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Second order convexity and a modified objective function method in mathematical programming

by

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Abstract: An approach to nonlinear constrained mathematical programming problems which makes use of a second order derivative is presented. By using a second order modified objective function method, a modified optimization problem associated with a primal mathematical programming problem is constructed. This auxiliary optimization problem involves a second order approximation of an objective function constituting the primal mathematical programming problem. The equivalence between the original mathematical programming problem and its associated modified optimization problem is established under second order convexity assumption. Several practical O.R. applications show that our method is efficient. Further, an iterative algorithm based on this approach for solving the considered nonlinear mathematical programming problem is given for the case when the functions constituting the problem are second order convex. The convergence theorems for the presented algorithm are established.

Keywords: mathematical programming, second order modified objective function optimization problem, second order convex function, second order optimality conditions.

1. Introduction

In the paper, we consider the nonlinear constrained mathematical programming problem

$$\begin{aligned} & f_0(x) \to \min \\ \text{subject to } & f_i(x) \leq 0, \ \ i = 1, ..., m, \qquad \text{(P)} \\ & x \in X, \end{aligned}$$

where $f_i : X \to R$, $i = 0, 1, ..., m$, are twice continuously differentiable functions on a nonempty open set $X \subset R^n$.

Let

$$D := \{x \in X : f_i(x) \leq 0, \quad i \in J = \{1, ..., m\}\}$$

denote the set of all feasible solutions in (P).

Throughout the paper we also use the following sets of indices: $J(\overline{x}) := \{i \in J : f_i(\overline{x}) = 0\}$ and $J_0(\overline{x}) = \{0\} \cup J(\overline{x})$.

Convexity plays a central role in the analysis of mathematical programming problems. The problem of minimizing a convex function subject to convex constraints is of a fundamental importance in mathematical programming. However, for many optimization problems, notably in mathematical programming, the characterization of optimal solutions with the help of second order conditions was always of a great interest as a means of refining the first order optimality conditions (for example, the need of second order information appears in numerical algorithms).

Mond and Weir (1981) and Bector and Chandra (1985) independently introduced second order convex and generalized convex functions (Bector and Chandra called them bonvex and pseudo/quasi bonvex functions). Using second order conditions, duality results were established on various duality theorems, for example, by Bector and Bector (1986).

Considerable attention has been given recently to devising new methods which solve the original mathematical programming problem and its duals with the help of some associated optimization problem (see, for example, Antczak, 2004).

In this paper, we present a new approach for solving a constrained mathematical programming problem involving twice differentiable functions. The aim of the present paper is to show how one can obtain optimality conditions for a nonlinear constrained mathematical programming problem with strong nonlinear objective function by constructing for it an equivalent minimization problem with a second order modified objective function. This associated modified optimization problem is obtained by a second order modification of the objective function in the given mathematical programming problem at an arbitrary but fixed feasible point $\overline{x}$. To prove the equivalence between the original mathematical programming problem and its associated modified optimization problem we use the second order convexity assumption imposed on the functions involved in the original programming problem. Moreover, the equivalent optimization problem obtained in this approach is, in general, less complicated and its optimal solutions are connected to the optimal points of the original minimization problem. In this way, we obtain the associated modified optimization problem with the same optimality solution as the original mathematical programming problem and the optimality value equal to the optimality value in the original mathematical programming problem.

Further, some example of O.R. problem is given for which the presented second order modified objective function method can be applied to find an optimal solution. The second order objective function optimization problem is constructed for the considered O.R. problem.

Finally, based on the approach introduced in the paper, an iterative method for solving a nonlinearly constrained mathematical programming problem with second order convex functions is proposed. This method consists of solving a sequence of second order modified objective function optimization problems which are constructed at feasible points obtained from successive steps of the respective algorithm. Thus, the method generates an iterative sequence of feasible solutions which, under suitable conditions, converges to an optimal solution of the original mathematical programming problem. The superlinear convergence rate of this sequence is proved. Moreover, it turns out that, when the Hessian matrix of the objective function is assumed to satisfy the Lipschitz condition on the set of all feasible solutions, the convergence rate of this algorithm increases to quadratic.

2. Preliminaries

Throughout the paper we write $\nabla f(x)$ and $\nabla^2 f(x)$ for the gradient of f and for the Hessian of f evaluated at x, respectively. We recall some definitions and properties that will be used in the present paper.

Definition 1 *Let $f : X \to R$ be a differentiable function on a nonempty open set $X \subset R^n$. Then f is first order convex (or shortly, convex) at $u \in X$ on X, if the inequality*

$$f(x) - f(u) \geq (x-u)^T \nabla f(u) \tag{1}$$

holds for all $x \in X$. If the inequality (1) holds for each $u \in X$ then f is convex on X.

