Control and Cybernetics

vol. **36** (2007) No. 2

On stability of some lexicographic multicriteria Boolean problem

by

V.A. Emelichev, E.E. Gurevsky

Belarussian State University
av. Independence 4, 220050, Minsk, Belarus
e-mail: emelichev@tut.by

Abstract: We consider a multicriteria lexicographic Boolean problem of minimizing absolute deviations of linear functions from zero. We investigate the stability radius which can be understood as a limit level of independent perturbations of the parameters, for which new lexicographic optima do not appear. Lower and upper accessible bounds of the stability radius are obtained.

Keywords: vector optimization, lexicographic set, Boolean programming, stability radius.

1. Introduction

When solving practical optimization problems it is necessary to take into account various kinds of uncertainty such as lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors and etc. Therefore, widespread use of discrete optimization models in the last decades stimulated many experts to investigate various aspects of stability theory for correct and incorrect problems.

Stability of an optimization problem is usually defined as a property of continuity or semi-continuity of the set-valued mapping, which defines the choice function. Numerous works are devoted to analysis of conditions, under which a problem possesses one or another property of stability (see, e.g., Sawaragi et al., 1985). In the case where the feasible solution set is finite, the definition of (semi-)continuity becomes simpler, reducing to a property of invariance of the set of solutions under "small" perturbations of initial data. On the other hand, complexity of combinatorial structure of the feasible solution set makes a discrete problem unpredictable when its parameters change. Therefore, the "classical" approach to stability analysis is not applicable to problems of discrete optimization. In this connection a new "quantitative" direction of research has been formed. Namely, the object of investigations is the limit level of perturbations of problem parameters, saving mentioned invariance of the solution set. Such quantitative characteristic is called stability radius.

Interest of mathematicians in stability theory of multicriteria discrete optimization problems remains very high, as confirmed by the extensive publishing activity (see, e. g., Sotskov et al., 1995; Sergienko et al., 1995, 2003; Greenberg, 1998; Chakravarti and Wagelmans, 1998; Libura et al., 1998, 2004; Hoesel and Wagelmans, 1999). We continue to study various types of stability for the vector discrete optimization problems with different kinds of partial criteria and principles of optimality (see, e. g., Emelichev and Berdysheva, 1998; Emelichev and Krichko, 2001; Emelichev et al., 2002, 2004 – 2006; Bukhtoyarov et al., 2003). Here we investigate stability of the solution set to a multicriteria Boolean problem with a finite number of criteria being absolute values of linear functions ordered lexicographically under independent perturbations for the coefficients of these functions. Under stability we understand a discrete analogue of the Hausdorff upper semi-continuity of the set-valued optimal mapping determining the lexicographic choice function. Lower and upper accessible bounds for the stability radius are obtained. As corollaries we give some qualitative results.

2. Basic definitions and notations

Let m be the number of criteria, n be the number of variables, $x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n = \{0,1\}^n$, $|X| \geq 2$, A_i denotes the i-th row of matrix $A = [a_{ij}]_{m\times n} \in \mathbf{R}^{m\times n}$, $m \geq 1$, $n \geq 2$, $i \in N_m = \{1, 2, \ldots, m\}$, $b = (b_1, b_2, \ldots, b_m)^T \in \mathbf{R}^m$. Let us define a vector function on the feasible set X

$$f(x, A, b) = (|A_1x + b_1|, |A_2x + b_2|, \ldots, |A_mx + b_m|).$$

In the criterion space $\mathbf{R}^m$ we consider the binary relation of lexicographic order "$\prec$" defined as follows. For any different vectors $y = (y_1, y_2, \ldots, y_m)$ and $y' = (y'_1, y'_2, \ldots, y'_m)$, put

$$y \prec y' \iff y_k < y'_k,$$

where $k = \min\{i \in N_m : \ y_i \neq y'_i\}$.

By m-criterion (vector) lexicographic Boolean programming problem

$$Z^m(A, b) : \ \text{lex} \ \min\{f(x, A, b) : \ x \in X\}$$

we understand the problem of finding the lexicographic set

$$L^m(A, b) = \{x \in X : \ \forall\, x' \in X \quad (f(x', A, b) \ \not\prec \ f(x, A, b))\}.$$

The elements of the set $L^m(A, b)$ are called lexicographic optima.

