Control and Cybernetics

vol. **36** (2007) No. 3

Radiality and semismoothness

by

Abderrahim Jourani

Université de Bourgogne, Institut de Mathématiques de Bourgogne
CNRS 5584
B.P. 47870, 21078 Dijon, France
e-mail: jourani@u-bourgogne.fr.

Abstract: We provide sufficient conditions for radiality and semismoothness. In general Banach spaces, we show that calmness ensures Dini-radiality as well as Dini-convexity of solution set to inequality systems. In finite dimensional spaces, we introduce the concept of Clarke-radiality and semismoothness of order m and show that each subanalytic set satisfies these properties. Similar properties are obtained for locally Lipschitzian subanalytic functions.

Keywords: Dini-radiality, Clarke-radiality, semismoothness, subdifferential, calmness, subanalytic set.

1. Introduction

Consider a Banach space X and let S be a closed subset of X. The distance of point x from the set S is given by

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

In the setting of Hilbert spaces, we know that when S is convex then each point $x \in X$ has a unique projection $P(x)$ on S, that is,

$$d(x, S) = \|x - P(x)\|.$$

But without convexity, it may happen that this projection can not exist or $P(x)$ is a set-valued mapping. Shapiro (1994) showed that existence and uniqueness of the projection may be ensured by a special class of non necessarily convex sets, called nearly convex sets. Following Shapiro, S is *nearly convex* at $\bar{x} \in S$ if there exists $m > 0$ such that

$$d(x, y + K(S, y)) = O(\|x - y\|^m) \text{ as } x, y \to \bar{x} \text{ in } S$$

where $K(S, y)$ denotes the contingent (or Bouligand) cone to S at y.

Closely to this notion, Lewis considered the following weaker concept: The set S is said to be *nearly radial* at $\bar{x} \in S$ if

$$d(\bar{x}, x + K(S, x)) = o(\|\bar{x} - x\|) \text{ as } x \to \bar{x} \text{ in } S.$$

He uses it in the study of robust regularization of real-valued functions.

An other concept treated in the present paper is semismoothness. This notion was introduced by Mifflin (1977) for studying necessary and sufficient conditions in constrained optimization problems. It was also used in several areas, namely for studying successive relaxation method in constrained optimization (see Miffin, 1977a; Konov, 1983 and references therein), or for giving sufficient conditions for calmness (see Henrion, Outrata, 2001 and references therein).

Our aim in this paper is to introduce new notions of radiality and semismoothness. One of them is weaker that the previous ones since it is expressed in terms of the (lower) Dini directional derivative, called Dini-radiality. The other one is given in terms of the Clarke's tangent cone, called Clarke's radiality.

In general Banach spaces, we show that calmness ensures Dini-radiality as well as Dini-convexity of the solution set to inequality systems. In finite dimensional spaces, we show that each subanalytic set is Clarke's radial and is semismooth. Similar properties are established for locally Lipschitzian subanalytic functions.

2. Notions of nonsmooth analysis

This section contains some background material on nonsmooth analysis. We give only concise definitions that will be needed in the paper. For more detailed information on the subject our references are Clarke (1983), Clarke et al. (1998), Mordukhovich (2005).

Let X be a Banach space, X^* be its topological dual with pairing $\langle \cdot, \cdot \rangle$ and $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function such that $f(\bar{x}) < +\infty$. The Clarke's subdifferential of f at $\bar{x}$ is the set

$$\partial_c f(\bar{x}) = \{x^* \in X^* : \quad \langle x^*, h \rangle \leq f^0(\bar{x}, h) \, \forall h \in X\}$$

where $f^0(\bar{x}, h)$ denotes the Clarke's directional derivative of f at $\bar{x}$ in the direction h, that is,

$$f^0(\bar{x}, h) = \limsup_{\substack{t \to 0^+ \\ x \to \bar{x}}} \frac{f(x + th) - f(x)}{t}.$$

The (lower) Dini directional derivative of f at $\bar{x}$ in the direction h is

$$f^-(\bar{x}, h) = \liminf_{\substack{t \to 0^+ \\ u \to h}} \frac{f(\bar{x} + tu) - f(\bar{x})}{t}.$$

Let $C \subset X$, with $x_0 \in C$. The contingent cone $K(C, x_0)$ to C at x_0 is the set given by the following Kuratowski limit:

$$K(C, x_0) = \limsup_{t \to 0^+} \frac{C - x_0}{t}$$

and the Clarke's tangent cone $T_c(C, x_0)$ to C at x_0 is the set of $v \in X$ for which $d_C^0(x_0, v) = 0$.