Definition 2 *Let $f : X \to R$ be a twice differentiable function on a nonempty open set $X \subset R^n$. Then f is said to be second order convex at $u \in X$ on X if the following inequality*

$$f(x) - f(u) \geq (x-u)^T \left[\nabla f(u) + \nabla^2 f(u) y\right] - \frac{1}{2} y^T \nabla^2 f(u) y \tag{2}$$

holds for all $y \in R^n$ and for all $x \in X$. If the inequality (2) holds for each $u \in X$ then f is said to be second order convex on X. If the inequality in (2) is sharp for each $x \in X$, $x \neq u$ then the function f is said to be second order strictly convex function on X.

Lemma 1 *Let $f : X \to R$ be a second order convex function defined on a nonempty convex set $X \subset R^n$ and $u \in X$ be a minimum point of f on X. Then the following inequality*

$$(x-u)^T \nabla f(u) \geq 0$$

holds for all $x \in X$.

Lemma 2 *Let $f : X \to R$ be a second order strictly convex function defined on a nonempty convex set $X \subset R^n$ and $u \in X$. Then there exists $M > m > 0$ such that the inequality*

$$m\|y\| \leq y^T \nabla^2 f(u) y \leq M\|y\|$$

holds for all $y \in R^n$.

Definition 3 *We define the Lagrange function or the Lagrangian $L : D \times R_+ \times R_+^m \to R$ in the considered mathematical programming problem (P) as follows*

$$L(x, \lambda, \xi) := \lambda f_0(x) + \sum_{i=1}^{m} \xi_i f_i(x).$$

Definition 4 *A set*

$$C(\overline{x}) := \left\{ d \in R^n : d^T \nabla f_i(\overline{x}) \leq 0,\ i \in J_0(\overline{x}) \right\}$$

is said to be the set of critical directions at $\overline{x}$.

Definition 5 *A point $\overline{x} \in D$ is said to be an optimal point in (P) if, for all $x \in D$,*

$$f_0(x) \geq f_0(\overline{x}).$$

It is well-known (see, for example, Bazaraa, Sherali, Shetty, 1991) that the Karush-Kuhn-Tucker optimality conditions are the first-order necessary optimality conditions for $\overline{x}$ to be an optimal solution in the considered mathematical programming problem.

Theorem 1 *(Bazaraa, Sherali, Shetty, 1991) Let $\overline{x}$ be an optimal solution in (P) and some suitable constraint qualification be satisfied at $\overline{x}$. Then, there exist $\overline{\lambda} \in R_+$ and $\overline{\xi} \in R_+^m$, such that*

$$\nabla L\left(\overline{x}, \overline{\lambda}, \overline{\xi}\right) = 0, \tag{3}$$

$$\overline{\xi}_i f_i(\overline{x}) = 0, \quad i \in J(\overline{x}), \tag{4}$$

$$\overline{\lambda} > 0,\ \overline{\xi} \geq 0. \tag{5}$$

It is also a well-known fact that under the suitable condition of the regularity of constraints, for example *Linear Independence Constraint Qualification* (LICQ) (Bazaraa, Sherali, Shetty, 1991), without loss of generality, it can be assumed that $\overline{\lambda} = 1$.

For optimization problems with twice differentiable functions, it is known (for example, see Ben-Tal, 1980) that the second-order optimality conditions (in the so-called dual form) are necessary for $\overline{x}$ to be an optimal solution in the considered mathematical programming problem.

Theorem 2 *(Ben-Tal, 1980) Let* $\overline{x}$ *be an optimal solution in (P) and the suitable constraint qualification be satisfied at* $\overline{x}$. *Then, for every* $d \in C(\overline{x})$, *there exist* $\overline{\lambda} \in R_+$ *and* $\overline{\xi} \in R_+^m$ *such that*

$$\nabla L\left(\overline{x}, \overline{\lambda}, \overline{\xi}\right) = 0, \tag{6}$$

$$d^T \nabla^2 L\left(\overline{x}, \overline{\lambda}, \overline{\xi}\right) d \geq 0, \tag{7}$$

$$\overline{\xi}_i f_i\left(\overline{x}\right) = 0, \quad i \in J\left(\overline{x}\right), \tag{8}$$

$$\overline{\lambda} d^T \nabla f_0\left(\overline{x}\right) = 0, \tag{9}$$

$$\overline{\xi}_i d^T \nabla f_i\left(\overline{x}\right) = 0, \quad i \in J_0\left(\overline{x}\right), \tag{10}$$

$$\overline{\lambda} > 0, \, \overline{\xi} \geq 0. \tag{11}$$

3. Second order modified objective function method

Let $\overline{x}$ be the given feasible solution in (P). We construct the following optimization problem ($\mathrm{P}^2(\overline{x})$), that is, an optimization problem with a second order modified objective function, given by

$$\begin{array}{c} f_0(\overline{x}) + (x-\overline{x})^T \nabla f_0(\overline{x}) + \frac{1}{2}(x-\overline{x})^T \nabla^2 f_0(\overline{x})(x-\overline{x}) \to \min \\ \text{subject to} \quad f_i(x) \leq 0, \quad i = 1, ..., m, \\ x \in X, \end{array} \tag{$P^2(\overline{x})$}$$

where f_i, $i = 0, 1, ..., m$, X are defined as in problem (P). We will call $(P^2(\overline{x}))$ the associated second order modified objective function optimization problem or the associated optimization problem with a second order modified objective function.

