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Stability of the convex combination of polynomials

by

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Abstract: In this paper, we consider the convex combination of polynomials. We provide a necessary and sufficient condition for Hurwitz stability of the convex combination of m real polynomials ($m \geq 3$) whose degrees may be different and both necessary, and necessary and sufficient conditions for Hurwitz and Schur stability of the convex combination of two complex polynomials. We show also that the convex combination of two polynomials whose degrees are respectively odd and even, is never Schur stable. We give a few examples completing the results.

Keywords: Hurwitz stability, Schur stability, roots of polynomials, convex combination of polynomials.

1. Introduction

In this paper, we focus our attention on the root locations of the convex combination of polynomials. This problem is closely related to stability analysis of systems with uncertain physical parameters, and its presence is manifested through variability in the coefficients of the characteristic polynomial. System stability is equivalent to the condition that the characteristic polynomial has all its roots in a certain region of the complex plane. For continuos linear time-invariant systems it is the open left half of the complex plane and for discrete linear time-invariant systems it is the open unit disc centered at the origin of the complex plane. The robust stability problem for polynomials families, which is a very important issue in control theory, has been intensively studied in numerical linear algebra (see Ackermann et al., 1994; Barmish, 1994; Bhattacharyya, Chapellat, Keel, 1995; Białas, 2002, and the references therein).

The paper is organized as follows. In Section 2 we establish the notation used throughout the paper and recall the classical necessary and sufficient conditions

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for Hurwitz and Schur stability of the real polynomials. In Section 3 we give and prove the necessary and sufficient condition for Hurwitz stability of the convex combination of m real polynomials ($m \geq 3$) whose degrees may be different. It is some generalization of Bartlett, Hollot and Huang Lin (1988). We show also that the convex combination of two polynomials having, respectively, odd and even degrees is never Schur stable. In Section 4 we give both necessary and necessary and sufficient conditions for Hurwitz and Schur stability of the convex combination of two complex polynomials. These results extend the already known criteria given by Ackermann and Barmish (see Ackermann and Barmish, 1988) and Białas (1985, 2004).

2. Preliminaries

Let $\mathbb{R}$ (respectively $\mathbb{C}$) denote the set of real (respectively complex) numbers. The imaginary unit will be denoted by i. The set of all real (respectively complex) polynomials of degree less than or equal to n will be denoted by $\mathcal{P}_n(\mathbb{R})$ (respectively $\mathcal{P}_n(\mathbb{C})$). Every polynomial will be viewed as a vector in coefficients space, i.e.

$$\mathcal{P}_n(\mathbb{C}) \ni a_n x^n + \ldots + a_0 \rightsquigarrow [a_n, \ldots, a_0]^T \in \mathbb{C}^{n+1}. \tag{1}$$

Let $\triangle_{i,j}[A]$ denote the determinant of the square submatrix of order $j-i+1$ formed from the square matrix $A = [a_{ij}]$ of order n by deleting some of the rows and columns, namely

$$\triangle_{i,j}[A] = \det \begin{bmatrix} a_{ii} & \cdots & a_{ij} \\ \vdots & \ddots & \vdots \\ a_{ji} & \cdots & a_{jj} \end{bmatrix}$$

for $n \geq j \geq i \geq 1$. For convenience, $\triangle_{1,j}[A]$ will be denoted by $\triangle_j[A]$ and called the j–th leading principal minor of the matrix A.

We will say that a square matrix A of order n is *positive innerwise* if the following conditions

$$\triangle_{1,n}[A] > 0, \triangle_{2,n-1}[A] > 0, \triangle_{3,n-2}[A] > 0, \ldots, \triangle_{\frac{n+1}{2},\frac{n+1}{2}}[A] > 0, \text{if } n \text{ is odd},$$
$$\triangle_{1,n}[A] > 0, \triangle_{2,n-1}[A] > 0, \triangle_{3,n-2}[A] > 0, \ldots, \triangle_{\frac{n}{2},\frac{n+2}{2}}[A] > 0, \text{ if } n \text{ is even},$$

hold. When in the above conditions the symbol ">" is replaced with the symbol "$\neq$" then the matrix is called *non-zero innerwise.*

A polynomial is said to be *Hurwitz (Schur) stable* if all its roots are in the open left-half of the complex plane (in the open unit disc). A set of polynomials is said to be Hurwitz (Schur) stable if all its elements are Hurwitz (Schur) stable.

The matrix

$$H(f) = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & \dots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1-n} & a_{2-n} & a_{3-n} & \dots & \dots & \dots & a_0 \end{bmatrix}$$

where $a_j = 0$ if $j < 0$, is called the *Hurwitz matrix* associated with the polynomial (1).

Also, let

$$S_1(f) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_3 & a_2 \\ 0 & a_n & a_{n-1} & \dots & a_4 & a_3 \\ 0 & 0 & a_n & \dots & a_5 & a_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n & a_{n-1} \\ 0 & 0 & 0 & \dots & 0 & a_n \end{bmatrix}$$

and

$$S_2(f) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & a_0 & \dots & a_{n-5} & a_{n-4} \\ 0 & a_0 & a_1 & \dots & a_{n-4} & a_{n-3} \\ a_0 & a_1 & a_2 & \dots & a_{n-3} & a_{n-2} \end{bmatrix}$$

be two matrices associated with the polynomial (1) and let

$$S^-(f) = S_1(f) - S_2(f), \tag{2}$$

$$S^+(f) = S_1(f) + S_2(f). \tag{3}$$

It is known that

$$\det H(f) = (-1)^{n(n-1)/2} a_n^{n-1} a_0 \prod_{1 \le k < l \le n} (z_k + z_l) \tag{4}$$

and

$$\det S^-(f) = a_n^{n-1} \prod_{1 \le k < l \le n} (1 - z_k z_l), \tag{5}$$

where $z_1, \dots, z_n$ are the roots of the polynomial f (see Jury, 1974; Lancaster, 1969).