The Clarke's normal cone to C at x_0 is defined by

$$N_c(C, x_0) = \{x^* \in X^* : \quad \langle x^*, v\rangle \le 0 \quad \forall v \in T_c(C, x_0)\}.$$

It is obvious that for all $h \in X$ we have

$$f^-(x_0, h) \le f^0(x_0, h) \text{ and } T_c(C, x_0) \subset K(C, x_0).$$

For the reader's convenience, we will denote by

$$d_S^-(x, h) \text{ and } d_S^0(x, h)$$

the Dini-directional derivative and the Clarke directional derivative, respectively, of the distance function to the set S at x in the direction h.

Finally, we will denote by $B(x, r)$ the closed ball centred at x and of radius r and, if not specified, the norm in a product of two Banach spaces is defined by $\|(a, b)\| = \|a\| + \|b\|$.

3. Radiality of inequality systems in Banach spaces

To ensure existence and differentiability of metric projections in Hilbert spaces for non necessarily convex sets, Shapiro introduced the concept of near convexity or $o(1)$-convexity. A set A in some Banach space X is said to be nearly convex at $\bar{x} \in A$ if

$$d(y, x + K(A, x)) = o(\|x - y\|) \text{ as } x, y \to \bar{x} \text{ in } A. \tag{1}$$

The set A is nearly convex if it is nearly convex at every point in A.

In order to study robust regularization of real-valued functions, Lewis considered the following weaker concept:

Definition 1 *A set A in some Banach space X is said to be nearly radial at $\bar{x} \in A$ if*

$$d(\bar{x}, x + K(A, x)) = o(\|\bar{x} - x\|) \text{ as } x \to \bar{x} \text{ in } A. \tag{2}$$

The set A is nearly radial if it is nearly radial at every point in A.

Now, we introduce another concept of radiality and near convexity in terms of the lower Dini directional derivative of the distance function.

Definition 2 *1) A set A in some Banach space X is said to be nearly Dini-radial (for short D-radial) at $\bar{x} \in A$ if*

$$d_A^-(x, \bar{x} - x)) = o(\|\bar{x} - x\|) \text{ as } x \to \bar{x} \text{ in } A. \tag{3}$$

The set A is nearly D-radial if it is nearly D-radial at every point in A.
2) A is said to be nearly Dini-convex (for short D-convex) at $\bar{x} \in A$ if

$$d_A^-(x, y - x)) = o(\|y - x\|) \text{ as } y, x \to \bar{x} \text{ in } A. \tag{4}$$

The set A is nearly D-convex if it is nearly D-convex at every point in A.

Clearly, convex sets are nearly D-convex and hence D-radially convex.

Taking into account the following inequality

$$d_A^-(x, h) \le d(h, K(A, x)) \quad \forall h$$

we clearly see that near radiality (respectively convexity) implies near D-radiality (respectively D-convexity). As in finite dimension the previous inequality holds as equality, both concepts coincide.

In this section, we are concerned with near D-radiality and near D-convexity of the solution set S to the following inequality system

$$\text{Find } x \in A, \quad \text{satisfying} \quad F(x) \in B \tag{5}$$

where $A \subset X$, $B \subset Y$ are closed sets, $F : X \mapsto Y$ is a mapping and X and Y are Banach spaces.

We say that the system (5) is calm at some solution point $\bar{x} \in S$ if there exist two real numbers $a > 0$ and $r > 0$ such that

$$d(x, S) \le a d(F(x), B) \quad \forall x \in B(\bar{x}, r) \cap A. \tag{6}$$

Now, we can state the following result whose proof is very simple to obtain.