In the following theorems, we prove the equivalence between the original mathematical programming problem (P) and its associated second order modified objective function optimization problem ($\mathrm{P}^2(\overline{x})$) under second order convexity assumption imposed on the functions constituting the original optimization problem (P).

Theorem 3 *Let* $\overline{x}$ *be an optimal solution in the associated second order modified objective function optimization problem* $(P^2(\overline{x}))$ *and the suitable constraint qualification (Bazaraa, Sherali, Shetty, 1991) be satisfied at* $\overline{x}$. *Moreover, we assume that the objective function* f_0 *in the original mathematical programming problem (P) is second order convex at* $\overline{x}$ *on* D. *Then,* $\overline{x}$ *is also optimal in problem (P).*

Proof. We proceed by contradiction. Suppose that $\overline{x}$ is not optimal in (P). Then there exists $\widetilde{x} \in D$ such that

$$f_0\left(\widetilde{x}\right) < f_0\left(\overline{x}\right). \tag{12}$$

By assumption, f_0 is second order convex at $\overline{x}$ on D. Then, by Definition 2, the following inequality

$$f_0(\widetilde{x}) - f_0(\overline{x}) \geq (\widetilde{x} - \overline{x})^T \left[\nabla f_0(\overline{x}) + \nabla^2 f_0(\overline{x})y\right] - \frac{1}{2}y^T\nabla^2 f_0(\overline{x})y \tag{13}$$

holds for all $y \in R^n$. Hence, by (12) and (13), the inequality

$$(\widetilde{x} - \overline{x})^T \left[\nabla f_0(\overline{x}) + \nabla^2 f_0(\overline{x})y\right] - \frac{1}{2}y^T\nabla^2 f_0(\overline{x})y < 0 \tag{14}$$

holds for all $y \in R^n$. Therefore, it is also satisfied for $y = \widetilde{x} - \overline{x}$. Hence, by (14),

$$(\widetilde{x} - \overline{x})^T \nabla f_0(\overline{x}) + \frac{1}{2}(\widetilde{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\widetilde{x} - \overline{x}) < 0. \tag{15}$$

Thus, (15) implies the inequality

$$f_0(\overline{x}) + (\widetilde{x} - \overline{x})^T \nabla f_0(\overline{x}) + \tfrac{1}{2}(\widetilde{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\widetilde{x} - \overline{x}) < f_0(\overline{x}) + (\overline{x} - \overline{x})^T \nabla f_0(\overline{x}) + \tfrac{1}{2}(\overline{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\overline{x} - \overline{x}),$$

which is a contradiction to the optimality of $\overline{x}$ in the associated second order modified objective function optimization problem ($P^2(\overline{x})$). This means that $\overline{x}$ is optimal in (P). ■

THEOREM 4 *Let $\overline{x}$ be a feasible solution in the original nonlinear mathematical programming problem (P) and the second order necessary optimality conditions in the dual form be satisfied at $\overline{x}$. Moreover, we assume that f_i, $i \in J(\overline{x})$, are second order convex at $\overline{x}$ on D. Then, $\overline{x}$ is also optimal in an associated second order modified objective function optimization problem ($P^2(\overline{x})$).*

Proof. We proceed by contradiction. Suppose that $\overline{x}$ is not an optimal solution in the second order modified objective function optimization problem ($P^2(\overline{x})$). Then, there exists a feasible solution $\widetilde{x} \in D$ such that

$$f_0(\overline{x}) + (\widetilde{x} - \overline{x})^T \nabla f_0(\overline{x}) + \tfrac{1}{2}(\widetilde{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\widetilde{x} - \overline{x}) < f_0(\overline{x}) + (\overline{x} - \overline{x})^T \nabla f_0(\overline{x}) + \tfrac{1}{2}(\overline{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\overline{x} - \overline{x}).$$

Thus,

$$(\widetilde{x} - \overline{x})^T \nabla f_0(\overline{x}) + \frac{1}{2}(\widetilde{x} - \overline{x})^T \nabla^2 f_0(\overline{x})(\widetilde{x} - \overline{x}) < 0. \tag{16}$$

