23) follows directly from the proof of asymptotic rates in Lasiecka and Toundykov (2006). The argument is identical, with the only difference being that our inability to place fixed a priori bounds on the parameter K forces to restrict the decay result to a *bounded* interval $[0, T_1]$. The decay of finite energy is governed by ODE

$$\dot{S} + h^{-1}\left(S/C_{\mathrm{dec}}\right) = 0, \qquad S(0) = E(0). \tag{30}$$

The general algorithm for constructing map h is given in Lasiecka and Toundykov (2006). Let us summarize the special cases that are of interest to us:

I. **Linearly bounded dissipation at infinity.** In that case exponent q corresponds to exponent of the inverse of the *concave* function h in the scalar estimate

$$s^2 + g(s)^2 \le h(sg(s)) \qquad \forall |s| < 1. \tag{31}$$

a) **Sublinear damping at the origin.** When g satisfies (12) for $|s| < 1$ with sublinear exponent θ, the estimate (31) is sharp for $h(s) = 2s^{2\theta/(1+\theta)}$. Hence we can take $s^q = Ch^{-1}(s) = Cs^{(1+\theta)/(2\theta)}$, i.e. $q = (1+\theta)/(2\theta)$.

b) **Superlinear damping at the origin.** If g satisfies (13) for $|s| < 1$ with exponent $r > 1$, then (31) follows for $h(s) = 2s^{2/(1+r)}$. So we let $q = (r+1)/2$.

II. **Sublinear damping at infinity**

1. The optimal form for the function h in (30) was derived in Lasiecka and Toundykov (2006). Let us review the argument: one fixes a regularity index $p > 2$, assuming that $\|w_t\|_{L^p(\Omega)}$ is *uniformly bounded on the interval where we employ the decay estimate* (which, so far, is finite).
2. Fix some $\lambda \in [0, 1)$ and find a scalar function $h_\lambda(s)$ that satisfies: (i) $|s| \leq h_\lambda(g(s)s)$ for all $|s| > 1$; and (ii) the map $s \mapsto h_\lambda(s)^{2(1-\lambda)p/(p-2\lambda)}$ is *concave* on $\mathbb{R}^+$.
3. In the case when h_λ can be estimated by a monomial, we can directly define $s^q = C[h_\lambda^{2(1-\lambda)}]^{-1}(s)$. For a given p, the optimal value of λ is dictated by the desire to bring exponent q closer to 1 (which ultimately yields a more rapid decay). Direct computation (see Lasiecka and Toundykov, 2006, Lemma 3) verifies that identities (24) and (25) hold.

III. **Superlinear damping at infinity.** We can proceed as follows (Lasiecka and Toundykov, 2006):

1. Pick regularity index $p \geq 1 + r$ (where r is the super-linear exponent of the damping), asserting that $\|\nabla w\|_{L^p(\Omega)}$ remains uniformly bounded on the interval where we would like the finite-energy decay estimate to hold.
2. Fix some $\mu \in [0, 1]$ and find a scalar map $h_\mu(s)$ that satisfies: (i) $|g(s)| \leq h_\mu(g(s)s)$ for all $|s| > 1$; and (ii) the function $s \mapsto h_\lambda(s)^{2p/[p(2-\mu)-2(1-\mu)]}$ is concave on $\mathbb{R}^+$.
3. When h_λ can be estimated by a monomial, we similarly let $s^q := \text{const}\left[h_\mu^{(2-\mu)}\right]^{-1}(s)$. For a given p, the optimal value of μ follows when we try to bring exponent q closer to 1. Direct computation (see Lasiecka and Toundykov, 2006, Lemma 3) verifies (26) and (27).

Using this information, we need to estimate K by $\mathcal{E}(t)$ via the appropriate Sobolev embedding $H^1 \hookrightarrow L^p$. However, in the superlinear at infinity case we must first relate K, i.e. the $L^p(\Omega)$ estimate on ∇w, to the higher order energy $\|w_{tt}\| + \|\nabla w_t\|$. Computing $L^2(\Omega)$ norms in the strong formulation of the original equation (2) gives

$$\|w\|_{D(A)} \leq \|v_t\|_{L^2(\Omega)} + \|v\|^r_{L^{2r}} + C(E(0)).$$

From the Sobolev embedding $H^{\gamma\left(1-\frac{1}{r}\right)}(\Omega) \hookrightarrow L^{2r}(\Omega)$ (where $\gamma = 3/2$ if $n = 3$ and $\gamma = 1$ if $n = 2$) and interpolation estimates, we derive

$$\|w(t)\|_{D(A)} \leq C_1 \mathcal{E}(t)^{1/2} + C_\Omega E(t)^{(r-\gamma(r-1))/2} \mathcal{E}(t)^{\gamma(r-1)/2} + C(E(0)). \tag{32}$$

Note: because K is the *supremum* value over the observed time-interval, we cannot take advantage of the decay given by the factor $E(t)^{(r-\gamma(r-1))/2}$ in (32), and we can merely bound $E(t)$ by a constant dependent on the initial condition. For the same reason we will not gain anything by singling out finite energy $E(t)$ in the interpolation of $\|\nabla w\|$, so we just write $\|\nabla w\|_{L^p(\Omega)} \leq C(E(0))\|w\|^{\gamma(1-2/p)}_{D(A)}$; now apply this last relation, along with (32), to inequality (26) in order to obtain (29).

This completes the proof of Lemma 1. ∎

The next result is a direct combination of Lemma 1 and Corollary 1.