Now, we recall the classical necessary and sufficient conditions for Hurwitz and Schur stability of the real polynomial (see Białas, 2002; Gantmacher, 1959; Jury, 1974).

Theorem 1 (Routh–Hurwitz, Białas, 2002) *The real polynomial* $f(x) = a_n x^n + \ldots + a_0$ *(*$a_n > 0$*) is Hurwitz stable if and only if all the leading principal minors of the Hurwitz matrix* $H(f)$ *are positive, i.e.*

$$\triangle_1 [H(f)] > 0, \quad \triangle_2 [H(f)] > 0, \quad \ldots, \quad \triangle_n [H(f)] > 0.$$

Theorem 2 (Liénard–Chipart, Gantmacher, 1959) *Necessary and sufficient conditions for Hurwitz stability of the real polynomial* $f(x) = a_n x^n + \ldots + a_0$ *(*$a_n > 0$*) may be expressed in any one of the four following forms:*

(1) $a_0 > 0,\ a_2 > 0,\ \ldots;\ \triangle_1 [H(f)] > 0,\ \triangle_3 [H(f)] > 0, \ldots,$

(2) $a_0 > 0,\ a_2 > 0,\ \ldots;\ \triangle_2 [H(f)] > 0,\ \triangle_4 [H(f)] > 0, \ldots,$

(3) $a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \ldots;\ \triangle_1 [H(f)] > 0,\ \triangle_3 [H(f)] > 0, \ldots,$

(4) $a_0 > 0,\ a_1 > 0,\ a_3 > 0,\ \ldots;\ \triangle_2 [H(f)] > 0,\ \triangle_4 [H(f)] > 0, \ldots.$

Theorem 3 (Jury, 1974) *The real polynomial* $f(x) = a_n x^n + \ldots + a_0$ *(*$a_n > 0$*) is Schur stable if and only if the following conditions*

(i) $f(1) > 0$ *and* $(-1)^n f(-1) > 0$*;*

(ii) the matrices $S^-(f)$*,* $S^+(f)$ *are positive innerwise*

hold.

We also need the following notions:
$V_m = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_i \geq 0 \ (i = 1, \ldots, n), \sum_{i=1}^n \alpha_i = 1\}$;
$C(f_1, \ldots, f_m) = \{\sum_{i=1}^m f_i \alpha_i : (\alpha_1, \ldots, \alpha_m) \in V_m\}$, where $f_1, \ldots, f_m \in \mathcal{P}_n(\mathbb{C})$; a set $\mathcal{R}(P) = \{z \in \mathbb{C} : \exists f \in P : f(z) = 0\}$ is called *a roots space* of P, where $P \subset \mathcal{P}_n(\mathbb{C})$ is some set of polynomials; $\dim P$ denotes the dimension of the set $P \subset \mathbb{C}^{n+1}$ and should be understood as the dimension of the affine hull of P (written $\operatorname{aff}(P)$); ∂P denotes the relative boundary of P.

3. Stability analysis of the convex combination of real polynomials

In this section, we will consider the convex combinations of real polynomials. As mentioned in the introduction, in control theory these sets are usually identified with the characteristic polynomial of linear time-invariant system with uncertain physical parameters.

We begin with the theorem which gives a necessary and sufficient condition for Hurwitz and Schur stability of the convex combination of m ($m \geq 3$) real polynomials, not necessarily with equal degrees. It is a certain generalization of the so-called *Edge Theorem*, which states that the roots spaces of convex combination of polynomials of the same degree and of these polynomials that are contained in the edges determine identical domain in the complex plane (see Bartlett, Hollot and Huang Lin, 1988). It is not very difficult to give an example, which shows that in the more general case where the degrees of polynomials may not be equal, the Edge Theorem does not hold (see Barmish, 1994, pp. 159-160).

Before we give the main results of this section, we prove the following lemma which will be instrumental in the sequel.

LEMMA 1 *Let* $P \subset \mathcal{P}_n(\mathbb{R})$ *be a given compact set of polynomials and assume that* $\dim P \geq 3$*. Then* $\mathcal{R}(P) = \mathcal{R}(\partial P)$*.*

Proof. Let $z_* \in \mathcal{R}(P)$. Thus, there exists a polynomial f_* in P for which $f_*(z_*) = 0$. Let $P_n(z_*)$ denote a whole family of polynomials of degree less than or equal to n and having z_* as a root. It is easy to see that $P_n(z_*)$ is an affine subspace of $\mathbb{R}^{n+1}$ and its dimension is greater than or equal to $n-1$. More precisely, if z_* is real then $\dim P_n(z_*) = n$ or otherwise $\dim P_n(z_*) = n-1$. Since $\dim(P) = k \geq 3$ and $\text{aff}(P) \cap P_n(z_*) \neq \varnothing$ we obtain that $\text{aff}(P) \cap P_n(z_*)$ is at least one-dimensional affine subspace. It follows from the compactness of P that $\text{aff}(P) \cap P_n(z_*)$ must pierce the boundary of P. This completes the proof. ■

Now, we will give and prove the main results of this section.

First, we give a necessary and sufficient condition for Hurwitz and Schur stability of the convex combination of m $(m \geq 3)$ real polynomials whose degrees may be different.