By assumption, f_0 is a second order convex function on D. Hence, (16) implies

$$(\widetilde{x} - \overline{x})^T \nabla f_0(\overline{x}) < 0. \tag{17}$$

By assumption f_i, $i \in J(\overline{x})$, are second order convex at $\overline{x}$ on D. Then, by Definition 2, it follows that, for $i \in J(\overline{x})$, the inequality

$$f_i(\widetilde{x}) - f_i(\overline{x}) \geq (\widetilde{x} - \overline{x})^T \left[\nabla f_i(\overline{x}) + \nabla^2 f_i(\overline{x})y\right] - \frac{1}{2}y^T\nabla^2 f_i(\overline{x})y \tag{18}$$

holds for all $y \in R^n$. Then, for $y = \widetilde{x} - \overline{x}$ and $i \in J(\overline{x})$, we get

$$f_i(\widetilde{x}) \geq (\widetilde{x} - \overline{x})^T \nabla f_i(\overline{x}) + \frac{1}{2} (\widetilde{x} - \overline{x})^T \nabla^2 f_i(\overline{x}) (\widetilde{x} - \overline{x}) . \tag{19}$$

Also, by (18), for $y = 0$ and $i \in J(\overline{x})$, it follows that

$$f_i(\widetilde{x}) \geq (\widetilde{x} - \overline{x})^T \nabla f_i(\overline{x}) . \tag{20}$$

Since $\widetilde{x} \in D$, then we obtain from (19) and (20) for $i \in J(\overline{x})$, respectively,

$$(\widetilde{x} - \overline{x})^T \nabla f_i(\overline{x}) + \frac{1}{2} (\widetilde{x} - \overline{x})^T \nabla^2 f_i(\overline{x}) (\widetilde{x} - \overline{x}) \leq 0, \tag{21}$$

$$(\widetilde{x} - \overline{x})^T \nabla f_i(\overline{x}) \leq 0. \tag{22}$$

Thus, by (17) and (22) it follows that $\widetilde{x} - \overline{x} \in C(\overline{x})$, that is, $\widetilde{x} - \overline{x}$ is a critical direction at $\overline{x}$. By assumption, $\overline{x}$ is such a feasible solution in the original mathematical programming problem, at which the second order necessary optimality conditions in the dual form (6)-(11) are satisfied. Then, for every $d \in C(\overline{x})$, there exist $\overline{\lambda} \in R_+$ and $\overline{\xi} \in R^m_+$ such that the second order necessary optimality conditions in the dual form are satisfied at $\overline{x}$. Since $\widetilde{x} - \overline{x} \in C(\overline{x})$ then they are satisfied at $\overline{x}$ for $d = \widetilde{x} - \overline{x}$. Hence, using $\overline{\lambda} \in R_+$ and $\overline{\xi} \in R^m_+$, we obtain from (16) and (21), respectively,

$$(\widetilde{x} - \overline{x})^T \overline{\lambda} \nabla f_0(\overline{x}) + \frac{1}{2} (\widetilde{x} - \overline{x})^T \overline{\lambda} \nabla^2 f_0(\overline{x}) (\widetilde{x} - \overline{x}) < 0, \tag{23}$$

$$(\widetilde{x} - \overline{x})^T \overline{\xi}_i \nabla f_i(\overline{x}) + \frac{1}{2} (\widetilde{x} - \overline{x})^T \overline{\xi}_i \nabla^2 f_i(\overline{x}) (\widetilde{x} - \overline{x}) \leq 0, \quad i \in J(\overline{x}). \tag{24}$$

Thus, by the second order necessary optimality conditions (9) and (10), we obtain from (23) and (24), respectively

$$(\widetilde{x} - \overline{x})^T \overline{\lambda} \nabla^2 f_0(\overline{x}) (\widetilde{x} - \overline{x}) < 0, \tag{25}$$

$$(\widetilde{x} - \overline{x})^T \overline{\xi}_i \nabla^2 f_i(\overline{x}) (\widetilde{x} - \overline{x}) \leq 0, \quad i \in J(\overline{x}). \tag{26}$$

Hence, by Definition 3, we get the inequality

$$(\widetilde{x} - \overline{x})^T \nabla^2 L(\overline{x}, \overline{\lambda}, \overline{\xi}) (\widetilde{x} - \overline{x}) < 0,$$

which contradicts the second order necessary optimality condition (7). Thus, the conclusion of the theorem is proved and, therefore, $\overline{x}$ is also optimal in $(\mathrm{P}^2(\overline{x}))$. ■

Based on the theorem above, the following result is true:

THEOREM 5 *Let $\overline{x}$ be a optimal solution in the original nonlinear mathematical programming problem (P). Moreover, we assume that f_i, $i \in J_0(\overline{x})$, are second order convex at $\overline{x}$ on D. Then $\overline{x}$ is optimal in an associated second order modified objective function optimization problem ($P^2(\overline{x})$).*

Proof. Follows from Theorem 4. ■

4. Applications to O.R. problems

In this section, we discuss the potential applications of the introduced approach to solving certain O.R. problems.

Now, we present an example of economic optimization problems involving second order convex functions, which we solve by using the introduced second order modified objective method.