Corollary 2 (Global bounds on higher energy under algebraic decay) *Adopt the hypotheses of Theorem 2. Suppose the damping exhibits non-linearly bounded behavior (12) or (13) either at the origin or at infinity.*

Let q and p_K be given by Lemma 1. Then the higher energy $\mathcal{E}(t)$ stays globally bounded if

$$\boxed{\begin{array}{r} \left[\left(\dfrac{1}{q-1}\right)\left(\dfrac{1}{2} + \gamma k\left(\dfrac{1}{2} - \dfrac{2}{\pi}\right)\right)\right] > 1 \\ p_K \cdot q\left[\left(\dfrac{1}{q-1}\right)\left(\dfrac{1}{2} + \gamma k\left(\dfrac{1}{2} - \dfrac{2}{\pi}\right)\right)\right] + \left(\dfrac{1}{2} + \dfrac{\gamma k}{\pi}\right) \leq 1 \end{array}} \tag{33}$$

- *In 3 dimensions, $\gamma = 3/2$, $1 < r \leq 3$, and $\pi \in [\max\{2k, 2\}, 6]$.*
- *In 2 dimensions, $\gamma = 1$, $1 < r$, and $\pi \in [\max\{2k, 2\}, \infty[$*

Proof. Linearly bounded dissipation is much easier to handle, so this corollary concerns only the case when $q > 1$ in (22), which results in algebraic decay rates of the finite energy. Let $K(t)$ and p_K be given by Lemma 1 depending on the nature of the dissipation. Note that in the case of superlinear damping at infinity, we can ignore without loss of generality constant C_2 in (28): quantity $K(t)$ ultimately contributes to the exponent $\widehat{p}_K$ in (19) of Corollary 1, and according to the same Corollary we would like to mollify $\widehat{p}_K$. Hence $K(t) = C_2 = \text{const}$ requires weaker assumptions than imposed by the principal term in (28). Combine (21) with (22):

$$E(t) \leq C\left(\frac{K(t)^q}{(t/T-1)}\right)^{\frac{1}{q-1}} \leq C_T \bar{\mathcal{E}}(t)^{p_K \frac{q}{q-1}} \left(\frac{1}{t}\right)^{\frac{1}{q-1}}, \qquad t > T.$$

Thus, using the notation of Corollary 1

$$\widehat{p}_K = p_K \frac{q}{q-1} \quad \text{and} \quad b(t) = \begin{cases} \text{const} & t < T \\ t^{-1/(q-1)} & t \geq T \end{cases}.$$

Value of $b(t)$ near the origin could be set to $E(0)$, but it is irrelevant for the proof, as behavior on $[0, T]$ does not affect the integrability condition. Now substitute these identities into the hypothesis (19). ∎

We split the remaining discussion into cases based on the behavior of dissipation g.

2.3. Linearly-bounded damping at infinity

Throughout this subsection we assume that the damping is linearly bounded at infinity: (11) for $|s| > 1$. From Lemma 1 it then follows that the decay does not depend on the higher norms of the solutions, i.e. $p_K = 0$ in (19).

2.3.1. Linear bounds at the origin

If in addition to linear-like growth at infinity the damping also satisfies linear bounds near the origin, then (22) can be applied with $q = 1$, implying that the decay of finite energy is exponential. So, any power of function $b(t)$ (as in estimate (19) of Corollary 1) is integrable on $\mathbb{R}_+$. Thus, by Corollary 1 higher energy is bounded for any source up to and including the critical exponent level ($k + 1 = 3$ in 3 dimensions, or any $k > 0$ in 2 dimensions).

2.3.2. Sublinear damping at the origin

Suppose, in addition to linear bounds at infinity, that g behaves sub-linearly near zero: (12) for $|s| < 1$ with polynomial-type growth of order $0 < \theta < 1$. From Lemma 1 we have $p_K = 0$ and $q = (1 + \theta)/(2\theta)$. Substitute p_K and q into (33) of Corollary 2 to obtain conditions that would guarantee global bounds on higher energy:

$$\frac{2\theta}{(1-\theta)} \left(\frac{1}{2} + \gamma k \left(\frac{1}{2} - \frac{2}{\pi} \right) \right) > 1, \qquad \frac{1}{2} + \frac{\gamma k}{\pi} \leq 1\,.$$

Observe that as π increases, it is easier to satisfy both inequalities. In 3 dimensions, $\gamma = 3/2$ and the maximal value for π is 6. Simplifying, we get

$$2/(4 + k) < \theta, \qquad k \leq 2\,.$$

Note that the condition $k \leq 2$ coincides with the critical restriction on the growth of the source map. Thus, for instance, when the source attains the critical level: $f(s) \sim -s^3$, we have $k = 2$ and it suffices to assume $\theta > 1/3$.

The argument for $n = 2$ dimensions is analogous. Now we use $\gamma = 1$ and since π can be chosen arbitrarily large it suffices to ensure $1/(k + 2) < \theta$.

Note that in each case we have obeyed the condition $\pi \geq \max\{2k, 2\}$ originally imposed in Proposition 2.

2.3.3. Superlinear damping at the origin

Suppose, in addition to linear bounds at infinity, that g behaves super-linearly near zero: (12) for $|s| < 1|$ with polynomial bound of order $r > 1$.