THEOREM 4 *Let* $f_1, \ldots, f_m \in \mathcal{P}_n(\mathbb{R})$ $(m \geq 3)$ *be given polynomials of the form*

$$f_1(x) = a_{n_1}^{(1)} x^{n_1} + a_{n_1-1}^{(1)} x^{n_1-1} + \ldots + a_0^{(1)}$$
$$\vdots$$
$$f_m(x) = a_{n_m}^{(m)} x^{n_m} + a_{n_m-1}^{(m)} x^{n_m-1} + \ldots + a_0^{(m)}$$

whose leading coefficients are positive, i.e.

$$a_{n_1}^{(1)} > 0, \ldots, \; a_{n_m}^{(m)} > 0.$$

Then $C(f_1, \ldots, f_m)$ *is Hurwitz (Schur) stable if and only if the sets* $C(f_i, f_j, f_k)$ *are Hurwitz (Schur) stable for* $1 \leq i < j < k \leq m$*.*

Proof. The necessity is obvious. For the sufficiency, let $\dim C(f_1, \ldots, f_m) = k \geq 3$. Suppose that each of the sets $C(f_i, f_j, f_l)$ is Hurwitz (Schur) stable for $i, j, l \in \{1, \ldots, m\}$. By Lemma 1, it follows that the set $C(f_1, \ldots, f_m)$ is Hurwitz (Schur) stable if and only if the set $\partial C(f_1, \ldots, f_m)$ is Hurwitz (Schur) stable. Note that for $N \geq 2$ we have

$$\partial C(f_1, \ldots, f_N) \subset \bigcup_{1 \leq i_1 < \ldots < i_{N-1} \leq N} C\left(f_{i_1}, \ldots, f_{i_{N-1}}\right).$$

It means that the set $C(f_1, \ldots, f_m)$ is Hurwitz (Schur) stable if and only if the $k-1$-dimensional sets $C\left(f_{i_1}, \ldots, f_{i_{m-1}}\right)$ are Hurwitz (Schur) stable for

$1 \leq i_1 < \ldots < i_{m-1} \leq m$. Proceeding in this way we will finally obtain that the set $C(f_1, \ldots, f_m)$ is Hurwitz (Schur) stable if and only if all the 2-dimensional boundary sets are Hurwitz (Schur) stable. This completes the proof. ■

3.1. Hurwitz stability of the convex combination of real polynomials

In order to apply Theorem 4, given above, to test the Hurwitz and Schur stability of the convex combinations of polynomials, we need to have some necessary and sufficient conditions for the stability of the convex combinations of three polynomials. These conditions are given below.

THEOREM 5 *Let $f_1, f_2, f_3 \in \mathcal{P}_n(\mathbb{R})$ be given Hurwitz stable polynomials. Then the set $C(f_1, f_2, f_3)$ is Hurwitz stable if and only if any one of the two following conditions*

(1) $\triangle_i [H(\alpha f_1 + \beta f_2 + \gamma f_3)] \neq 0$ *for* $i = 1, 3, \ldots$;

(2) $\triangle_i [H(\alpha f_1 + \beta f_2 + \gamma f_3)] \neq 0$ *for* $i = 2, 4, \ldots$

holds for every $(\alpha, \beta, \gamma) \in V_3$.

Proof. Note that Hurwitz stability of the polynomials f_1, f_2, f_3 implies that all their coefficients are positive. Hence, all the coefficients of the polynomial $\alpha f_1 + \beta f_2 + \gamma f_3$ are also positive for every $(\alpha, \beta, \gamma) \in V_3$. Moreover, the determinants $\triangle_i [H(\alpha f_1 + \beta f_2 + \gamma f_3)]$ are all continuous functions of (α, β, γ). The Liénard–Chipart's theorem and the assumption imply that they are positive for some $(\alpha_0, \beta_0, \gamma_0) \in V_3$. Hence, they will be positive for every $(\alpha, \beta, \gamma) \in V_3$ if and only if they will not vanish for every $(\alpha, \beta, \gamma) \in V_3$. An application of Liénard–Chipart's theorem completes the proof. ■

Note, that combining Theorem 5 with Theorem 4 we obtain an algorithm for testing the Hurwitz stability of the convex combination of polynomials whose degrees do not have to be equal. The following example illustrates the above result.

EXAMPLE 1 *Consider the three Hurwitz stable polynomials*

$$f(x) = x^3 + 4x^2 + 5x + 2$$
$$g(x) = x^2 + 2x + 1$$
$$h(x) = x^2 + x + 2$$

and examine the Hurwitz stability of their convex combination.

For $\alpha \neq 0$ *the Hurwitz matrix associated with the polynomial* $\alpha f + \beta g + (1 - \alpha - \beta) h$ *has the form*

$$H(\alpha f + \beta g + (1 - \alpha - \beta) h) = \begin{bmatrix} 3\alpha + 1 & \alpha & 0 \\ 2 - \beta & 4\alpha + \beta + 1 & 3\alpha + 1 \\ 0 & 0 & 2 - \beta \end{bmatrix}.$$

Since

$$\triangle_2\left[\alpha f+\beta g+(1-\alpha-\beta)\,h\right]=12\alpha^2+4\alpha\beta+5\alpha+\beta+1,$$

we obtain that $\triangle_2$ *is non-zero for any nonnegative* α,β.