Thus, all the criteria are ordered with respect to their importance (Chervak, 2002; Sergienko, 1988).

The following properties are evident.

Property 1 *The solution $x^0 \in X$ is a lexicographic optimum of problem $Z^m(A, b)$, if*

$$\forall\, x \in X\backslash\{x^0\} \quad \left(|A_1x + b_1| > |A_1x^0 + b_1|\right).$$

Property 2 *The solution x is not a lexicographic optimum of problem $Z^m(A, b)$, if there exists a solution $x^0 \in X$ such that*

$$|A_1x + b_1| > |A_1x^0 + b_1|.$$

Note that the vector function $f(x, A, b)$ is a measure of inconsistency (deviations) of the system of linear Boolean equations

$$Ax + b = \mathbf{0}^{(m)}, \quad x \in X, \tag{1}$$

where $\mathbf{0}^{(m)} = (0, 0, \ldots, 0)^T \in \mathbf{R}^m$.

It is easy to see that system of equations (1) is consistent if and only if the set of efficient vector evaluations

$$f(L^m(A, b)) = \{y \in \mathbf{R}^m : \ y = f(x, A, b),\ x \in L^m(A, b)\}$$

contains only vector $\mathbf{0}^{(m)}$.

For each number $k \in \mathbf{N}$, we endow the space $\mathbf{R}^k$ with two metrics l_1 and l_∞, i. e. we define norms of a vector $z = (z_1, z_2, \ldots, z_k) \in \mathbf{R}^k$ as follows:

$$||z||_1 = \sum_{j \in N_k} |z_j|, \quad ||z||_\infty = \max_{j \in N_k} |z_j|.$$

Under a norm of matrix we understand the norm of vector, composed of all its elements.

For any number $\varepsilon > 0$ we define the set of perturbing pairs:

$$\Omega(\varepsilon) = \{(A', b') \in \mathbf{R}^{m\times(n+1)} : \ \max\{||A'||_\infty, ||b'||_\infty\} < \varepsilon\}.$$

The problem $Z^m(A+A', b+b')$, where $(A', b') \in \Omega(\varepsilon)$, is called the perturbed problem.

Following Emelichev, Berdysheva (1998), Emelichev et al. (2002), where the lower and the upper bounds of the stability radius for the vector lexicographic integer linear programming problem are obtained, we call the problem $Z^m(A, b)$ stable, if the following statement holds:

$$\Xi = \{\varepsilon > 0 : \forall\ (A', b') \in \Omega(\varepsilon)\ (L^m(A + A', b + b') \subseteq L^m(A, b))\} \neq \emptyset.$$

Thus, the stability of problem $Z^m(A, b)$ can be viewed as discrete upper Hausdorff semi-continuity at point (A, b) of the following set-valued optimal mapping (see, e. g., Tanino and Sawaragi, 1980)

$$L^m : \ \mathbf{R}^{m\times(n+1)} \to 2^X,$$

which assigns $L^m(A,b)$ to the pair (A,b).

Under the stability radius of problem $Z^m(A,b)$ we understand the number

$$\rho^m(A,b) = \begin{cases} \sup\ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset. \end{cases}$$

In other words, the stability radius $\rho^m(A,b)$ is a supremum of the magnitude of perturbations (A',b') of a given pair (A,b) for which $L^m(A+A',b+b') \subseteq L^m(A,b)$.

If $L^m(A,b) = X$, the problem $Z^m(A,b)$ is stable with $\rho^m(A,b) = \infty$. The problem $Z^m(A,b)$ is called non-trivial if the set $\overline{L}^m(A,b) = X \backslash L^m(A,b)$ is non-empty.

Let us use the notations

$$\text{sg } t = \begin{cases} 1, & \text{if } t \geq 0, \\ -1, & \text{if } t < 0, \end{cases}$$

$$Q = \{-1, 1\}$$

and the implication

$$(\exists q \in Q \quad \forall q' \in Q \quad (qt > q't')) \Rightarrow |t| > |t'|, \tag{2}$$

which holds for any numbers $t,\ t' \in \mathbf{R}$.