According to Lemma 1, we have $p_K = 0$ and $q = (1+r)/2$. Substitute these relations into Corollary 2:

$$\frac{2}{(r-1)}\left(\frac{1}{2} + \gamma k\left(\frac{1}{2} - \frac{2}{\pi}\right)\right) > 1, \qquad \frac{1}{2} + \frac{\gamma k}{\pi} \leq 1\,.$$

Large values of π weaken the restrictions on k and r. Thus, in $n = 3$ dimensions ($\gamma = 3/2$) we take the critical Sobolev exponent $\pi = 6$ and find

$$r < (4 + k)/2, \qquad k \leq 2$$

(again, $k \leq 2$ is redundant, since it is assumed a priori in 3D). For example, when the source is critical: $f(s) \sim -s^3$, it suffices to have at any sub-cubic dissipation: $r < 3$.

The derivation for $n = 2$ dimensions is similar: set $\gamma = 1$, then we may take $\pi \to \infty$ and solve for strict inequalities, obtaining $r < 2 + k$.

Again, in both 2- and 3-dimensional settings we have been consistent with the requirement $\pi \geq 2k$.

2.4. Linearly-bounded damping at the origin

Throughout this subsection we will assume that $g(s)$ is linearly bounded in the vicinity of $s = 0$, i.e. (11) holds for $|s| < 1$.

2.4.1. Sublinear damping at infinity

Suppose, in addition to linear bounds at the origin, g satisfies (12) for $|s| > 1$ with exponent $0 < \theta < 1$. As before, start with case $n = 3$. From the estimates (24) and (25) of Lemma 1, we get

$$p_K = \frac{\gamma(1-\theta)(p-2)}{2(p-1-\theta)}, \quad \text{and} \quad q = \frac{p-\theta-1}{p-2}\,.$$

Substitute these identities into (33):

$$\left[\frac{(p-2)}{(1-\theta)}\left(\frac{1}{2} + \gamma k\left(\frac{1}{2} - \frac{2}{\pi}\right)\right)\right] > 1 \tag{34}$$

$$\frac{\gamma(1-\theta)}{2}\left[\frac{(p-2)}{(1-\theta)}\left(\frac{1}{2} + \gamma k\left(\frac{1}{2} - \frac{2}{\pi}\right)\right)\right] + \left(\frac{1}{2} + \frac{\gamma k}{\pi}\right) \leq 1\,, \tag{35}$$

where $p > 2$ can in general be arbitrarily large, but recall that in 3 dimensions we are working under restriction $p \leq 6$ in order to be consistent with Sobolev embeddings used to compute p_K in (24).

We also remind that γ is a constant originated from Sobolev embeddings: $\gamma = 3/2$ in 3 dimensions and $\gamma = 1$ in 2 dimensions. The above system of inequalities can be resolved as follows:

1. If we could bring the LHS of (34) arbitrarily close (from above) to 1, then inequality (35) could be replaced by a strict estimate
$$\frac{\gamma(1-\theta)}{2} + \left(\frac{1}{2} + \frac{\gamma k}{\pi}\right) < 1 \tag{36}$$
which is easier to satisfy for large values of π.
2. Thus, optimal ranges for θ and k can be obtained if we manage to bring the LHS of (34) close to 1, yet retain the ability to maximize π.
For large π (e.g. $\pi = 6$ in 3D and any $\pi < \infty$ in 2D) this optimal setup easily follows if we merely pick a suitable p that makes LHS of (34) equal $1+\delta$ for any given (small) $\delta > 0$. Such a p is always available: one can check that regardless of the dimension, the necessary range is $2 < p < 4 - 2\theta$, provided $\pi \geq 4$.
3. Thus, we can simply substitute (34) into (35) and solve the result (36) for maximal possible π. In 3 dimensions we set $\pi = 6$, $\gamma = 3/2$, which yields
$$(k+1)/3 < \theta, \qquad k \leq 2.$$
For instance, if $g(s)$ grows as $\sqrt[3]{s}$ at infinity, then we require the source $f(s)$ to be strictly sub-quadratic: $|f(s)| \leq |s|^{2-\delta}$, any $\delta > 0$.
4. When $n = 2$ we may select $\pi > \max\{2k, 2\}$ so large that the term $\gamma k/\pi$ in (36) can be disregarded. From there we readily obtain $\theta > 0$, i.e. in 2 dimensions, under sublinear damping at infinity we may choose any source as long as the damping is not saturated.

2.4.2. Superlinear damping at infinity

Suppose g satisfies (13) with exponent $1 < r$.

According to the results (27) and (29) of Lemma 1 we have
$$p_K = \max\left\{\frac{1}{2},\, \frac{\gamma}{2}(r-1)\right\} \cdot \gamma \left(1 - \frac{2}{p}\right) \frac{p(r-1)}{r(p-1)-1}, \quad \text{and} \quad q = \frac{r(p-1)-1}{r(p-2)}.$$

For convenience define
$$\mathcal{M}(r) := \max\{1,\, \gamma(r-1)\}\,; \;\; \mathcal{N}(k,p,r;\pi) := \left[\frac{r(p-2)}{(r-1)}\left(\frac{1}{2} + \gamma k\left(\frac{1}{2} - \frac{2}{\pi}\right)\right)\right].$$

Substitute the above identities into (33). After a few cancellations we get:

$$\mathcal{N}(k,p,r;\pi) > 1 \tag{37}$$

$$\mathcal{M}(r)\frac{\gamma}{2}\left(\frac{r-1}{r}\right)\mathcal{N}(k,p,r;\pi) + \left(\frac{1}{2}+\frac{\gamma k}{\pi}\right) \leq 1. \tag{38}$$