Moreover, for $\alpha=0$ *the Hurwitz matrix associated with the polynomial* $\beta g+(1-\beta)\,h$ *is of the form*

$$H\left(\beta g+(1-\beta)\,h\right)=\begin{bmatrix}\beta+1 & 1\\ 0 & 1\end{bmatrix}.$$

Obviously,

$$\triangle_1\left[\beta g+(1-\beta)\,h\right]=\beta+1$$

and it is non-zero for every $\beta\in[0,1]$. *By Theorem 5, the set* $C\,(f,g,h)$ *is Hurwitz stable. As illustrated in Fig. 1, the roots space of* $C(f,g,h)$ *is contained*

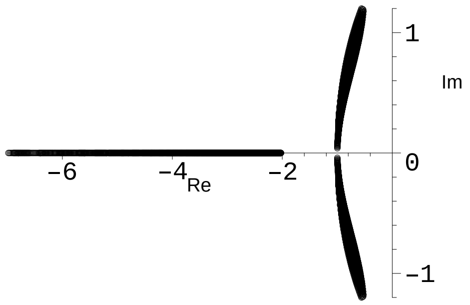

Figure 1. Root locations of the set $C(f,g,h)$ for Example 1

in the open left-half of the complex plane what confirms our result.

3.2. Schur stability of the convex combination of real polynomials

This subsection deals with the Schur stability of the convex combination of real polynomials (a complex case will be considered in the following section). At the beginning we have to note that it is easy to modify Theorem 5 given in the previous subsection to get the necessary and sufficient condition for Schur stability of the set $C\,(f_1,f_2,f_3)$ for any Schur stable polynomials $f_1,f_2,f_3\in\mathcal{P}_n(\mathbb{R})$.

But first, we will give some negative result, namely, we will show that if the degrees of two real Schur stable polynomials are not simultaneously odd nor even then their convex combination is not Schur stable.

To do this, let

$$\begin{aligned}f\,(x)&=a_nx^n+\ldots+a_0,\ a_n>0,\\ g\,(x)&=b_mx^m+\ldots+b_0,\ b_m>0\end{aligned}\tag{6}$$

be two real Schur stable polynomials. Assume that the degree of f is odd while that of g is even and consider the function φ of $\alpha \in [0,1]$ defined as follows

$$\varphi(\alpha) = \alpha f(-1) + (1-\alpha) g(-1).$$

Since f and g are Schur stable, it follows from Jury's criterion that $\varphi(0) \cdot \varphi(1) < 0$. Hence, there exists $\alpha_0 \in (0,1)$ for which $\varphi(\alpha_0) = 0$. It means that the set $C(f,g)$ includes an unstable element, namely $\alpha_0 f + (1-\alpha_0) g$. So, the set $C(f,g)$ is not Schur stable.

Our result is summarized in the following theorem:

THEOREM 6 *Assume that $f, g \in \mathcal{P}_n(\mathbb{R})$ are given Schur stable polynomials of the form (6). Assume that n is odd and m is even. Then the convex combination of the polynomials f and g is not Schur stable.*

Now, we will formulate the necessary and sufficient condition for Schur stability of the set $C(f_1, f_2, f_3)$ for any Schur stable polynomials $f_1, f_2, f_3 \in \mathcal{P}_n(\mathbb{R})$.

THEOREM 7 *Let $f_1, f_2, f_3 \in \mathcal{P}_n(\mathbb{R})$ be given Schur stable polynomials whose degrees are all simultaneously odd or even.*

Then $C(f_1, f_2, f_3)$ is Schur stable if and only if the matrices $S^-(\alpha f_1 + \beta f_2 + \gamma f_3)$ and $S^+(\alpha f_1 + \beta f_2 + \gamma f_3)$ are positive innerwise for every $(\alpha, \beta, \gamma) \in V_3$, i.e. the following conditions

- $\triangle_{1+i,N-1-i}[S^-(\alpha f_1 + \beta f_2 + \gamma f_3)] \neq 0$ *for* $i = 0, \ldots, \frac{N-2}{2}$ $(i = 0, \ldots, \frac{N-3}{2})$ *if N is even (odd);*
- $\triangle_{1+i,N-1-i}[S^+(\alpha f_1 + \beta f_2 + \gamma f_3)] \neq 0$ *for* $i = 0, \ldots, \frac{N-2}{2}$ $(i = 0, \ldots, \frac{N-3}{2})$ *if N is even (odd)*

hold for every $(\alpha, \beta, \gamma) \in V_3$, where $N := \deg(\alpha f_1 + \beta f_2 + \gamma f_3)$.

Proof. The assumption implies that the conditions

$$(\alpha f_1 + \beta f_2 + \gamma f_3)(1) > 0$$

and

$$(-1)^N (\alpha f_1 + \beta f_2 + \gamma f_3)(-1) > 0$$

hold for every $(\alpha, \beta, \gamma) \in V_3$. Moreover, the determinants occurring in the theorem are all continuous functions of $(\alpha, \beta, \gamma) \in V_3$ and, by Jury's theorem, are positive for some $(\alpha_0, \beta_0, \gamma_0) \in V_3$. Hence, they will be positive for every $(\alpha, \beta, \gamma) \in V_3$ if and only if they do not vanish for every $(\alpha, \beta, \gamma) \in V_3$. An application of Jury's theorem completes the proof. ■

4. Stability analysis of the convex combination of two complex polynomials

In this section, we will consider families of polynomials with complex coefficients. In stability theory, such families arise, for example, when considering the θ–*Hurwitz stability* of convex combinations of real polynomials, i.e. stability with respect to the rotated Hurwitz stability region $S_\theta = \left\{z \in \mathbb{C} : \mathfrak{Re}(ze^{-i\theta}) < 0\right\}$ (see e.g. Bhattacharyya, Chapellat, Keel, 1995, pp. 324-325).

Now, we will give necessary and necessary and sufficient conditions for Hurwitz and Schur stability of convex combination of two complex polynomials.

4.1. Necessary conditions for Hurwitz and Schur stability

Let $f, g \in \mathcal{P}_n(\mathbb{C})$ be given as follows:

$$\begin{aligned} f(z) &= a_n z^n + \ldots + a_0,\ a_n \neq 0, \\ g(z) &= b_m z^m + \ldots + b_0,\ b_m \neq 0. \end{aligned} \tag{7}$$

First, we will give and prove the necessary conditions for Hurwitz and Schur stability of the set $C(f,g)$. We begin with Hurwitz case.