A common problem in organizations is determining how much of a needed item should be kept on hand. For producers, the problem may relate to how many units of each raw material should be kept available. This problem is identified with an area called inventory control, or inventory management. Concerning the question of how much "inventory" to keep on hand, there may be costs associated with having too little or too much inventory on hand.

Now, we consider more precisely an example of an economic problem of a similar nature.

EXAMPLE 1 (**Police Patrol Allocation**) *A police department has determined that the average daily crime rate in the city depends upon the number of officers assigned to each shift. Specifically, the function describing this relationship is*

$$N = 200 - 5xe^{-0.02x},$$

where N equals the average daily crime rate and x equals the number of officers assigned to each shift. Police analysts indicate that the function f is valid for x assumed to satisfy the condition: $f_1(x) = x^2 - 50x \leq 0$ (this condition follows, for example, from the possibility of employing officers in this department). It is not difficult to see that the feasible solution $\overline{x} = 50$ is optimal in the considered optimization problem.

Thus, we obtain the following constrained mathematical programming problem:

$$\begin{array}{r} f_0(x) = 200 - 5xe^{-0.02x} \to \min \\ f_1(x) = x^2 - 50x \leq 0 \end{array} \quad \text{(P)}$$

Since both the objective function f_0 and the constraint function f_1 are second order convex at $\overline{x}$ then this optimization problem can be solved by using the second order modified objective function method. Then, we construct the following optimization problem ($P^2(\overline{x})$) for the considered Police Patrol Allocation problem:

$$\begin{array}{r} 200 + 0.1e^{-1}(x-50)^2 \to \min \\ x^2 - 50x \leq 0. \end{array} \quad (\mathrm{P}^2(50))$$

Thus, we obtain a quadratic optimization problem. We are in position to solve the above optimization problem immediately. It is not difficult to see that the feasible solution $\overline{x} = 50$ *is optimal in this optimization problem.*

5. Computational algorithm

In this section, based on the second order modified objective function approach, we discuss the potential algorithms for solving the nonlinear optimization problems in which functions involved are second order convex on the set of all feasible solutions.

As follows from the preceding section, using the second order objective function approach, in place of solving the original mathematical programming with nonlinear objective function, we can solve some auxiliary optimization problem with a second order modified objective function. Hence, in place of solving the original mathematical programming problem (P) with a strongly nonlinear objective function, we have to solve a quadratic optimization problem. Based on Theorems 3 and 5, we conclude that the original programming problem (P) and its associated second order modified objective function optimization problem are equivalent in the following sense: $\overline{x}$ is optimal in the original programming problem (P) if and only if $\overline{x}$ is optimal in the second order modified optimization problem ($\mathrm{P}^2(\overline{x})$). However, the second order modified objective function optimization problem ($\mathrm{P}^2(\overline{x})$) is easier to solve.

This conclusion follows from the fact that, over the years, a large number of methods have been developed for solving quadratic optimization problems. These methods can be divided into two categories: finite methods and iterative methods. Finite methods solve a given optimization problem by some kind of pivoting procedures and terminate in finite time (see, for example, Dantzing, 1963; Cottle and Dantzing, 1968; Zangwill, 1969). On the other hand, iterative methods generate an infinite sequence, which converges to a limit point that solves a given optimization problem (see, for example, Bazaraa, Sherali, Shetty, 1991; Fletcher, 2000; Pang, 1984).

In this section, we shall present an algorithm of an iterative method for the solution of the considered nonlinear mathematical programming problem which is based on solving the second order modified objective function optimization problem. This algorithm reduces the solution of the original mathematical programming with a nonlinear objective function to a sequence of such auxiliary optimization problems.

To prove convergence of the presented algorithm to an optimal point in the original mathematical programming problem (P), we shall use the concept of R-superlinear and R-quadratic convergence rate as defined in Ortega, Rheinboldt (1970).

Definition 6 *The sequence* $\{x_k\}$ *in* R^n *is said to converge to* $\overline{x}$ *with R-superlinear rate if, for each* $\delta \in (0, 1)$*, no matter how small, there exist* $\alpha > 0$*,*

$p \geq 0$ such that the following inequality

$$\|x_k - \overline{x}\| \leq \alpha\delta^k$$

holds for all $k \geq p$.

DEFINITION 7 *The sequence $\{x_k\}$ in R^n is said to converge to $\overline{x}$ with R-quadratic rate if there exist $\delta \in (0,1)$, $\alpha > 0$, $p \geq 0$ such that the following inequality*

$$\|x_k - \overline{x}\| \leq \alpha\delta^{2^k}$$

holds for all $k \geq p$.

Now, we present an algorithm for solving the considered nonlinear mathematical programming problem (P) with second order convex functions. Based on the second order modified objective function approach, to solve the original mathematical programming problem (P), this method consists of generating the sequence of second order modified objective function optimization problems $(P^2(x_k))$. At each step an optimization problem of such a type is solved by means of a finite algorithm to obtain the next point x_{k+1} and hence the next auxiliary optimization problem $(P^2(x_{k+1}))$.