Another condition stated in Lemma 1 was that superlinear dissipation required the additional regularity p to be at least $1+r$:

$$r-1 \leq p-2. \tag{39}$$

We will recast the system of inequalities (37) - (39) in the following way:

i) All the terms in (38) are non-negative, so, in particular, it has to hold when $\mathcal{N}$ reaches its minimum at $p = 1+r$. Thus, substituting (39) into (38), and simplifying, gives a necessary condition:

$$k\left[\mathcal{M}(r)\gamma^2(r-1)(\pi-4)+4\gamma\right] \leq \pi\left[2-\mathcal{M}(r)\gamma(r-1)\right]. \tag{40}$$

ii) Once we know that (38) is not contradicted by $\inf_{p,\pi}\mathcal{N}$, we may need to further increase $\mathcal{N}(k,p,r;\pi)$ (by adjusting p and/or π) until inequality (37) is also satisfied. This procedure is consistent if we do not violate what was previously achieved in step (i), namely that (38) *continues to hold* even when $\mathcal{N} = 1+\delta$ for some small $\delta > 0$. Equivalently, use the strict inequality:

$$\mathcal{M}(r)\frac{\gamma}{2}\left(\frac{r-1}{r}\right) + \left(\frac{1}{2}+\frac{\gamma k}{\pi}\right) < 1 \iff k < \frac{\pi}{2\gamma}\left(1-\mathcal{M}(r)\gamma\frac{(r-1)}{r}\right). \tag{41}$$

iii) Finally, in the end we will check whether our choice of π guarantees that $\mathcal{N} > 1$ for some suitable p.

A convenient observation is that

$$\mathcal{M}(r)\gamma^2(r-1)(\pi-4)+4\gamma \geq 0. \tag{42}$$

The reason for this claim is that the only other alternatives are:

- Suppose $\mathcal{M}(r)\gamma^2(r-1)(\pi-4)+4\gamma < 0$, and $2-\mathcal{M}(r)\gamma(r-1) \geq 0$. In this case we necessarily conclude $\pi < 2$, which is outside the admissible range.
- Alternatively, one could have $\mathcal{M}(r)\gamma^2(r-1)(\pi-4)+4\gamma < 0$, and $2-\mathcal{M}(r)\gamma(r-1) < 0$, but substituting these relations into (40) implies that we must place a bound on k *from below*, thus excluding the fundamental case of a linear source. For this reason we shall not consider this scenario.

Thus, we may solve an equivalent problem:

A) From (40), (41), with (42) we readily obtain

$$\boxed{k \leq \frac{\pi\left[2-\mathcal{M}(r)\gamma\,(r-1)\right]}{\mathcal{M}(r)\gamma^2\,(r-1)\,(\pi-4)+4\gamma}, \quad k < \frac{\pi}{2\gamma}\left(1-\mathcal{M}(r)\gamma\frac{(r-1)}{r}\right)} \tag{43}$$

B) To guarantee $k \geq 0$, we must have

$$\boxed{\begin{array}{l} \mathcal{M}(r)\gamma(r-1) \leq 2 \quad (\text{unless } \pi = 2 \text{ and } \gamma = 1) \\ \mathcal{M}(r)\gamma(r-1) < r \end{array}} \tag{44}$$

Case $\pi \searrow 2$, $\gamma = 1$ is special because it yields cancellation, which removes the corresponding singularity at $\mathcal{M}(r)\gamma(r-1) = 2$.

C) In the end we will need to check that we have been consistent with (42), and that the choice of π does not contradict existence of p that could yield (37). For instance, note that both conditions would hold if π is large enough (e.g. $\pi \geq 4$).

Two-dimensional case

Set $\gamma = 1$, and recall that $\max\{2k, 2\} \leq \pi < \infty$.

1. Suppose $1 < r \leq 2$, then $\mathcal{M}(r) = 1$ and both conditions in (44) hold. Inequalities in (43) give:

$$k \leq \frac{\pi(3-r)}{\pi(r-1)+4(2-r)}, \qquad k < \frac{\pi}{2r}.$$

   Since $r - 2 \geq 0$ then both bounds improve as $\pi \to \infty$, thus it is sufficient to have

$$\boxed{k < (3-r)/(r-1) \quad \text{if} \quad r < 2; \qquad \text{and} \qquad k \leq 1 \quad \text{if} \quad r = 2} \tag{45}$$

   Note also that for large values of π, we can always find $p \geq 1 + r$ so that (37) holds.

2. Suppose $2 < r$, then $\mathcal{M}(r) = r - 1$. From (43) derive

$$k \leq \frac{\pi\left(2-(r-1)^2\right)}{(r-1)^2\pi + 4(1-(r-1)^2)} \tag{46}$$

$$k < (\pi/2)\left(1-(r-1)^2/r\right). \tag{47}$$

   Since $(r-1)^2 > 1$, then the RHS of the first estimate decreases with π, whereas the RHS of the second one increases. This behavior reflects in some sense the trade-off between the exponents of $E(t)$ and $\mathcal{E}(t)$ in (15). The largest range for k can be found when the two bounds coincide: equating the RHSs we solve for π to find

$$\pi = \frac{2r(2r-5)}{r^2-3r+1} \quad \text{if} \quad r < 1+\sqrt{2}, \qquad \pi = 2 \quad \text{if} \quad r \geq 1+\sqrt{2}. \tag{48}$$