THEOREM 8 *Assume that $f, g \in \mathcal{P}_n(\mathbb{C})$ are the polynomials defined by (7).*

(i) Let $n = m$ and let

$$a_n^{-1} b_n \notin (-\infty, 0). \tag{8}$$

If the convex combination $C(f,g)$ is Hurwitz stable then the matrix $H^{-1}(f)\,H(g)$ has no eigenvalue in $(-\infty, 0]$.

(ii) Let $n > m$.

If the convex combination $C(f,g)$ is Hurwitz stable then the matrix $H^{-1}(f)\,H(\widetilde{g})$ has no eigenvalue in $(-\infty, 0)$, where $\widetilde{g}(z) = 0z^n + \ldots + 0z^{m+1} + g(z)$.

Proof. (i) It is obvious that the condition (8) guarantees that the leading coefficient of the polynomial $\alpha f + (1-\alpha)\,g$ is non-zero for every $\alpha \in [0,1]$. Moreover, the Hurwitz stability of the polynomial f implies that the matrix $H^{-1}(f)$ exists. It follows from the assumption and from (4) that for every $\alpha \in [0,1]$ we have

$$\det\left(H\left(\alpha f + (1-\alpha)\,g\right)\right) \neq 0.$$

Also, for $\alpha \neq 1$ we have that

$$\det\left(H\left(\alpha f + (1-\alpha)\,g\right)\right) = \det\left(\alpha H(f) + (1-\alpha)\,H(g)\right) \neq 0$$

if and only if

$$\det\left(\frac{\alpha}{1-\alpha} I + H^{-1}(f)\,H(g)\right) \neq 0.$$

This completes the proof of part (i).

(ii) Following a similar line of argument as in the proof of (i), we have that

$$\det\left(H\left(\alpha f+(1-\alpha)\widetilde{g}\right)\right)\neq 0 \tag{9}$$

for every $\alpha\in(0,1]$. Similarly, (9) can be expressed equivalently by

$$\det\left(\frac{\alpha}{1-\alpha}I+H^{-1}(f)\,H(\widetilde{g})\right)\neq 0,$$

for every $\alpha\in(0,1)$. This completes the proof of Theorem 8. ■

Now, we will give the necessary condition for Schur stability of the set $C(f,g)$.

THEOREM 9 *Assume that $f,g\in\mathcal{P}_n(\mathbb{C})$ are the polynomials defined by (7).*

(i) Let $n=m$ and let the condition (8) holds.
If the convex combination $C(f,g)$ is Schur stable then the matrix $(S^-)^{-1}(f)\,S^-(g)$ has no eigenvalue in $(-\infty,0]$.

(ii) Let $n>m$.
If the convex combination $C(f,g)$ is Schur stable then the matrix $(S^-)^{-1}(f)\,S^-(\widetilde{g})$ has no eigenvalue in $(-\infty,0)$, where $\widetilde{g}(z)=0z^n+\ldots+0z^{m+1}+g(z)$.

Proof. The proof of this theorem is similar to that of Theorem 8 and hence is omitted here (instead of using Hurwitz matrix H, we use the matrix S^-). ■

It appears to be interesting to note that in contrast to the real case, the above conditions are necessary but not sufficient for Hurwitz and Schur stability of the convex combination of two complex polynomials. The following examples confirm this fact.

EXAMPLE 2 *Let $f,g\in\mathcal{P}_2(\mathbb{C})$ be two Hurwitz stable polynomials given as follows:*

$$f(z)=z^2+(5-i)\,z+4-4i,$$
$$g(z)=z^2+(2-9i)\,z-17-9i.$$

Then the Hurwitz matrices associated with f and g have the form

$$H(f)=\begin{bmatrix}5-i & 1\\ 0 & 4-4i\end{bmatrix}\ \text{and}\ H(g)=\begin{bmatrix}2-9i & 1\\ 0 & -17-9i\end{bmatrix},$$

respectively. The simple calculations show that

$$H^{-1}(f)\,H(g)=\begin{bmatrix}\frac{19}{85}+\frac{43}{85}i & -\frac{147}{6290}+\frac{53}{1258}i\\ 0 & -\frac{16}{185}+\frac{52}{185}i\end{bmatrix}$$

what means that the matrix $H^{-1}(f)\,H(g)$ has not eigenvalue in $(-\infty,0]$. On the other hand, the polynomial $0.5f+0.5g$ has the root with positive real part. It means that the set $C(f,g)$ is not Hurwitz stable.

Example 3 *For the Schur stability case, let* $f, g \in \mathcal{P}_2(\mathbb{C})$ *be given as follows:*

$$f(z) = (z + 0.6i)^3 = z^3 + 1.8iz^2 - 1.08z - 0.216\,i$$
$$g(z) = (z - 0.6i)^3 = z^3 - 1.8iz^2 - 1.08z + 0.216\,i.$$

It is obvious that f *and* g *are Schur stable.*

In view of (2) and (3), the matrices $S^-(f)$ *and* $S^-(g)$ *have the form*

$$S^-(f) = \begin{bmatrix} 1 & 2.016i \\ 0.216\,i & 2.08 \end{bmatrix}$$

and

$$S^-(g) = \begin{bmatrix} 1 & -2.016i \\ -0.216\,i & 2.08 \end{bmatrix},$$

respectively.