Sukharev, Timokhov and Fedorov (1986) and Fletcher (2000) considered some finite method for solving a differentiable quadratic optimization problem with linear inequality constraints. Based on this method, we propose an iteration method for solving a nonlinear optimization problem with inequality constraints. This method consists of solving the following auxiliary optimization problem $(P^2(x_k))$

$$F_k(x) := f_0(x_k) + (x - x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x - x_k)^T \nabla^2 f_0(x_k)(x - x_k) \to \min$$
$$f_i(x) \leq 0, \ \ i \in J, \qquad (P^2(x_k))$$

to obtain the next point x_{k+1} in the iteration sequence generated in this method. At this point, we construct a successive optimization problem $(P^2(x_k))$ with a second order modified objective function.

Note that, since f_0 is assumed to be a twice continuously differentiable function, then $\nabla^2 f_0(x_k)$ is the continuous, symmetric $n \times n$ Hessian matrix.

PROPOSITION 1 *Let the objective function f_0 in the original mathematical programming problem be a second order convex function on the set of all feasible solutions D. Then:*

i) the objective function F_k in the second order modified objective optimization problem $\left(P^2(x_k)\right)$ is a second order convex function at x_k on the set of all feasible solutions D.

ii) $$F_k(x_k) \geq F_k(x_{k+1}) + (x_k - x_{k+1})^T \nabla F_k(x_{k+1}) + \tfrac{1}{2}(x_k - x_{k+1})^T \nabla^2 F_k(x_{k+1})(x_k - x_{k+1}).$$

iii) $(x_k - x_{k+1})^T \nabla F_k(x_{k+1}) \geq 0.$

iv) Moreover, the objective function F_k in the second order modified objective optimization problem $\left(P^2(x_k)\right)$ is a second order convex function on the set of all feasible solutions D.

Proof. Proofs of Propositions i) and ii) follow by Definition 2 and the definition of the objective function F_k in the second order modified objective function optimization problem $(P^2(x_k))$.

Proof of Proposition iii) follows from Proposition 1 i) and ii).

Proof iv) Let z, x be any feasible solutions in problem (P), that is, $z, x \in D$. Then, the following inequality

$$\frac{1}{2}\Big[(z-x_k)^T \nabla^2 f_0(x_k)(z-x_k) - 2(z-x_k)^T \nabla^2 f_0(x_k) y + y^T \nabla^2 f_0(x_k) y\Big] \geq 0$$

holds. Since f_0 is a C^2 function then the matrix $\nabla^2 f_0(x_k)$ is symmetric. Hence, the inequality above gives

$$\tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(z-x_k) - (z-x_k)^T \nabla^2 f_0(x_k) y + \tfrac{1}{2} y^T \nabla^2 f_0(x_k) y \geq \\ \tfrac{1}{2}(z-x)^T \nabla^2 f_0(x_k)(x-x_k) - \tfrac{1}{2}(x-x_k)^T \nabla^2 f_0(x_k)(z-x),$$

and, so

$$\tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(z-x) \geq \tfrac{1}{2}(z-x)^T \nabla^2 f_0(x_k)(x-x_k) + \\ (z-x)^T \nabla^2 f_0(x_k) y - \tfrac{1}{2} y^T \nabla^2 f_0(x_k) y.$$

Thus,

$$\tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(z-x_k) \geq \tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(x-x_k) \\ +\tfrac{1}{2}(z-x)^T \nabla^2 f_0(x_k)(x-x_k) + (z-x)^T \nabla^2 f_0(x_k) y - \tfrac{1}{2} y^T \nabla^2 f_0(x_k) y,$$

and, so

$$\tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(z-x_k) \geq \tfrac{1}{2}(x-x_k)^T \nabla^2 f_0(x_k)(x-x_k) \\ +\tfrac{1}{2}(z-x) \nabla^2 f_0(x_k)(x-x_k) + \tfrac{1}{2}(z-x) \nabla^2 f_0(x_k)(x-x_k) \\ +(z-x) \nabla^2 f_0(x_k) y - \tfrac{1}{2} y \nabla^2 f_0(x_k) y.$$

Hence,

$$f_0(x_k) + (z-x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(z-x_k)^T \nabla^2 f_0(x_k)(z-x_k) \geq \\ f_0(x_k) + (x-x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(x-x_k)^T \nabla^2 f_0(x_k)(x-x_k) + \\ (z-x)^T \nabla f_0(x_k) + (z-x)^T \nabla^2 f_0(x_k)(x-x_k) + \\ (z-x)^T \nabla^2 f_0(x_k) y - \tfrac{1}{2} y^T \nabla^2 f_0(x_k) y.$$