Theorem 1 *Let $\bar{x} \in S$. Suppose that*
i) the mapping F is continuously differentiable at $\bar{x}$;
ii) the system (5) is calm at $\bar{x}$;
iii) A is convex.
Then
α) the near D-radiality of B at $F(\bar{x})$ implies the near D-radiality of S at $\bar{x}$.
β) the near D-convexity of B at $F(\bar{x})$ implies the near D-convexity of S at $\bar{x}$.

Proof. Since the map F is continuously differentiable at $\bar{x}$, there exist $k_F > 0$ and $s > 0$ such that

$$\|F(x) - F(\bar{x})\| \le k_F \|x - \bar{x}\| \quad \forall x \in B(\bar{x}, s). \tag{7}$$

Let $\delta = \min(r, s)$ where r satisfies relation (6). Then for all $x \in B(\bar{x}, \delta) \cap S$ and $t \in]0, 1[$, we have $x + t(\bar{x} - x) \in A \cap B(\bar{x}, \delta)$ and

$$d(x + t(\bar{x} - x), S) \le a d(F(x + t(\bar{x} - x), B).$$

Taylor expansion of F at $\bar{x}$ leads to

$$F(x + t(\bar{x} - x)) = F(x) + tD_x F(x)(\bar{x} - x) + o(t).$$

Combining the two last inequalities one obtains

$$d_S^-(x, \bar{x} - x) \leq a d_B^-(F(x), D_x F(x)(\bar{x} - x)) \text{ as } x \to \bar{x} \text{ in } S.$$

Since

$$D_x F(x)(\bar{x} - x)) = F(\bar{x}) - F(x) + o(\bar{x} - x) \text{ as } x \to \bar{x}$$

and

$$d_B^-(F(x), F(\bar{x}) - F(x) + o(\bar{x} - x)) \leq d_B^-(F(x), F(\bar{x}) - F(x)) + o(\|\bar{x} - x\|)$$

we have

$$\begin{aligned} d_S^-(x, \tfrac{\bar{x}-x}{\|\bar{x}-x\|}) &\leq a d_B^-(F(x), \tfrac{F(\bar{x})-F(x)}{\|F(\bar{x})-F(x)\|}) \times \tfrac{\|F(\bar{x})-F(x)\|}{\|\bar{x}-x\|} + \tfrac{o(\|\bar{x}-x\|)}{\|\bar{x}-x\|} \\ &\leq a k_F d_B^-(F(x), \tfrac{F(\bar{x})-F(x)}{\|F(\bar{x})-F(x)\|}) + \tfrac{o(\|\bar{x}-x\|)}{\|\bar{x}-x\|} \end{aligned}$$

Now, since B is nearly D-radial at $F(\bar{x})$ then S is nearly D-radial at $\bar{x}$. ■

As a consequence, we obtain the following corollary:

COROLLARY 1 *Consider the system (5) with A and B convex. Suppose that F is continuously differentiable at $\bar{x} \in S$ and*

$$0 \in \mathit{core}[D_x F(\bar{x})(A - \bar{x}) - (B - F(\bar{x}))].$$

Then S is nearly D-convex at $\bar{x}$.

Proof. It suffices to see (Cominetti, 1990) that the core condition implies calmness of the system (5) and to use Theorem 1. ■

In the case of $A = X$ and $B = \{0\}$, we obtain the following corollary:

COROLLARY 2 *Suppose that F is continuously differentiable at $\bar{x}$, with $F(\bar{x}) = 0$, and that $D_x F(\bar{x})X = Y$. Then, the solution set to the equation*

$$F(x) = 0 \tag{8}$$

is nearly D-convex at $\bar{x}$.

4. Radiality and semismoothness in finite dimension

Throughout this section the space $\mathbb{R}^n$ will be endowed with the usual scalar product, denoted by $\langle \cdot, \cdot \rangle$, and the associated Euclidean norm, denoted by $\| \cdot \|$.

Our aim in this section is to show that the broad class of subanalytic sets satisfies radiality and semismoothness. In fact, we will establish a more general result including both previous concepts.