Simple calculations show that the eigenvalues of the matrix

$$\left(S^-\right)^{-1}(f)\, S^-(g) = \begin{bmatrix} 0.65378 & -3.334i \\ -0.17174i & 0.65378 \end{bmatrix}$$

are $\lambda_0 = \overline{\lambda_1} \approx 0.65378 + 0.75669i$. *On the other hand, the polynomial* $1/3f + 2/3g$ *has the root with modulus greater than one. It means that the set* $C(f, g)$ *is not Schur stable.*

4.2. Necessary and sufficient conditions for Hurwitz and Schur stability

In this subsection we will give and prove the necessary and sufficient conditions for Hurwitz and Schur stability of the convex combination of two complex polynomials.

Let $f \in \mathcal{P}_n(\mathbb{C})$ be a polynomial of the form $f(z) = a_n z^n + \ldots + a_0$. Let $\overline{f} \in \mathcal{P}_n(\mathbb{C})$ denote a complex conjugate of the polynomial f, i.e. $\overline{f}(z) = \overline{a_n} z^n + \ldots + \overline{a_0}$.

We begin with the lemma which will be useful for proving the subsequent theorems.

Lemma 2 *Let* $f, \overline{f} \in \mathcal{P}_n(\mathbb{C})$ *be a complex conjugate pair of polynomials as defined above. Then* f *is Hurwitz (Schur) stable if and only if the real polynomial* $f \cdot \overline{f} \in \mathcal{P}_{2n}(\mathbb{R})$ *is Hurwitz (Schur) stable.*

Proof. The proof follows immediately from the fact that $f(z) = 0$ if and only if $\overline{f}(\overline{z}) = 0$. ■

Note that the polynomial $f \cdot \overline{f} \in \mathcal{P}_{2n}(\mathbb{R})$ can be expressed in the form

$$f(z) \cdot \overline{f}(z) = \sum_{k=0}^{2n} c_k z^k,$$

where $c_k \in \mathbb{R}$ for $k \in \{0, \ldots, 2n\}$ and are defined as follows

$$c_k = \sum_{l=0}^{2n} a_l \overline{a_{k-l}}, \tag{10}$$

where $a_j = 0$ if $j < 0$ or $j > n$.

Consider now two complex polynomials $f, g \in \mathcal{P}_n(\mathbb{C})$ defined by (7). Let $\varphi_{\alpha,f,g}$ be the polynomial given by

$$\varphi_{\alpha,f,g}(z) = \alpha f(z) + (1-\alpha) g(z),$$

for some $\alpha \in [0,1]$. Then the polynomial $\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}}$ can be written as

$$\left(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}}\right)(z) = \sum_{k=0}^{2n} c_k z^k,$$

where, according to (10), the real coefficients c_k $(k = 0, \ldots, 2n)$ have the form

$$c_k = \sum_{l=0}^{2n} \left[\alpha a_l + (1-\alpha) b_l\right] \overline{\left[\alpha a_{k-l} + (1-\alpha) b_{k-l}\right]}, \tag{11}$$

where $a_j = 0$ if $j < 0$ or $j > n$; $b_j = 0$ if $j < 0$ or $j > m$.

4.2.1. The Hurwitz stability case

We are now ready to formulate and prove the necessary and sufficient conditions for Hurwitz stability of the set $C(f,g)$ where $f, g \in \mathcal{P}_n(\mathbb{C})$.

THEOREM 10 *Assume that $f, g \in \mathcal{P}_n(\mathbb{C})$ are two Hurwitz stable polynomials defined by (7).*

*(i) Let $n = m$ and assume that (8) holds.
Then the set $C(f,g)$ is Hurwitz stable if and only if all the leading principal minors of the Hurwitz matrix $H(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ are non-zero for every $\alpha \in [0,1]$, i.e. the conditions*

$$\triangle_1\left[H\left(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}}\right)\right] \neq 0, \ldots, \triangle_{2n}\left[H\left(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}}\right)\right] \neq 0,$$

hold for every $\alpha \in [0,1]$.

(ii) Let $n > m$. Then the set $C(f,g)$ is Hurwitz stable if and only if all the leading principal minors of the Hurwitz matrix $H(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ are non-zero for every $\alpha \in (0,1]$.

Proof. (i) The proof of (i) is analogous to that of Theorem 5. It is obvious that all the leading principal minors of the Hurwitz matrix $H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)$ are continuous functions of $\alpha\in[0,1]$. Moreover, from Routh–Hurwitz theorem it follows that they are all positive for some $\alpha_0\in[0,1]$. Hence, they will be positive for every $\alpha\in[0,1]$ if and only if they do not vanish for every $\alpha\in[0,1]$. The Routh–Hurwitz theorem completes the proof of case (i).

(ii) Notice that for $\alpha=0$ the matrix $H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)$ has dimension $m\times m$, while for $\alpha\neq 0$ its dimension is $n\times n$. Since the polynomial $\varphi_{0,f,g}\cdot\overline{\varphi_{0,f,g}}$ is Hurwitz stable it follows that it suffices to check the Hurwitz stability of $\varphi_{\alpha,f,g}$ for $\alpha\in(0,1]$. Now, using the same kind of argument as above we end the proof of case (ii). ■

Note that if $f,g\in\mathcal{P}_n(\mathbb{R})$ are two real Hurwitz stable polynomials then to check the Hurwitz stability of their convex combination by applying Theorem 10, instead of the Hurwitz matrix $H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)$, we can examine the matrix $H\left(\varphi_{\alpha,f,g}\right)$. It is computationally more efficient because of the lower dimension of the matrix $H\left(\varphi_{\alpha,f,g}\right)$.

We now give and prove the other, often more convenient, necessary and sufficient condition for Hurwitz stability of the set $C(f,g)$.