3. Basic results

Suppose

$$\varphi^m(A,b) = \min_{x \in \overline{L}^m(A,b)} \ \max_{x' \in L^m(A,b)} \ \min_{q \in Q} \frac{|A_1(x+qx') + b_1(1+q)|}{||x+qx'||_1 + 1 + q}. \tag{3}$$

THEOREM 1 *If $Z^m(A,b)$ is non-trivial, the following estimates hold:*

$$\varphi^m(A,b) \leq \rho^m(A,b) \leq \max\{||A_1||_\infty, |b_1|\}.$$

Proof. It is easy to see that $\varphi := \varphi^m(A,b) \geq 0$.

At first we prove the inequality $\rho^m(A,b) \geq \varphi$. Without loss of generality, we suppose that $\varphi > 0$ (otherwise the inequality $\rho^m(A,b) \geq \varphi$ is evident). Consider any $(A',b') \in \Omega(\varphi)$. By (3) for any $x \in \overline{L}^m(A,b)$ there exists $x^0 \in L^m(A,b)$ such that for any $q \in Q$ we have

$$\psi < \varphi \leq \alpha(x, x^0, q), \tag{4}$$

where

$$\alpha(x, x^0, q) = \frac{|A_1(x+qx^0) + (1+q)b_1|}{||x+qx^0||_1 + 1 + q},$$

$$\psi = \max\{||A'||_\infty, ||b'||_\infty\}.$$

This means that

$$|A_1(x + qx^0) + (1 + q)b_1| > 0,\ q \in Q,$$

and in view of inclusion $x^0 \in L^m(A, b)$, we obtain

$$|A_1x + b_1| > |A_1x^0 + b_1|.$$

Hence

$$\beta(q) := A_1(\sigma x + qx^0) + b_1(\sigma + q) > 0,\ q \in Q,$$

where $\sigma = \text{sg}\ (A_1x + b_1)$.

Therefore, taking into account (4) we have

$$\begin{aligned}
&\sigma((A_1 + A'_1)x + b_1 + b'_1) + q((A_1 + A'_1)x^0 + b_1 + b'_1) = \\
&= \beta(q) + \sigma(A'_1(x + \sigma qx^0) + (1 + \sigma q)b'_1) \geq \\
&\geq \beta(q) - (||A'||_\infty ||x + \sigma qx^0||_1 + ||b'||_\infty(1 + \sigma q)) \geq \\
&\geq \beta(q) - \psi(||x + \sigma qx^0||_1 + 1 + \sigma q) > \\
&> |A_1(x+\sigma qx^0)+(1+\sigma q)b_1| - \alpha(x, x^0, \sigma q) \cdot (||x+\sigma qx^0||_1+1+\sigma q) = 0,\ q \in Q.
\end{aligned}$$

Thus, we obtain

$$\sigma((A_1 + A'_1)x + b_1 + b'_1) > q((A_1 + A'_1)x^0 + b_1 + b'_1),\ q \in Q.$$

By (2) we find

$$|(A_1 + A'_1)x + b_1 + b'_1| > |(A_1 + A'_1)x^0 + b_1 + b'_1|.$$

In view of Property 2 we conclude, that x is not a solution to $Z^m(A+A', b+b')$.

Therefore, the following formula is true:

$$\forall (A', b') \in \Omega(\varphi)\ \ (L^m(A + A', b + b') \subseteq L^m(A, b)),$$

which proves the inequality $\rho^m(A, b) \geq \varphi$.

Let us prove the upper bound:

$$\rho^m(A, b) \leq \zeta := \max\{||A_1||_\infty, |b_1|\}. \tag{5}$$

Take any $\varepsilon > \zeta$ and $x^0 \in \overline{L}^m(A, b)$. To prove the inequality (5) we construct a pair $(A^*, b^*) \in \Omega(\varepsilon)$ such that $x^0 \in L^m(A + A^*, b + b^*)$.

We show that the pair $(A^*, b^*) \in \Omega(\varepsilon)$ defined as

$$a^*_{ij} = \begin{cases} -a_{ij} + \delta, & \text{if } i = 1,\ x^0 = \mathbf{0}^{(n)},\ j \in N_n, \\ -a_{ij} - \frac{\delta}{||x^0||_1}, & \text{if } i = 1,\ x^0 \neq \mathbf{0}^{(n)},\ x^0_j = 1, \\ -a_{ij} + \delta, & \text{if } i = 1,\ x^0 \neq \mathbf{0}^{(n)},\ x^0_j = 0, \\ 0 & \text{in other cases,} \end{cases}$$

$$b_i^* = \begin{cases} -b_i, & \text{if } i = 1,\ x^0 = \mathbf{0}^{(n)}, \\ -b_i + \delta, & \text{if } i = 1,\ x^0 \neq \mathbf{0}^{(n)}, \\ 0 & \text{in other cases,} \end{cases}$$

where

$$0 < \delta < \varepsilon - \zeta,$$

satisfies the requirement $x^0 \in L^m(A + A^*, b + b^*)$.