For this reason, we introduce the following definition:

Definition 3 *Let $A \subset \mathbb{R}^n$ be a closed set containing $\bar{x}$ and let $m > 0$.*
1) We say that A is nearly Clarke-radial (for short C-radial) of order m at $\bar{x}$ if

$$d(\bar{x} - x, T_c(A, x)) = o(\|\bar{x} - x\|^m) \text{ as } x \to \bar{x} \text{ in } A. \tag{9}$$

2) We say that A is semismooth of order m at $\bar{x}$ if for each $x_k \to \bar{x}$ in A we have

$$\lim_{k \to \infty} \langle x_k^*, \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|^m} \rangle = 0 \quad \forall x_k^* \in \partial_c d(x_k, A).$$

All the definitions used here concerning semianalytic and subanalytic sets are borrowed from the paper by Edward Bierstone and Pierre D. Milman (1988).

Let M be a real analytic manifold. If U is an open set of M, let $\mathcal{A}(U)$ denote the ring of real analytic functions.

A subset A of M is *semianalytic* if each $a \in M$ has a neighbourhood V such that

$$V \cap A = \cup_{i=1}^{p} \cap_{j=1}^{q} \{x : \quad f_{ij}(x) \sigma_{ij} 0\}$$

where $f_{ij} \in \mathcal{A}(V)$ and $\sigma_{ij} \in \{=, >\rangle$.

We have the following representation of semianalytic sets.

Proposition 1 (Bierstone and Milman, 1988) *1) Every open semianalytic subset X of M is a finite union of semianalytic sets of the form*

$$\{x \in M : \quad f_i(x) > 0, \quad i = 1, \cdots, k\}$$

where $f_i \in \mathcal{A}(X)$.

2) Every closed semianalytic subset X of M is a finite union of semianalytic sets of the form

$$\{x \in M : \quad f_i(x) \geq 0, \quad i = 1, \cdots, k\}$$

where $f_i \in \mathcal{A}(X)$.

These sets are not stable under linear projection, that is the linear projection of a semianalytic set needs not to be semianalytic (see Bierstone and Milman, 1988). This is the reason why we consider a larger class of subsets, called subanalytic, satisfying this property. A subset X of M is *subanalytic* if each point of X admits a neighbourhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set, i. e., there is a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $X \cap U = \pi(A)$, where $\pi : M \times N \mapsto M$ is the projection.

Some very interesting properties of these sets are listed in the following proposition.

PROPOSITION 2 (Bierstone and Milman, 1988) *1) The closure of a subanalytic set is a subanalytic set.*
2) The complement of subanalytic set is a subanalytic set.
3) The distance function to a subanalytic set is subanalytic.
4) The projection of a relatively compact subanalytic set is subanalytic.
5) A finite union of subanalytic sets is subanalytic.
6) A finite intersection of subanalytic sets is subanalytic.

Other characterizations of subanalytic sets can be found in Bierstone and Milman (1988).

Now we may state the following theorem:

THEOREM 2 *Let $A \subset \mathbb{R}^n$ be a closed subanalytic set containing $\bar{x}$. Then there exist $\alpha > 0$, $r > 0$ and $s > 1$ such that*

$$|\langle w, \bar{x} - x\rangle| \leq \alpha \|w\| \|\bar{x} - x\|^s \quad \forall x \in B(\bar{x}, r) \cap A \text{ and } w \in N_c(A, x).$$

Consequently, A is nearly C-radial and semismooth of order m (for some $m > 1$) at $\bar{x}$.

Proof. Consider the function $g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$g(x, w) = \Psi_A(x) - \langle w, x - \bar{x}\rangle$$

where $\Psi_A(x) = 0$ if $x \in A$ and $+\infty$ otherwise. Since tangency and normality are local notions, we may assume that A is a compact subanalytic set and hence the function g is globally subanalytic. By Corollary 16 from Bolte et al. (2007), there exist $a > 0$, $1 > r > 0$ and $\theta \in]0, 1[$ such that

$$|g(x, w) - g(\bar{x}, 0)|^\theta \leq ad((0, 0), \partial_c g(x, w)) \ \ \forall (x, w) \in [B(\bar{x}, r) \times B(0, r)] \cap \mathrm{dom} g$$

or equivalently

$$|\langle w, x - \bar{x}\rangle|^\theta \leq a[d(w, N_c(A, x)) + \|x - \bar{x}\|] \ \ \forall x \in B(\bar{x}, r) \cap A, \ \ \forall w \in B(0, r).$$