THEOREM 11 *Assume that $f,g\in\mathcal{P}_n(\mathbb{C})$ are two Hurwitz stable polynomials defined by (7).*

(i) Let $n=m$, let c_k $(k=0,\ldots,2n)$ be real numbers defined by (11) and assume that (8) holds.
Then the set $C(f,g)$ is Hurwitz stable if and only if any one of the four following conditions

(a) $c_{2n}\neq 0,\ c_{2n-2}\neq 0,\ c_{2n-4}\neq 0,\ \ldots;$

$$\triangle_1\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ \triangle_3\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ldots;$$

(b) $c_{2n}\neq 0,\ c_{2n-2}\neq 0,\ c_{2n-4}\neq 0,\ \ldots;$

$$\triangle_2\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ \triangle_4\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ldots;$$

(c) $c_{2n}\neq 0,\ c_{2n-1}\neq 0,\ c_{2n-3}\neq 0,\ \ldots;$

$$\triangle_1\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ \triangle_3\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ldots;$$

(d) $c_{2n}\neq 0,\ c_{2n-1}\neq 0,\ c_{2n-3}\neq 0,\ \ldots;$

$$\triangle_2\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ \triangle_4\left[H\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)\right]\neq 0,\ldots$$

hold for every $\alpha\in[0,1]$.

(ii) Let $n>m$. Then the set $C(f,g)$ is Hurwitz stable if and only if any one of the above conditions holds for every $\alpha\in(0,1]$.

Proof. (i) Note that the determinants $\triangle_j [H(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})]$ $(j = 1, \ldots, 2n)$ and the coefficients c_j $(j = 0, \ldots, 2n)$ occurring in the theorem are all continuous functions of $\alpha \in [0, 1]$. Moreover, from Liénard–Chipart's theorem and from the assumption it follows that they are all positive for some $\alpha_0 \in [0, 1]$. Hence, they will be positive for every $\alpha \in [0, 1]$ if and only if they do not vanish for every $\alpha \in [0, 1]$. Liénard–Chipart's theorem completes the proof.

(ii) The proof of case (ii) is similar to that of (i) and hence is omitted here.

■

We now give the example which illustrates the above result.

EXAMPLE 4 *Consider again two complex polynomials defined in Example 2, namely*

$$f(z) = z^2 + (5 - i)z + 4 - 4i,$$
$$g(z) = z^2 + (2 - 9i)z - 17 - 9i.$$

We now check the Hurwitz stability of the set $C(f, g)$ by applying Theorem 11.

A simple calculation shows that $\varphi_{\alpha,f,g}(z) = z^2 + a_1 z + a_0$, where

$$a_1 = 3\alpha + 2 + (8\alpha - 9)i,$$
$$a_0 = 21\alpha - 17 + (5\alpha - 9)i.$$

Hence, by (11), we have that

$$(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})(z) = z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0,$$

where

$$c_3 = 6\alpha + 4,$$
$$c_2 = 73\alpha^2 - 90\alpha + 51,$$
$$c_1 = 206\alpha^2 - 252\alpha + 94,$$
$$c_0 = 466\alpha^2 - 804\alpha + 370.$$

Then, the Hurwitz matrix $H(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ has the form

$$\begin{bmatrix} 6\alpha + 4 & 1 & 0 & 0 \\ 206\alpha^2 - 252\alpha + 94 & 73\alpha^2 - 90\alpha + 51 & 6\alpha + 4 & 1 \\ 0 & 466\alpha^2 - 804\alpha + 370 & 206\alpha^2 - 252\alpha + 94 & 73\alpha^2 - 90\alpha + 51 \\ 0 & 0 & 0 & 466\alpha^2 - 804\alpha + 370 \end{bmatrix}.$$

Since one of the roots of $\triangle_3 [H(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})]$ is $\alpha_0 \approx 0.254\,33$, it follows from Theorem 11 that the set $C(f, g)$ is not Hurwitz stable.

We shall note that the method for checking the Hurwitz stability of the convex combinations of complex polynomials given in Theorem 11 is much more practical than the one given in Theorem 10. This fact follows from the observation that Theorem 11 requires the calculation of half of the number of determinants that Theorem 10 requires.

4.2.2. The Schur stability case

Now, we give and prove the necessary and sufficient condition for Schur stability of the set $C(f,g)$ where $f,g \in \mathcal{P}_n(\mathbb{C})$.

THEOREM 12 *Assume that $f,g \in \mathcal{P}_n(\mathbb{C})$ are two Schur stable polynomials defined by (7).*

(i) Let $n = m$ and assume that (8) holds. Then the set $C(f,g)$ is Schur stable if and only if $(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})(1) \neq 0$, $(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})(-1) \neq 0$ for every $\alpha \in [0,1]$ and the matrices $S^-(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$, $S^+(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ are non-zero innerwise for every $\alpha \in [0,1]$, i.e. the conditions

- $\triangle_{1+i,n-1-i}[S^-(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})] \neq 0$ *for* $i = 0,\ldots,\frac{n-2}{2}$ *(*$i=0,\ldots,\frac{n-3}{2}$*) if n is even (odd);*
- $\triangle_{1+i,n-1-i}[S^+(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})] \neq 0$ *for* $i=0,\ldots,\frac{n-2}{2}$ *(*$i=0,\ldots,\frac{n-3}{2}$*) if n is even (odd)*

hold for every $\alpha \in [0,1]$.

(ii) Let $n > m$. Then the set $C(f,g)$ is Schur stable if and only if the above conditions hold for every $\alpha \in (0,1]$.