Indeed, it is easy to see that the following assertion is true:

$$(A_1 + A_1^*)x^0 + (b_1 + b_1^*) = 0. \tag{6}$$

Let us show that

$$\forall\ x \in X \backslash \{x^0\} \quad (|(A_1 + A_1^*)x + b_1 + b_1^*| > 0). \tag{7}$$

To do this, we introduce the notations

$$N(x, x^0) = |\{j \in N_n :\ x_j = 1\ \&\ x_j^0 = 0\}|,$$
$$M(x, x^0) = |\{j \in N_n :\ x_j = 1\ \&\ x_j^0 = 1\}|.$$

It is easy to see that

$$M(x, x^0) \leq ||x^0||_1. \tag{8}$$

For any solution $x \in X \backslash \{x^0\}$ we consider two possible cases.

Case 1: $x = \mathbf{0}^{(n)}$. Then $x^0 \neq \mathbf{0}^{(n)}$. Therefore we obtain

$$(A_1 + A_1^*)x + b_1 + b_1^* = b_1 + b_1^* = \delta > 0.$$

Case 2: $x \neq \mathbf{0}^{(n)}$. We consider two possible sub-cases.

2.1: $x^0 = \mathbf{0}^{(n)}$. Then we have

$$(A_1 + A_1^*)x + b_1 + b_1^* = (A_1 + A_1^*)x = ||x||_1 \cdot \delta > 0.$$

2.2: $x^0 \neq \mathbf{0}^{(n)}$. In this sub-case we obtain

$$(A_1 + A_1^*)x + b_1 + b_1^* = (A_1 + A_1^*)x + \delta = N(x, x^0) \cdot \delta - M(x, x^0) \cdot \frac{\delta}{||x^0||_1} + \delta.$$

The right side of these equalities is denoted by ω, i. e. $\omega := N(x, x^0) \cdot \delta - M(x, x^0) \cdot \frac{\delta}{||x^0||_1} + \delta$. If $N(x, x^0) \neq 0$, then taking into account (8) we find that $\omega \geq \delta > 0$. If $N(x, x^0) = 0$, then it is easy to see, that $M(x, x^0) < ||x^0||_1$, and therefore we have $\omega > 0$.

In conclusion, the inequality (7) holds true.

Therefore, in view of (6), we find

$$\forall\, x \in X\backslash\{x^0\} \quad \left(|(A_1 + A_1^*)x + b_1 + b_1^*| > |(A_1 + A_1^*)x^0 + b_1 + b_1^*|\right).$$

From here on the basis of Property 1 we conclude that the solution x^0 is a lexicographic optimum of perturbed problem $Z^m(A + A', b + b')$. Thus,

$$\forall\, \varepsilon > \zeta \;\; \exists\, (A^*, b^*) \in \Omega(\varepsilon) \;\; (L^m(A + A^*, b + b^*) \nsubseteq L^m(A, b)).$$

Hence, for any number $\varepsilon > \zeta$, we have $\rho^m(A, b) < \varepsilon$ i. e. inequality (5) is true. ∎

THEOREM 2 *If $L^m(A, b) = \{x^*\}$, that we have*

$$\rho^m(A, b) = \min_{x \in X\backslash\{x^*\}} \gamma(x, x^*), \tag{9}$$

where

$$\gamma(x, x^*) = \min\{\alpha(x, x^*, q) : \; q \in Q\}.$$

Proof. Taking into account the inequality $\rho^m(A, b) \geq \varphi$ (Theorem 1) it remains to show that

$$\rho^m(A, b) \leq \xi,$$

where ξ is the right-hand side of equality (9). To do this we show that the following formula

$$\forall\, \varepsilon > \xi \;\; \exists\, (A', b') \in \Omega(\varepsilon) \;\; \exists\, x' \in X\backslash\{x^*\} \;\; (x' \in L^m(A + A', b + b')) \tag{10}$$

is valid.