Let $x \in B(\bar{x}, r) \cap A$ and $w \in N_c(A, x) \backslash \{0\}$. Then $r\frac{w}{\|w\|} \in N_c(C, x) \cap B(0, r)$ and hence

$$|\langle r\frac{w}{\|w\|}, x - \bar{x}\rangle|^\theta \leq a\|x - \bar{x}\|.$$

The proof is then completed by taking into account the relation $d(h, T(C, x)) = \sup\{\langle x^*, h\rangle : x^* \in N_c(A, x) \cap B(0, 1)\}$ for all h. ■

Corollary 3 *Let $A \subset \mathbb{R}^n$ be a closed subanalytic set containing $\bar{x}$. Then there exists $m > 1$ such that*

$$d(\bar{x}, x + T_c(A, x)) + d(x, \bar{x} + T_c(A, x)) = o(\|\bar{x} - x\|^m) \text{ as } x \to \bar{x} \text{ in } A.$$

As $d_A^0(x, h) \leq d(h, T_c(A, x))$, we obtain :

Corollary 4 *Let $A \subset \mathbb{R}^n$ be a closed subanalytic set containing $\bar{x}$. Then there exists $m > 1$ such that*

$$d_A^0(x, \bar{x} - x) + d_A^0(x, x - \bar{x}) = o(\|\bar{x} - x\|^m) \text{ as } x \to \bar{x} \text{ in } A.$$

The following corollary is an extension of the Lewis's result from the semi-algebraic case to the subanalytic one.

Corollary 5 *Let $A \subset \mathbb{R}^n$ be a closed subanalytic set containing $\bar{x}$. Then there exists $m > 1$ such that*

$$d(\bar{x}, x + K(A, x)) + d(x, \bar{x} + K(A, x)) = o(\|\bar{x} - x\|^m) \text{ as } x \to \bar{x} \text{ in } A.$$

Proof. This follows from Corollary 3 because $T_c(A, x) \subset K(A, x)$. ■

An example of C-radial and semismooth sets is the skeleton of bounded sets. Let $\Omega \subset \mathbb{R}^n$ be an open set. The skeleton set of Ω is defined by

$$\mathcal{S}_\Omega = \{p \in \bar{\Omega} : \quad B(p, r) \subset \mathcal{C}(\Omega)\}$$

where $\mathcal{C}(\Omega)$ is the core of Ω, that is the set of closed maximal balls inscribed in the closure of Ω.

Corollary 6 *Suppose that Ω is connected and relatively compact semianalytic. Then $\mathcal{S}_\Omega$ is nearly C-radial and semismooth of order m (for some $m > 1$) at $\bar{x}$.*

Proof. It follows from Chazal and Soufflet (2004) that the set $\mathcal{S}_\Omega$ is subanalytic and Theorem 2 completes the proof. ■

Another example of a C-radial and semismooth set is the medial axis of bounded sets. Let $\Omega \subset \mathbb{R}^n$ be an open set. For each point $x \in \Omega$, we consider the set

$$N(x) = \{y \in \Omega^c : \|x - y\| = d(x, \Omega^c\}$$

where Ω^c denotes the complement of Ω.

The medial axis of Ω is the set given by

$$\mathcal{M}_\Omega = \{x \in \Omega : \quad \text{card}(N(x)) \geq 2\}.$$

Corollary 7 *Suppose that Ω is connected and relatively compact semianalytic. Then the closure of $\mathcal{M}_\Omega$ is nearly C-radial and semismooth of order m (for some $m > 1$) at $\bar{x}$.*

Proof. It follows from Chazal and Soufflet (2004) that the set $\mathcal{M}_\Omega$ is subanalytic and by Proposition 2, the closure of $\mathcal{M}_\Omega$ is also subanalytic. So, Theorem 2 completes the proof. ∎

5. Semismoothness of locally Lipschitzian functions

Similar definition of semismoothness can be given for arbitrary functions. We say that a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is semismooth at $\bar{x}$ if

$$\forall x_k \to \bar{x} \text{ and } v_k \in \partial_c f(x_k), \quad \lim_{k\to\infty} \frac{f(x_k) - f(\bar{x}) - \langle v_k, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|^m} = 0. \quad (10)$$