We write down the inequality above

$$\begin{aligned}&f_0(x_k) + (z - x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(z - x_k)^T \nabla^2 f_0(x_k)(z - x_k) \geq \\ &f_0(x_k) + (x - x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(x - x_k)^T \nabla^2 f_0(x_k)(x - x_k) + \\ &\quad (z - x)^T \left[\nabla f_0(x_k) + \nabla^2 f_0(x_k)(x - x_k) + \nabla^2 f_0(x_k)\, y\right] - \tfrac{1}{2} y^T \nabla^2 f_0(x_k)\, y.\end{aligned}$$

Then, by the definition of the objective function F_k in the second order modified objective function optimization problem $(P^2(x_k))$, we get

$$F_k(z) \geq F_k(x) + (z - x)^T \left[\nabla F_k(x) + \nabla^2 F_k(x) y\right] - \frac{1}{2} y^T \nabla^2 F_k(x) y.$$

Thus, by Definition 2, we conclude that F_k is second order convex on the set of all feasible solutions D. ∎

We shall establish that the convergence rate of the proposed algorithm is superlinear, and that it is at least quadratic if the Hessian of the objective function f_0 in the original mathematical programming problem (P) is assumed to satisfy the Lipschitz condition on the set of all feasible solutions.

THEOREM 6 *Let D be a convex compact subset of R^n and f_0 be a twice continuously differentiable function on D. Moreover, we assume that f_0 is a second order strictly convex function on D. Then, the iteration sequence $\{x_k\}$ generated by solving the sequence of optimization problems $\left(P^2(x_k)\right)$ satisfies the following conditions:*

i) *$f_0(x_{k+1}) \leq f_0(x_k)$ for any k,*

ii) *if f_0 is a second order strictly convex function on D then the iteration sequence $\{x_k\}$ converges superlinearly to $\overline{x}$,*

iii) *if f_0 is a second order strictly convex function on D and the matrix $\nabla^2 f_0$ satisfies the Lipschitz condition on D then the iteration sequence $\{x_k\}$ converges quadratically to $\overline{x}$.*

Proof. i) By assumption, f_0 is a second order convex function on D. Then, by Definition 2, the following inequality

$$f_0(x) \geq f_0(x_k) + (x - x_k)^T \left[\nabla f_0(x_k) + \nabla^2 f_0(x_k)\, y\right] - \tfrac{1}{2} y^T \nabla^2 f_0(x_k)\, y$$

holds for any $x \in D$ and all $y \in R^n$. Hence, for $x = x_{k+1}$ and $y = x_{k+1} - x_k$, it follows that

$$f_0(x) \geq f_0(x_k) + (x_{k+1} - x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k). \tag{27}$$

From the algorithm follows that x_{k+1} is an optimal point of the function $F_k(x)$. Then, by Proposition 1 iii),

$$(x_k - x_{k+1})^T \nabla F_k(x_{k+1}) \geq 0.$$

Thus,

$$(x_{k+1} - x_k)^T \nabla f_0(x_k) + (x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k) \leq 0. \tag{28}$$

Since f_0 is a second order convex function, then (28) gives

$$(x_{k+1} - x_k)^T \nabla f_0(x_k) \leq 0.$$

The inequality above implies that

$$f_0(x_{k+1}) \leq f_0(x_k).$$

This means that the iterative sequence $\{f_0(x_k)\}$ is a monotonic decreasing sequence. Hence, the proof is complete. ■

Proof. ii) Let x_{k+1} be optimal in $(P^2(x_k))$. By assumption, the objective f_0 is a second order strictly convex function and the set of all feasible solutions D in the original optimization problem (P) is convex. Then, by Proposition 1 iii), the inequality

$$(x_k - x_{k+1})^T \nabla F_k(x_{k+1}) \geq 0$$

holds. Thus,

$$(x_{k+1} - x_k)^T \nabla f_0(x_k) + (x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k) \leq 0. \tag{29}$$

By definition, we have

$$\begin{aligned} F_k(x_{k+1}) = f_0(x_k) + (x_{k+1} - x_k)^T \left[\nabla f_0(x_k) + \nabla^2 f_0(x_k)(x_{k+1} - x_k)\right] \\ -\tfrac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k). \end{aligned}$$

By Lemma 2, it follows that $(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k) \geq m\,\|x_{k+1}-x_k\|^2$. Then, by (29),

$$F_k(x_{k+1}) - f_0(x_k) \leq -\frac{m}{2}\|x_{k+1} - x_k\|^2. \tag{30}$$

By assumption, f_0 is a second order strictly convex function on the set of all feasible solutions D. Then, by Definition 2, the inequality

$$\begin{aligned} f_0(x_{k+1}) \geq f_0(x_k) + (x_{k+1} - x_k)^T \left[\nabla f_0(x_k) + \nabla^2 f_0(x_k)(x_{k+1} - x_k)\right] \\ -\tfrac{1}{2} y^T \nabla^2 f_0(x_k)\, y \end{aligned}$$