According to the definition of number $\xi \geq 0$, there exists a solution $x^0 \in X\backslash\{x^*\}$ such that

$$\gamma(x^0, x^*) = \xi. \tag{11}$$

Let us use the notations

$$\sigma^* = \text{sg}\;(A_1 x^* + b_1),$$
$$\sigma^0 = \text{sg}\;(A_1 x^0 + b_1).$$

It is easy to see that at least one of the numbers $N(x^0, x^*)$ and $N(x^*, x^0)$ is positive and the following assertions hold:

$$\max\{N(x^*, x^0),\; N(x^0, x^*)\} \leq ||x^0 + x^*||_1, \tag{12}$$
$$N(x^*, x^0) + N(x^0, x^*) = ||x^0 - x^*||_1. \tag{13}$$

To build the needed perturbing pair $(A', b') \in \Omega(\varepsilon),\ \varepsilon > \xi$, we consider three possible cases.

Case 1: $\alpha(x^0, x^*, -1) < \alpha(x^0, x^*, 1)$. Then

$$0 < \alpha(x^0, x^*, 1) \leq \frac{|A_1x^* + b_1| + |A_1x^0 + b_1|}{||x^* + x^0||_1 + 2} \tag{14}$$

and according to (11) there exists a number $\delta < \varepsilon$ such that

$$0 \leq \alpha(x^0, x^*, -1) = \xi < \delta < \alpha(x^0, x^*, 1). \tag{15}$$

From here, setting the elements of the perturbing pair (A', b') as follows:

$$a'_{ij} = \begin{cases} \sigma^*\delta, & \text{if } i = 1,\ x_j^* = 1,\ x_j^0 = 0, \\ -\sigma^0\delta, & \text{if } i = 1,\ x_j^* = 0,\ x_j^0 = 1, \\ 0 & \text{in other cases,} \end{cases} \tag{16}$$

$$b' = \mathbf{0}^{(m)},$$

and taking into account (13) we have

$$\sigma^*((A_1 + A'_1)x^* + b_1 + b'_1) - \sigma^0((A_1 + A'_1)x^0 + b_1 + b'_1) =$$
$$|A_1x^* + b_1| - |A_1x^0 + b_1| + \delta(N(x^0, x^*) + N(x^*, x^0)) \geq$$
$$\geq -|A_1(x^0 - x^*)| + \delta||x^0 - x^*||_1 >$$
$$> -|A_1(x^0 - x^*)| + \alpha(x^0, x^*, -1)||x^0 - x^*||_1 = 0, \tag{17}$$

$$\sigma^*((A_1 + A'_1)x^* + b_1 + b'_1) + \sigma^0((A_1 + A'_1)x^0 + b_1 + b'_1) =$$
$$= |A_1x^* + b_1| + |A_1x^0 + b_1| + \delta(N(x^*, x^0) - N(x^0, x^*)). \tag{18}$$

Besides that, it is obvious, that

$$\max\{||A'||_\infty, ||b'||_\infty\} = \delta, \tag{19}$$

i.e. $(A', b') \in \Omega(\varepsilon)$.

We denote the right-hand side of equality (18) by η. If $N(x^0, x^*) = 0$, then in view of (14) we have $\eta > 0$. If $N(x^0, x^*) > 0$, then by virtue of (12), (14) and (15) we derive

$$\delta N(x^0, x^*) < |A_1x^* + b_1| + |A_1x^0 + b_1|.$$

It follows that

$$\eta > \delta N(x^*, x^0) \geq 0.$$

So, $\eta > 0$, and we obtain

$$\sigma^*((A_1 + A'_1)x^* + b_1 + b'_1) > \sigma^0 q((A_1 + A'_1)x^0 + b_1 + b'_1),\ q \in Q.$$