Proof. (i) Note that, by assumption, the conditions

$$(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})(1) > 0$$

and

$$(-1)^{2n}(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})(-1) > 0$$

hold for every $\alpha \in [0,1]$. Moreover, the determinants occurring in the theorem are all continuous functions of $\alpha \in [0,1]$ and by Jury's theorem they are positive for some $\alpha_0 \in [0,1]$. Hence, they do be positive for every $\alpha \in [0,1]$ if and only if they will not vanish for every $\alpha \in [0,1]$. An application of Jury's theorem completes the proof.

(ii) Note that for $\alpha = 0$ the matrices $S^-(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ and $S^+(\varphi_{\alpha,f,g} \cdot \overline{\varphi_{\alpha,f,g}})$ have dimension $m-1 \times m-1$, while for $\alpha \neq 0$ their dimension is $n-1 \times n-1$. Since the polynomial $\varphi_{0,f,g} \cdot \overline{\varphi_{0,f,g}}$ is Schur stable it follows that it suffices to check the Schur stability of $\varphi_{\alpha,f,g}$ for $\alpha \in (0,1]$. Now, using the same kind of argument as above we end the proof of case (ii). ■

At the end, we give one more simple example illustrating our result.

EXAMPLE 5 *Consider two complex Schur stable polynomials defined as*

$$f(z) = 4z^2 - (4+2i)z + 1 + i \text{ and } g(z) = 4z^2 + (2+4i)z - 1 + i.$$

Simple calculations lead to the following form of the function φ

$$\varphi_{\alpha,f,g}(z) = 4z^2 + (2 - 6\alpha + (4 - 6\alpha)i)z - 1 + 2\alpha + i.$$

Hence

$$\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)(z)=c_4z^4+c_3z^3+c_2z^2+c_1z+c_0,$$

where

$$c_4=16,\ c_3=16-48\alpha,\ c_2=72\alpha^2-56\alpha+12,$$
$$c_1=-24\alpha^2+8\alpha+4,\ c_0=4\alpha^2-4\alpha+2.$$

It is easy to see that

$$\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)(-1)=100\alpha^2-20\alpha+10\neq 0$$

and

$$\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)(1)=52\alpha^2-100\alpha+50\neq 0$$

for every $\alpha\in[0,1]$*. Moreover, by (2) and (3), we have that*

$$S^-\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)=\begin{bmatrix}16 & -48\alpha+16 & -52\alpha+68\alpha^2+10\\ 0 & 4\alpha-4\alpha^2+14 & -56\alpha+24\alpha^2+12\\ 4\alpha-4\alpha^2-2 & -8\alpha+24\alpha^2-4 & 56\alpha-72\alpha^2+4\end{bmatrix}$$

and

$$S^+\left(\varphi_{\alpha,f,g}\cdot\overline{\varphi_{\alpha,f,g}}\right)=\begin{bmatrix}16 & -48\alpha+16 & -60\alpha+76\alpha^2+14\\ 0 & -4\alpha+4\alpha^2+18 & -40\alpha-24\alpha^2+20\\ -4\alpha+4\alpha^2+2 & 8\alpha-24\alpha^2+4 & -56\alpha+72\alpha^2+28\end{bmatrix}.$$

Hence

$$\triangle_{1,3}\left(S^-\right)=-1088\alpha^6+7616\alpha^5-21\,152\alpha^4+32\,256\alpha^3-31\,120\alpha^2+12\,528\alpha+1560,$$
$$\triangle_{2,2}\left(S^-\right)=-4\alpha^2+4\alpha+14,$$
$$\triangle_{1,3}\left(S^+\right)=-1216\alpha^6+8000\alpha^5-12\,512\alpha^4-18\,432\alpha^3+42\,320\alpha^2-19\,120\alpha+6920,$$
$$\triangle_{2,2}\left(S^+\right)=4\alpha^2-4\alpha+18.$$

Calculating the roots of the above functions, we obtain the following results:

$$\mathcal{R}\left(\triangle_{1,3}\left(S^-\right)\right)\approx\{-0.098,1.098,0.5\pm1.34i,2.5\pm0.5i\};$$
$$\mathcal{R}\left(\triangle_{2,2}\left(S^-\right)\right)\approx\{-1.436,2.436\};$$
$$\mathcal{R}\left(\triangle_{1,3}\left(S^+\right)\right)\approx\{-1.713,1.344,0.209\pm0.398i,3.265\pm1.25i\};$$
$$\mathcal{R}\left(\triangle_{2,2}\left(S^+\right)\right)\approx\{0.5\pm2.06i\}.$$

By Theorem 12, it follows that the set $C(f,g)$ *is Schur stable. As illustrated in Fig. 2, the roots space of* $C(f,g)$ *is contained in the unit circle, what confirms our result.*

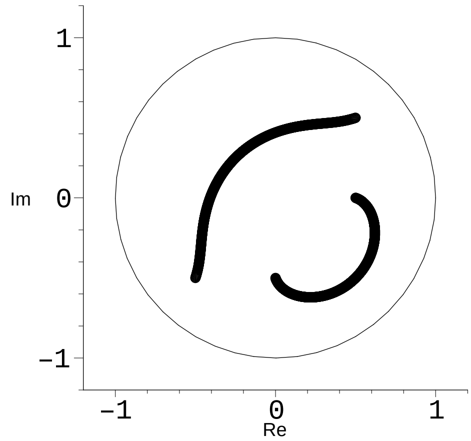

Figure 2. Root locations of the set $C(f, g)$ for Example 5

5. Concluding remarks

We have considered convex combinations of m ($m \geq 3$) real polynomials and of two complex polynomials with generally different degrees. Both the necessary and the necessary and sufficient conditions for Hurwitz and Schur stability of these sets have been established.

The important problems of control theory such as examining the stability of polytopes of complex polynomials of different degrees or of sets of complex polynomials with multilinear or polynomial uncertainty structure are still open.

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