Recall that a priori we have $\pi \geq \max\{2, 2k\}$ so we must "truncate" π from below for $r \geq 1+\sqrt{2}$. Requirements (44) give two upper bounds: $r \leq 1+\sqrt{2}$ and $r < (3+\sqrt{5})/2$, from which we would normally choose the (stronger) $1+\sqrt{2}$ bound. However, observe that the optimal value $\pi = 2$ found in (48) (which precisely comes into force as r exceeds $1+\sqrt{2}$) yields cancellation of singularities in (46). Thus, in the view of (48) we can state

$$\boxed{r < (3+\sqrt{5})/2}$$

Substituting π into either (46) or (47) gives

$$\boxed{k < 5 - 2r, \qquad 2 < r < 1+\sqrt{2}} \tag{49}$$

Note that on the interval $2 < r < 1+\sqrt{2}$ we have $\pi \geq \max\{2k, 2\}$, as required. Substituting k and π into (37), likewise, shows that a sufficiently large p would always satisfy (37). Also for $r \in]2, 1+\sqrt{2}[$ and π as in (48), inequality (42) holds as well. So all the necessary side-conditions are met. The value $\pi = 2$ produces

$$\boxed{k < 1 - (r-1)^2/r, \qquad 1+\sqrt{2} \leq r < (3+\sqrt{5})/2} \tag{50}$$

Again it is not hard to verify that all the a priori conditions on π and p are satisfied.

Inequalities (45) - (49) confirm the statement of Theorem 2 for superlinear damping at infinity in 2 dimensions.

<u>Three-dimensional case</u>

Recall that in three dimensions we set $\gamma = 3/2$, $\pi \in [\max\{2, 2k\}, 6]$ and $p \in [1+r, 6]$.

1. Suppose $r \leq 5/3$, then $\mathcal{M}(r) = 1$ and inequalities (43) give:

$$k \leq \frac{\pi(2-(3/2)(r-1))}{(9/4)(r-1)\pi + 6 - 9(r-1)}, \qquad k < \frac{\pi}{3}\left(1 - \frac{3(r-1)}{2r}\right)$$

since $6 - 9(r-1) \geq 0$, then both bounds improve as π increases. Take the maximal value $\pi = 6$ to find

$$k \leq (14-6r)(3r+1), \qquad k < (3-r)/r.$$

Since $r > 1$, the first of the two bounds is always smaller, and we end up with

$$\boxed{1 < r \leq 5/3 \quad \Longrightarrow \quad k \leq (14-6r)/(3r+1)} \tag{51}$$

2. Suppose $r > 5/3$, then $\mathcal{M}(r) = (3/2)(r-1)$. In this case (43) is equivalent to:

$$k \leq \left(\frac{2}{3}\right) \frac{\pi(8-9(r-1)^2)}{9(r-1)^2\pi + 4\left[4-9(r-1)^2\right]}, \quad \text{and} \quad k < \frac{\pi}{3}\left(1 - \frac{9(r-1)^2}{4r}\right). \tag{52}$$

Inequalities (44) translate into

$$r \leq 1 + 2\sqrt{2}/3 \quad \text{and} \quad \boxed{r < \left(11 + 2\sqrt{10}\right)/9 \approx 1.925}$$

from which we choose the second (stronger) bound. Since $4-9(r-1)^2 < 0$ for $r > 5/3$, then we again have a tradeoff: the first inequality in (52) improves as π increases, whereas the second one becomes more stringent. Solving for optimal π (keeping in mind $\pi \leq 6$) yields:

$$\pi = \frac{4(9r^2 - 20r + 5)}{9r^2 - 22r + 9} \ \text{ if } \ \frac{5}{4} < r \leq \frac{17}{9}; \ \pi = 6 \ \text{ if } \ \frac{17}{9} < r < \frac{11 + 2\sqrt{10}}{9}. \tag{53}$$

The fact that π had to be truncated down to 6 past $r = 17/9$, means that the second inequality in (52) at $\pi = 6$ provides the stronger bound. Substituting the expression for π into (52) gives

$$\boxed{\begin{array}{rcl} 5/3 < r \leq 17/9 & \Longrightarrow & k < (20/3) - (5/3r) - 3r \\ 17/9 < r < (11+2\sqrt{10})/9 & \Longrightarrow & k < 2 - [9(r-1)^2/(2r)] \end{array}} \tag{54}$$

Finally, let us comment that π according to (53) never goes below 5 for our range of r. One can easily check that in this case condition (37) holds (for a suitable p in the range $[1+r, 6]$); for the same reason inequality (42) is satisfied as well, and π always exceeds $2k$.

Inequalities (51) - (54) confirm the statement of Theorem 2 for superlinear damping at infinity in 3 dimensions. The proof of Theorem 2 is now complete.

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3. APPENDIX: Local (in time) bounds on higher energy

In this section we establish *local* in time a priori bounds on the higher norms of the solution, as claimed in Proposition 1. In fact, we will prove a more general result, for systems with mixed boundary conditions when the functions in the domain of the corresponding Laplacian may develop singularities at the junction of distinct boundary dynamics.

Proposition a-3 *Let g to be a continuous monotone increasing function with $g(0) = 0$. Assume the source map $f \in C^1(\mathbb{R})$ satisfies*

$$|f'(s)| \leq C_f(1 + |s|^2).$$

The boundary conditions may be of mixed type: e.g. $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2$ and

$$\left(\alpha_i \frac{\partial w}{\partial \nu} + \beta_i w\right)\bigg|_{\Gamma_i} = 0, \qquad i = 1, 2, \tag{a-55}$$

where $\bigcup \Gamma_i$ is a (non-overlapping) covering of Γ. We assert that there is either a Dirichlet, or a Robin-type segment of positive measure.

Pick smooth initial condition: $\{w_0, w_1\} \in D(A) \times D(A^{1/2})$, where $A = -\Delta$ defined on $L^2(\Omega)$ functions, which possess corresponding distributional derivatives and satisfy the above boundary conditions.