Therefore, using (2), we find

$$|(A_1 + A_1')x^* + b_1 + b_1'| > |(A_1 + A_1')x^0 + b_1 + b_1'|. \tag{20}$$

Case 2: $\alpha(x^0, x^*, -1) > \alpha(x^0, x^*, 1)$. Then according to (11) there exists a number $\delta < \varepsilon$, such that

$$0 \le \alpha(x^0, x^*, 1) = \xi < \delta < \alpha(x^0, x^*, -1).$$

From here, setting the elements of the perturbing pair (A', b') as follows:

$$a'_{ij} = \begin{cases} -\sigma^0\delta, & \text{if } i = 1,\ j \in N_n, \\ 0, & \text{if } i \neq 1,\ j \in N_n, \end{cases}$$
$$b'_i = \begin{cases} -\sigma^0\delta, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1, \end{cases}$$

we derive

$$\begin{aligned}
&-\sigma^0((A_1 + A_1')x^* + b_1 + b_1') - \sigma^0((A_1 + A_1')x^0 + b_1 + b_1') = \\
&= -\sigma^0(A_1(x^0 + x^*) + 2b_1) + \delta(||x^*||_1 + ||x^0||_1 + 2) \\
&> -|A_1(x^0 + x^*) + 2b_1| + \alpha(x^0, x^*, 1)(||x^0 + x^*||_1 + 2) = 0,
\end{aligned}$$

$$\begin{aligned}
&-\sigma^0((A_1 + A_1')x^* + b_1 + b_1') + \sigma^0((A_1 + A_1')x^0 + b_1 + b_1') = \\
&= \sigma^0 A_1(x^0 - x^*) - \delta(||x^0||_1 - ||x^*||_1) = \\
&= |A_1(x^0 - x^*)| - \delta(||x^0||_1 - ||x^*||_1) > \\
&> |A_1(x^0 - x^*)| - \alpha(x^0, x^*, -1)||x^0 - x^*||_1 = 0,
\end{aligned}$$

besides this, equality (19) is true, i. e. $(A', b') \in \Omega(\varepsilon)$.

Therefore, the following inequalities hold:

$$-\sigma^0((A_1 + A_1')x^* + b_1 + b_1') > \sigma^0 q((A_1 + A_1')x^0 + b_1 + b_1'),\ q \in Q.$$

Wherefrom by virtue of (2) we obtain inequality (20).

Case 3: $\alpha := \alpha(x^0, x^*, -1) = \alpha(x^0, x^*, 1) = \gamma(x^0, x^*)$. Then

$$\alpha \le \frac{|A_1 x^* + b_1| + |A_1 x^0 + b_1|}{||x^* + x^0||_1 + 2}. \tag{21}$$

Consider two possible variants.

At first $\alpha = 0$. Then the following equalities are evident:

$$A_1 x^0 + b_1 = A_1 x^* + b_1 = 0. \tag{22}$$

If $N(x^*, x^0) > 0$, then setting the elements of the perturbing pair (A', b') by

$$a'_{ij} = \begin{cases} \delta, & \text{if } i = 1,\ x^*_j = 1,\ x^0_j = 0, \\ 0 & \text{in other cases,} \end{cases}$$
$$b' = \mathbf{0}^{(m)},$$

where

$$0 \leq \xi < \delta < \varepsilon,$$

we obtain, that equality (19) is true, and according to (22) inequality (20) is also true.

If $N(x^*, x^0) = 0$, then $N(x^0, x^*) > 0$, i. e. there exists an index $p \in N_n$, such that

$$x^0_p = 1, \quad x^*_p = 0.$$

Then, setting the elements of the perturbing pair (A', b') by

$$a'_{ij} = \begin{cases} -\delta/2, & \text{if } (i,j) = (1,p), \\ 0, & \text{if } (i,j) \neq (1,p), \end{cases}$$
$$b'_i = \begin{cases} \delta, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1, \end{cases}$$

where

$$0 \leq \xi < \delta < \varepsilon,$$

we have that in view of (22) inequality (20) is true and $(A', b') \in \Omega(\varepsilon)$.

Let now $\alpha > 0$. Suppose

$$\alpha = \xi < \delta < \varepsilon. \tag{23}$$

We build a perturbing pair (A', b') by formula (16). Then the relations (17), (18) and (19) are true. As in case 1, we show, that $\eta > 0$. If $N(x^0, x^*) = 0$, then according to (21) we obtain $\eta > 0$. If $N(x^0, x^*) > 0$, then, taking into account (12) and (21), we may impose the following condition on the number δ in addition to condition (23):

$$\delta N(x^0, x^*) < |A_1 x^* + b_1| + |A_1 x^0 + b_1|.$$

Therefore (see case 1) we have $\eta > \delta N(x^*, x^0) \geq 0$. Hence, in this case inequality (20) is true.