We recall that f *is subanalytic if its graph is a subanalytic set.*

Remark 1 *It follows from the monotonicity lemma (Van Den Dries, Miller, 1996), that if f is subanalytic locally Lipschitzian function at $\bar{x}$, then f has a directional derivative at $\bar{x}$, that is, the following limit exists (in $\mathbb{R}$) for all $h \in \mathbb{R}^n$*

$$\lim_{t\to 0^+} \frac{f(\bar{x} + th) - f(\bar{x})}{t}.$$

Theorem 3 *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a subanalytic function which is locally Lipschitzian at $\bar{x}$. Then f is semismooth of order m at $\bar{x}$, for some $m > 1$.*

Proof. We claim that there exists $m > 1$ such that for each $x_k \to \bar{x}$ and $v_k \in \partial_c f(x_k)$, for all k, we have

$$\limsup_{k\to\infty} \left|\frac{f(x_k) - f(\bar{x}) - \langle v_k, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|^m}\right| = 0.$$

Suppose that our claim is false. Then there exists an increasing function $\varphi : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\limsup_{k\to\infty} \left|\frac{f(x_k) - f(\bar{x}) - \langle v_k, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|^{1+\frac{1}{k}}}\right| = \lim_{k\to\infty} \left|\frac{f(x_{\varphi(k)}) - f(\bar{x}) - \langle v_{\varphi(k)}, x_{\varphi(k)} - \bar{x}\rangle}{\|x_{\varphi(k)} - \bar{x}\|^{1+\frac{1}{\varphi(k)}}}\right| > 0.$$

Since f is locally Lipschitzian at $\bar{x}$, the sequence $(v_{\varphi(k)})$ is bounded and we can assume that $v_{\varphi(k)} \to \bar{v}$, with $\bar{v} \in \partial_c f(\bar{x})$. Now consider the function $g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$g(x, v) = f(x) - \langle v, x - \bar{x}\rangle.$$

Since f is subanalytic and locally Lipschitzian at $\bar{x}$, then so is g. As

$$\partial_c g(x,v) = \{\partial_c f(x) - v\} \times \{\bar{x} - x\rangle$$

it follows that $(\bar{x}, \bar{v})$ is a Clarke's critical point of g.

By Bolte et al. (2007), there exist $a > 0$, $r > 0$ and $\theta \in]0,1[$ such that

$$|g(x,v) - g(\bar{x},\bar{v})|^\theta \leq a d((0,0), \partial_c g(x,v)) \quad \forall x \in B(\bar{x}, r), \quad \forall v \in B(\bar{v}, r)$$

or, equivalently,

$$|f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle|^\theta \leq a[d(v, \partial_c f(x) + \|x - \bar{x}\|] \;\; \forall x \in B(\bar{x}, r), \;\; \forall v \in B(\bar{v}, r).$$

For k sufficiently large we get

$$|f(x_{\varphi(k)}) - f(\bar{x}) - \langle v_{\varphi(k)}, x_{\varphi(k)} - \bar{x}\rangle|^\theta \leq a\|x_{\varphi(k)} - \bar{x}\|$$

which ensures that

$$\lim_{k\to\infty} |\frac{f(x_{\varphi(k)}) - f(\bar{x}) - \langle v_{\varphi(k)}, x_{\varphi(k)} - \bar{x}\rangle}{\|x_{\varphi(k)} - \bar{x}\|^{1+\frac{1}{\varphi(k)}}}| = 0$$

and this contradiction completes the proof. ■

We also have the following equivalent version:

THEOREM 4 *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a subanalytic function which is locally Lipschitzian at $\bar{x}$. Then there exists $m > 1$ such that*

$$\lim_{x\to\bar{x}} \sup_{v\in\partial_c f(x)} |\frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|^m}| = 0.$$

Consequently,

$$\lim_{x\to\bar{x}} \frac{f(x) - f(\bar{x}) - f^0(x, x - \bar{x})}{\|x - \bar{x}\|^m} = 0$$

and

$$\lim_{x\to\bar{x}} \frac{f^0(x, x - \bar{x}) + f^0(x, \bar{x} - x)}{\|x - \bar{x}\|^m} = 0.$$