Then, wave equation (2) with mixed boundary conditions (a-55) has a strong solution which has the following regularity:

$$\{w, w_t\} \in L^\infty\left(0, T; D(A) \times D(A^{1/2})\right) \qquad \forall T < \infty.$$

Proof. We note right away that result of Proposition a-3 follows immediately in some special cases:

I. If $f = 0$ and damping g does not grow too rapidly at infinity, i.e. $g : H^1(\Omega) \to L^2(\Omega)$, then the domain of the corresponding nonlinear generator, $\mathcal{A}$, of the semi-flow for (2) is given by $D(\mathcal{A}) = D(A) \times D(A^{1/2})$. The flow in this case is non-expansive and a smooth initial state produces a trajectory which stays within the domain in a bounded fashion:

$$\|\{w, w_t\}\|_{D(\mathcal{A})} \leq \|\{w_0, w_1\}\|_{D(\mathcal{A})}.$$

II. Adding a Lipschitz source will yield the same bound but with an exponential weight:

$$\|\{w, w_t\}\|_{D(\mathcal{A})} \leq e^{\omega t}\|\{w_0, w_1\}\|_{D(\mathcal{A})} \tag{a-56}$$

where ω depends on f (e.g. Barbu, 1993, p. 204 and Theorem 1.5, p. 216).

III. If the source is locally Lipschitz, then due to the a priori bounds on the finite-energy $E(t)$ (see Lasiecka and Toundykov, 2006) for the dissipative wave equation, one can establish analogous exponential estimate, with ω dependent on the norm of the initial data.

However, when growth of g exceeds the polynomial bound of order $p^*/2$, where p^* is the critical Sobolev embedding exponent $H^1(\Omega) \to L^2(\Omega)$ (e.g. 3 in our 3-dimensional setup), then the domain of the semigroup generator should be described in a more precise fashion:

$$D(\mathcal{A}) = \big\{\{w, v\} \ : \ v \in D(A^{1/2}),\ Aw + \chi g(v) \in L^2(\Omega)\big\}. \tag{a-57}$$

In particular, it is not enough to remain in $D(\mathcal{A})$ to guarantee $w \in D(A)$. Thus, we actually begin with a more regular initial data: in $D(A) \times D(A^{1/2})$, and show that this additional regularity is still preserved by the dynamics.

In the general case, when $f \neq 0$, and when g does not necessarily map $D(A^{1/2})$ into $L^2(\Omega)$, the desired estimate can be obtained formally if we multiply the equation by $-\Delta w_t$ and integrate by parts. To make the argument rigorous, one could either consider spectral approximations of Δw_t, or invoke the theory of nonlinear m-accretive operators (Barbu, 1993; Showalter, 1997). The subsequent argument follows the latter approach and was originally proposed to us by Viorel Barbu. In essence, we will approximate damping g and the multiplier Δw_t by appropriate Yosida approximations.

Yosida approximations of A and g
Operator $A = -\Delta$ is m-accretive (of subgradient type) on $L^2(\Omega)$. For $\lambda > 0$ define the approximate identity $J_\lambda := (I + \lambda A)^{-1}$, and introduce the Yosida approximation of A:

$$A_\lambda = \lambda^{-1}\big(I - J_\lambda\big) = J_\lambda A\,.$$

Each A_λ is a bounded linear m-accretive on $L^2(\Omega)$ and $A_\lambda u \to Au$ in $L^2(\Omega)$ as $\lambda \searrow 0$, for all $u \in D(A)$ (e.g. see Barbu, 1993, Proposition 3.5, p. 104).

A priori we do not assume g to be differentiable, however, its Yosida approximation is Lipschitz continuous. In particular, g can be extended to an m-accretive operator $B : D(A^{1/2}) \to [D(A^{1/2})]'$, with Yosida approximation B_μ, which is a Nemytski operator corresponding to a monotone Lipschitz function $g_\mu(s) = g \circ (I + \mu g)^{-1}(s)$.

Approximate equations
Consider an approximation of (2)

$$w^\mu_{tt} + Aw^\mu + \chi(x)B_\mu(w^\mu_t) = f(w^\mu). \tag{a-58}$$

Function g_μ is linearly bounded at infinity, hence the domain of the semigroup generator coincides with $D(A) \times D(A^1/2)$. Because the finite energy of the system is always globally bounded (with respect to $E(0)$), f can be replaced by an appropriately chosen Lipschitz perturbation (dependent on the initial energy), e.g. see Chueshov, Eller and Lasiecka (2002, Theorem 7.2). From the theory of ω-m-monotone operators it now follows that (e.g. see Barbu, 1993, (1.15) p. 204 and Theorem 1.5 p. 216)

$$\|\{w^\mu, w^\mu_t\}\|_{D(\mathcal{A})} = \|Aw^\mu\| + \|A^{1/2}w^\mu_t\| \le e^{\omega t}\left(\|Aw_0\| + \|A^{1/2}w_1\|\right)$$

(we could take $\omega = 0$ in the purely monotone case: $f \equiv 0$).