So in each of three possible cases we have built a perturbing pair $(A', b') \in \Omega(\varepsilon)$ such that inequality (20) is true. This inequality, in view of Property 2, shows us that the solution x^* is not a lexicographic optimum of the perturbed problem $Z^m(A+A', b+b')$. Therefore, taking into account $L^m(A+A', b+b') \neq \emptyset$ we see that there exists a solution $x' \neq x^*$ such that $x' \in L^m(A+A', b+b')$.

Summarizing the above, we conclude that formula (10) is true. Therefore we have $\rho^m(A, b) \leq \xi$. ■

4. Corollaries

The theorems imply several corollaries.

Let us introduce the set

$$S^m(A, b) = \{x \in X : \ \forall\, x' \in X \ \ (|A_1 x + b_1| \leq |A_1 x' + b_1|)\}. \tag{24}$$

It is evident that the inclusion $L^m(A, b) \subseteq S^m(A, b)$ holds for any parameters $A \in \mathbf{R}^{m \times n},\ b \in \mathbf{R}^m$.

Corollary 1 *If the equality*

$$L^m(A, b) = S^m(A, b) \tag{25}$$

holds, then problem $Z^m(A, b)$ is stable.

Proof. To prove Corollary 1 it is enough to consider the case where problem $Z^m(A, b)$ is non-trivial. Let equality (25) be true. Then in view of (24), for any solution $x \in \overline{L}^m(A, b)$ there exists a lexicographic optimum $x' \in L^m(A, b)$ such that

$$|A_1 x + b_1| > |A_1 x' + b_1|.$$

Therefore we conclude that

$$\begin{aligned}
&\forall\, q \in Q \ \ \forall \ \ x \in \overline{L}^m(A, b) \ \ \exists\, x' \in L^m(A, b) \\
&|A_1(x + qx') + b_1(1 + q)| \geq |A_1 x + b_1| - |A_1 x' + b_1| > 0, \\
&||x + qx'||_1 + 1 + q > 0.
\end{aligned}$$

From here we have $\varphi^m(A, b) > 0$. It means (from Theorem 1) that problem $Z^m(A, b)$ is stable. ■

Since equality $L^1(A, b) = S^1(A, b)$ is true, Corollary 1 implies

Corollary 2 *Scalar (single-criterion) problem $Z^1(A, b)$ is stable for any $A \in \mathbf{R}^n,\ b \in \mathbf{R}$.*

Corollary 3 *Let $L^m(A, b) = \{x^0\}$. Problem $Z^m(A, b)$ is stable if and only if $S^m(A, b) = \{x^0\}$.*

Proof. Sufficiency follows from Corollary 1.

We prove the *necessity* by contradiction. Let the inequality $S^m(A,b) \neq \{x^0\}$ be true. Then there exists a solution x^*, belonging to the set $S^m(A,b)\backslash\{x^0\}$. It is easy to see that in this case the following equality holds:

$$|A_1x^0 + b_1| = |A_1x^* + b_1|.$$

Therefore, there exists a number $q \in Q$ such that

$$A_1(x^* + qx^0) + b_1(1+q) = 0.$$

From here, by virtue of Theorem 2 (in view of (9)) we conclude that $\rho^m(A,b)=0$. The last equality contradicts the stability of problem $Z^m(A,b)$. ■

5. Examples

Let us illustrate some of the obtained results.

Example 1 Let $m = 2$, $n = 3$, $X = \{x^1, x^2, x^3\}$, $x^1 = (1,1,0)^T$, $x^2 = (0,1,1)^T$, $x^3 = (1,0,0)^T$,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Then

$$f(x^1, A, b) = (1,1),$$
$$f(x^2, A, b) = (0,0),$$
$$f(x^3, A, b) = (2,1).$$

Therefore, $L^2(A,b) = \{x^2\}$ and in view of Theorem 2 we obtain $\rho^2(A,b) = \frac{1}{6}$.

Example 2 Let $m = 2$, $n = 3$, $X = \{x^1, x^2, x^3\}$, $x^1 = (1,1,0)^T$, $x^2 = (0,1,1)^T$, $x^3 = (1,0,0)^T$,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Then

$$f(x^1, A, b) = (0,1),$$
$$f(x^2, A, b) = (0,0),$$
$$f(x^3, A, b) = (1,1).$$

Therefore $L^2(A,b) = \{x^2\} \neq \{x^1, x^2\} = S^2(A,b)$ and by Corollary 3 we have $\rho^2(A,b) = 0$, i.e. the problem $Z^2(A,b)$ is not stable.