Proof. We claim that there exists $m > 1$ such that

$$\limsup_{x\to\bar{x}} \sup_{v\in\partial_c f(x)} |\frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|^m}| = 0.$$

So suppose the contrary, that there exist a sequence $x_k \to \bar{x}$ and a real number $\alpha > 0$ such that

$$\lim_{k\to\infty} \sup_{v\in\partial_c f(x_k)} |\frac{f(x_k) - f(\bar{x}) - \langle v, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|^{1+\frac{1}{k}}}| > \alpha.$$

So, there exist $k_0 \in \mathbb{N}$ and a sequence (v_k) such that for all $k \geq k_0$ we have

$$v_k \in \partial_c f(x_k) \text{ and } |\frac{f(x_k) - f(\bar{x}) - \langle v, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|^{1+\frac{1}{k}}}| > \frac{\alpha}{2}$$

and this is a contradiction with Theorem 3. ■

Taking into account Remark 1 and Theorem 4, we obtain the following result:

Corollary 8 *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a subanalytic function which is locally Lipschitzian at $\bar{x}$. Then for all $h \in \mathbb{R}^n$*

$$f'(\bar{x}, h) := \lim_{t\to 0^+} \frac{f(\bar{x} + th) - f(\bar{x})}{t} = \lim_{t\to 0^+} f^0(\bar{x} + th, h).$$

Remark 2 *The last result does not assert that a locally subanalytic function f at $\bar{x}$ is Clarke regular, that is*

$$f'(\bar{x}, h) = f^0(\bar{x}, h) \quad \forall h.$$

For example, the distance function to the subanalytic set

$$A = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \quad y^2 \leq x^4\}$$

is not Clarke regular at 0.

Remark 3 *After we have sent the paper to the journal, we received the paper Bolte, Daniilidis and Lewis (2007) where the authors showed that tame functions are semismooth.*

Acknowledgment

The author wishes to express his appreciation to Professor Aris Daniilidis for careful reading and many helpful comments that improved the original manuscript.

References

Bierstone, E. and Milman, P.D. (1988) Semianalytic and subanalytic sets. *Publications de l'IHES* **67**, 5-42.

Bolte, J., Daniilidis, A., Lewis, A. and Shiota, M. (2007) Clarke subgradients of stratifiable functions. *SIAM J. Optim.* **18**, 556-572.

Bolte, J., Daniilidis, A. and Lewis, A. (2007) Tame functions are semismooth. To appear in *Math. Prog.*

Chazal, F. and Soufflet, R. (2004) Stability and finiteness properties of medial axis and skeleton. *J. Cont. Dynamical Syst* **10**, 149-170.

Clarke, F. (1983) *Optimization and Nonsmooth Analysis.* Wiley Interscience, New York.

Clarke, F.H., Ledyaev, Yu.S., Stern, R.J. and Wolenski, P.R. (1998) *Nonsmooth Analysis and Control Theory.* Springer, New York.

Cominetti, R. (1990) Metric regularity, tangent sets, and second-order optimality conditions. *Appl. Math. Optim.* **21** (3), 265-287.

Henrion, R. and Outrata, J. (2001) A subdifferential condition for calmness of multifunctions. *J. Math. Anal. Appl.* **258**, 110-130.

Konov, I.V. (1983) Subgradient successive relaxation method for solving optimization problems. *VINITI*, 531-83.

Lewis, A.S. (2002) Robust regularization. Technical report, Simon Fraser University, September 2002.

Mifflin, R. (1977a) Semismooth and semiconvex functions in constrained optimization. *SIAM J. Cont. Optim.* **15**, 959-972.

Mifflin, R. (1977b) An algorithm for constrained optimization with semismooth functions. *Math. Oper. Res.* **2**, 191-207.

Mordukhovich, B.S. (2005) *Variational Analysis and Generalized Differentiation*, Vol. 1: *Basic Theory*, Vol. 2: *Applications.* Springer, Berlin.

Shapiro, A. (1994) Existence and differentiability of metric projections in Hilbert spaces. *SIAM J. Optim.* **4**, 130-141.

Van Den Dries, L. and Miller, C. (1996) Geometries categories and o-minimal structures. *Duke Math. J.* **84**, 497-540.