Energy identity for Yosida approximation
Multiply the approximate equation (a-58) by $A_\lambda w_t^\mu \in L^2(\Omega)$. Examine the result term by term:

$$(w_{tt}^\mu, A_\lambda w_t^\mu) = \frac{d}{dt}\frac{1}{2}\left\|A_\lambda^{1/2} w_t^\mu\right\|^2, \qquad (Aw^\mu, A_\lambda w_t^\mu) = \frac{d}{dt}\frac{1}{2}\left\|A^{1/2}A_\lambda^{1/2} w^\mu\right\|^2.$$

Next, we claim that

$$(B_\mu u, A_\lambda u) \geq 0 \qquad \forall u \in L^2(\Omega). \tag{a-59}$$

Proceed in two steps:

i. First let us show that $(B_\mu u, Au) \geq 0$ for all $u \in D(A)$. Recall that g_μ is Lipschitz, hence a.e. differentiable and $(g_\mu(u), Au) = \int_\Omega g_\mu'(u)|\nabla u|^2 + BT$. where the boundary terms are given by

$$BT = -\int_\Gamma \frac{\partial u}{\partial \nu} g_\mu(u).$$

From (a-55) we know that any given segment $\Gamma_i \subset \Gamma$ of the boundary satisfies conditions of either Dirichlet, Neumann, or Robin type:

- [**Dirichlet**] $u\big|_{\Gamma_i} = 0 \implies g_\mu(u)\big|_{\Gamma_i} = 0 \implies BT\big|_{\Gamma_i} = 0$
- [**Neumann**] $\frac{\partial u}{\partial \nu}\big|_{\Gamma_i} = 0 \implies BT\big|_{\Gamma_i} = 0$
- [**Robin**] $\frac{\partial u}{\partial \nu}\big|_{\Gamma_i} = -\beta u$, hence $BT\big|_{\Gamma_i} = \int_{\Gamma_i} \beta g_\mu(u) u \geq 0$, by monotonicity of g_μ.

Thus, $BT \geq 0$ and, consequently $(g_\mu(u), Au) \geq 0$.

ii. Finally, to verify (a-59) recall that by monotonicity of B_μ we have $(A_\lambda u, B_\mu u - B_\mu J_\lambda u) = \frac{1}{\lambda}(u - J_\lambda u, B_\mu u - B_\mu J_\lambda) \geq 0$. From here:

$$\begin{aligned}(A_\lambda u, B_\mu u) =&(A_\lambda u, B_\mu u - B_\mu J_\lambda u) + (A_\lambda u, B_\mu J_\lambda u) \geq \\ \geq&(A_\lambda u, B_\mu J_\lambda u) = (AJ_\lambda u, B_\mu J_\lambda u) \geq 0\end{aligned}$$

since $J_\lambda u \in D(A)$ for any $u \in L^2(\Omega)$.

These steps confirm (a-59).

Thus, multiplication of the approximate equation (a-58) by $A_\lambda w_t^\mu$ produces the following relation:

$$\frac{d}{dt}\frac{1}{2}\left\|A_\lambda^{1/2} w_t^\mu\right\|^2 + \frac{d}{dt}\frac{1}{2}\left\|A^{1/2}A_\lambda^{1/2} w^\mu\right\|^2 \leq (f(w^\mu), A_\lambda w_t^\mu)$$

for any $T < \infty$. Equivalently

$$\boxed{\begin{aligned}\left\|A_\lambda^{1/2} w_t^\mu(T)\right\|^2 + \left\|A^{1/2} A_\lambda^{1/2} w^\mu(T)\right\|^2 \leq \left\|A_\lambda^{1/2} w_1\right\|^2 + \left\|A^{1/2} A_\lambda^{1/2} w_0\right\|^2 \\ + 2\int_0^T (A_\lambda^{1/2} f(w^\mu), A_\lambda^{1/2} w_t^\mu)\end{aligned}} \tag{a-60}$$

Passing to the limit

Note that the solution w^μ to the approximate equation depends on μ, but not on λ. Since A is an m-accretive operator, then $\|A_\lambda u\| \leq \|Au\|$ for $u \in D(A)$ (e.g. see Barbu, 1993, Proposition 3.2, p. 101). Using linear interpolation one can show that as $\lambda \to 0^+$, $A_\lambda^{1/2} u \to A^{1/2} u$ for $u \in D(A^{1/2})$. We get

$$\left\|A^{1/2} w_t^\mu(T)\right\|^2 + \|Aw^\mu(T)\|^2 \leq \left\|A^{1/2} w_1\right\|^2 + \|Aw_0\|^2 + 2\int_0^T \Big(A^{1/2} f(w^\mu), A^{1/2} w_t^\mu\Big) \tag{a-61}$$

By the Trotter-Kato theorem for nonlinear equations (e.g. Barbu, 1993, p. 231)

$$(w^\mu, w_t^\mu) \to (w, w_t) \quad \text{strongly in} \quad L^\infty(0,T;\mathcal{H}) \quad \text{as} \quad \mu \searrow 0 \tag{a-62}$$

where $\mathcal{H} = D(A^{1/2}) \times L^2(\Omega)$. However, before we pass to the limit in μ we will need additional a priori bounds on the source term. Without loss of generality assume $|f'(s)| \leq |s|^2$ (an additional constant summand can be handled by the same argument). We will also need the following interpolation result: $\|u\|_{L^\infty}^2 \leq \|Au\| \, \|A^{1/2}u\|$ (see Brenner and Scott, 1994, p. 39).