Acknowledgments

The authors are grateful to an anonymous referee for a very careful reading of the paper and for numerous specific suggestions that improved the presentation tremendously.

References

Bukhtoyarov, S., Emelichev, V. and Stepanishina, Y. (2003) Stability of discrete vector problems with the parametric principle of optimality. *Cybernetics and Systems Analysis* **39** (4), 604–614.

Chakravarti, N. and Wagelmans, A. (1998) Calculation of stability radius for combinatorial optimization. *Operations Research Letters* **23** (1), 1–7.

Chervak, Yu. (2002) Unimprovable choice (in Ukrainian). Uzhgorod National University, Uzhgorod.

Emelichev, V. and Berdysheva, R. (1998) On the radii of steadiness, quasi-steadiness, and stability of a vector trajectory problem on lexicographic optimization. *Discrete Math. Appl.* **8** (2), 135–142.

Emelichev, V. and Krichko, V. (2001) Stability radius of an efficient solution to a vector quadratic problem of Boolean programming. *Comput. Math. and Mathem. Physics.* **41** (2), 322–326.

Emelichev, V., Girlich, E., Nikulin, Yu. and Podkopaev, D. (2002) Stability and regularization of vector problems of integer linear programming. *Optimization* **51** (4), 645 – 676.

Emelichev, V., Krichko, V. and Nikulin, Yu. (2004) The stability radius of an efficient solution in minimax Boolean programming problem. *Control and Cybernetics* **33** (1), 127–132.

Emelichev, V., Kuz'min, K. and Leonovich, A. (2004) Stability in the combinatorial vector optimization problems. *Automatic and Remote Control* **65** (2), 227–240.

Emelichev, V., Kuz'min, K. and Nikulin, Yu. (2005) Stability analysis of the Pareto optimal solution for some vector Boolean optimization problem. *Optimization* **54** (6), 545–561.

Emelichev, V. and Kuz'min, K. (2005) The stability radius of efficient solution to a vector problem of Boolean programming in the l_1 metric. *Doklady Mathematics* **71** (2), 266–268.

Emelichev, V. and Kuz'min, K. (2006) Finite cooperative games with a parametric concept of equilibrium under uncertainty conditions. *Journal of Computer and Systems Sciences International* **45** (2), 276–281.

Greenberg, H. J. (1998) An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization. In: D.L. Woodruff, ed., *Advances in Computational and Stochastic Optimization, Logic Programming and Heuristic Search.* Kluwer Acad. Publ., Boston, 97–148.

Hoesel, S. and Wagelmans, A. (1999) On the complexity of postoptimality analysis of 0-1 programs. *Discrete Appl. Math.* **91**, 251–263.

Libura, M., van der Poort, E., Sierksma, G. and van der Veen, J. (1998) Stability aspects of the traveling salesman problem based on k-best solutions. *Discrete Appl. Math.* **87**, 159–185.

Libura, M. and Nikulin, Yu. (2004) Stability and accuracy functions in multicriteria combinatorial optimization problem with $\sum$–MINMAX and $\sum$–MINMIN partial criteria. *Control and Cybernetics* **33** (3), 511–524.

Sergienko, I., Kozeratskaya, L. and Lebedeva, T. (1995) *Investigation of stability and parametric analysis of discrete optimization problems* (in Russian). Kiev, Naukova Dumka.

Sergienko, I. and Shilo, V. (2003) *Discrete optimization problems. Problems, decision methods, investigations* (in Russian). Kiev, Naukova Dumka.

Sergienko, I. (1988) *Mathematical models and methods of solving discrete optimization problems* (in Russian). Kiev, Naukova Dumka.

Sotskov, Yu., Leontev, V. and Gordeev, E. (1995) Some concepts of stability analysis in combinatorial optimization. *Discrete Appl. Math.* **58** (2), 169–190.

Tanino, T. and Sawaragi, Y. (1980) Stability of nondominated solutions in multicriteria decision-making. *Journal of Optimization Theory and Applications* **30** (2), 229–253.

Sawaragi, Y., Nakayama, H. and Tanino, T. (1985) *Theory of Multi-Objective Optimization.* Academic Press, Orlando.