Suppose $u \in D(A)$ and $v \in D(A^{1/2})$, then sequentially derive:

$$\begin{aligned}|(f(u), A_\lambda v)| &\leq \|A^{1/2} f(u)\| \cdot \|A^{1/2} v\| \leq \varepsilon_0 \|\nabla f(u)\|^2 + C_{\Omega,\varepsilon_0} \|A^{1/2} v\|^2 \\ &= \varepsilon_0 \int_\Omega |f'(u)|^2 \cdot |\nabla u|^2 + C_{\Omega,\varepsilon_0} \|A^{1/2} v\|^2 \\ &\leq \varepsilon_0 \|u\|_{L^\infty}^4 \|\nabla u\|^2 + C_{\Omega,\varepsilon_0} \|A^{1/2} v\|^2 \\ &\leq \varepsilon_0 \|Au\|^2 \|A^{1/2} u\|^2 \|\nabla u\|^2 + C_{\Omega,\varepsilon_0} \|A^{1/2} v\|^2.\end{aligned} \tag{a-63}$$

Now set $u = w^\mu$ and $v = w_t^\mu$. Observe that $\|A^{1/2} w\|$ and $\|A^{1/2} w_t\|$ are locally bounded in time, which follows from (a-56) and the definition of the evolution generator (a-57) (note that we cannot yet make any claims about $\|Aw\|$); using this a priori bound and convergence (a-62), we conclude that $\|A^{1/2} w^\mu\|$ and $\|A^{1/2} w_t^\mu\|$ can be dominated by some constant $C_T = C(T, E(0), \|A^{1/2} w_1\|)$ on the interval $[0,T]$. Thus, from (a-63) with a sufficiently small ε_0 (dependent on C_T) we have

$$|(f(w^\mu), A_\lambda w_t^\mu)| \leq \varepsilon \|Aw^\mu\|^2 + C_{T,\varepsilon}. \tag{a-64}$$

From (a-61) and (a-64) conclude that

$$\left\|A^{1/2}w_t^{\mu}(T)\right\|^2+\|Aw^{\mu}(T)\|^2 \leq \left\|A^{1/2}w_1\right\|^2+\|Aw_0\|^2+\varepsilon \sup_{t\in[0,T]}\|Aw^{\mu}(t)\|^2+C_{T,\varepsilon}.$$

The RHS of this estimate is monotone in T, hence we can take supremum over $t \in [0,T]$ on the left and for a sufficiently small ε absorb the term $\varepsilon \sup_{t\in[0,T]} \|Aw^{\mu}(t)\|^2$ into the LHS:

$$(1-\varepsilon) \sup_{t\in[0,T]} \left(\left\|A^{1/2}w_t^{\mu}(t)\right\|^2 + \|Aw^{\mu}(t)\|^2\right) \leq \left\|A^{1/2}w_1\right\|^2 + \|Aw_0\|^2 + C_{T,\varepsilon}.$$

The RHS is independent of μ, so with the help of this a priori bound and (a-62) we can let $\mu \to 0^+$, which implies the statement of Proposition a-3. ■

References

Adams, R.A. (1975) *Sobolev Spaces.* Academic Press, New York-London.

Barbu, V. (1993) *Analysis and Control of Nonlinear Infinite Dimensional Systems.* Academic Press Inc., Boston, MA.

Bardos, C., Lebeau, G. and Rauch, J. (1992) Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30**, 1024 – 1065.

Brenner, S.C. and Scott, L.R. (1994) *The Mathematical Theory of Finite Element Method.* Springer.

Chueshov, I., Eller, M. and Lasiecka, I. (2002) On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation. *Comm. Partial Differential Equations* **27**, 1901 – 1951.

Grisvard, P. (1985) *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston, MA.

Grisvard, P. (1989) Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités. *J. Math. Pures et Appl.* **68**, 215 – 259.

Georgiev, V. and Todorova, G. (1994) Existence of solutions of the wave equation with nonlinear damping and source term. *J. Differential Equations* **109**, 295–308.

Haraux, A. (1987) Semi-linear Hyperbolic Problems in Bounded Domains. *Math. Rep.* **3**, i–xxiv.

Haraux, A. (1981) *Nonlinear Evolution Equations - Global Behaviour of Solutions.* Springer Verlag.

Komornik, V. (1994) *Exact Controllability and Stabilization, The Multiplier Method.* Masson-John Wiley, Paris.

Lagnese, J. (1983) Decay of the solution of the wave equation in a bounded region with boundary dissipation. *J. Diff. Equations* **50**, 163–182.

Lasiecka, I. (2002) *Mathematical Control Theory of Coupled PDEs.* SIAM.

LASIECKA, I. and TATARU, D. (1993) Uniform boundary stabilization of semilinear wave equation with nonlinear boundary dissipation. *Diff. and Integral Equations* **6**, 507–533.

LASIECKA, I., TRIGGIANI, R. and ZHANG, X. (2000) Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot. In: *Differential Geometric Methods in the Control of Partial Differential Equations (Boulder, CO, 1999)*, AMS, Providence, RI, 227-325.

LASIECKA, I. and TOUNDYKOV, D. (2006) Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms. *Nonlinear Analysis* **64**, 1757 –1797.

SERRIN, J., TODOROVA, G. and VITILLARO, E. (2003) Existence for a nonlinear wave equation with damping and source terms. *Integral and Differential Equations* **16**, 13–50.

SHOWALTER, R. E. (1997) *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations.* AMS, Providence, RI.

TOUNDYKOV, D. (2007) Optimal decay rates for solutions of nonlinear wave equation with localized nonlinear dissipation of unrestricted growth and critical exponents source terms under mixed boundary conditions. *Nonlinear Analysis T.M.A.* **67**, 512 – 544.

VANCOSTENOBLE, J. and MARTINEZ, P. (2000) Optimality of energy estimates for the wave equation with nonlinear boundary velocity damping. *SIAM J. Control Optim.* **39**, 776–797.