holds for all $y \in R^n$. Hence, it is also satisfied for $y = x_{k+1} - x_k$. Thus, the inequality above yields

$$\begin{aligned} f_0(x_{k+1}) \geq f_0(x_k) + (x_{k+1} - x_k)^T \nabla f_0(x_k) \\ +\tfrac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k), \end{aligned} \tag{31}$$

and, so

$$f_0(x_{k+1}) \geq F_k(x_{k+1}). \tag{32}$$

By Theorem 6 i), it follows that $f_0(x_k) \geq f_0(x_{k+1})$. Then, by (31) and (32),

$$f_0(x_k) - F_k(x_{k+1}) \geq f_0(x_{k+1}) - F_k(x_{k+1}) \geq 0.$$

Thus, we have

$$F_k(x_{k+1}) - f_0(x_k) = (x_{k+1} - x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x_k - x_{k+1})^T \nabla^2 f_0(x_k)(x_{k+1} - x_k)$$

and, so

$$F_k(x_{k+1}) - f_0(x_k) \geq (x_k - x_{k+1})^T \nabla f_0(x_k). \tag{33}$$

Now, we use the Lagrange formula for the function $\Psi(x) = (x_{k+1} - x_k)^T \nabla f_0(x)$. Then, we obtain

$$(x_{k+1} - x_k)^T [\nabla f_0(x_k) - \nabla f_0(x_{k-1})] = (x_{k+1} - x_k)^T \nabla^2 f_0(\widetilde{x}_k)(x_k - x_{k-1}),$$

where $\widetilde{x}_k = x_{k-1} + \theta_k (x_k - x_{k-1})$, $\theta_k \in (0,1)$. Note that $\widetilde{x}_k \in D$ since by assumption D is a convex set. Hence,

$$(x_{k+1} - x_k)^T \nabla f_0(x_k) \geq (x_{k+1} - x_k)^T \nabla f_0(x_{k-1}) + (x_{k+1} - x_k)^T \nabla^2 f_0(\widetilde{x}_k)(x_k - x_{k-1}). \tag{34}$$

By (33) and (34),

$$F_k(x_{k+1}) - f_0(x_k) \geq (x_{k+1} - x_k)^T \nabla f_0(x_{k-1}) + (x_{k+1} - x_k)^T \nabla^2 f_0(\widetilde{x}_k)(x_k - x_{k-1}).$$

Thus,

$$F_k(x_{k+1}) - f_0(x_k) \geq (x_{k+1} - x_k)^T \left[\nabla f_0(x_{k-1}) + \nabla^2 f_0(x_{k-1})(x_k - x_{k-1})\right] + (x_{k+1} - x_k)^T \left[\nabla^2 f_0(\widetilde{x}_k) - \nabla^2 f_0(x_{k-1})\right](x_k - x_{k-1}). \tag{35}$$

Since x_k is optimal in the second order modified objective function optimization problem $(P^2(x_{k-1}))$ then, by Lemma 1,

$$(x_{k+1} - x_k)^T \left[\nabla f_0(x_{k-1}) + \nabla^2 f_0(x_{k-1})(x_k - x_{k-1})\right] \geq 0.$$

Thus, (35) gives

$$F_k(x_{k+1}) - f_0(x_k) \geq (x_{k+1} - x_k)^T \left[\nabla^2 f_0(\widetilde{x}_k) - \nabla^2 f_0(x_{k-1})\right](x_k - x_{k-1}),$$

and by (30),

$$f_0(x_k) - F_k(x_{k+1}) \leq \|x_{k+1} - x_k\| \left\|\nabla^2 f_0(x_{k-1}) - \nabla^2 f_0(\widetilde{x}_k)\right\| \|x_k - x_{k-1}\|. \tag{36}$$

Using (30) together with (36), we obtain

$$\frac{m}{2}\|x_{k+1} - x_k\|^2 \leq \|x_{k+1} - x_k\| \left\|\nabla^2 f_0(x_{k-1}) - \nabla^2 f_0(\widetilde{x}_k)\right\| \|x_k - x_{k-1}\|$$

and, so

$$\|x_{k+1} - x_k\| \leq \frac{2\left\|\nabla^2 f_0(x_{k-1}) - \nabla^2 f_0(\widetilde{x}_k)\right\|}{m}\|x_k - x_{k-1}\|. \tag{37}$$

We denote by ε_k the sequence $\varepsilon_k = \frac{2\left\|\nabla^2 f_0(x_{k-1}) - \nabla^2 f_0(\widetilde{x}_k)\right\|}{m}$. Then,

$$\|x_{k+1} - x_k\| \leq \varepsilon_k \|x_k - x_{k-1}\|. \tag{38}$$

Since f_0 is twice continuously differentiable on the set D then, by the definition of the point $\widetilde{x}_k$, it follows that $\varepsilon_k \to 0$ for $k \to \infty$.

Now, we show that the sequence $\{x_k\}$ generated by the algorithm converges superlinearly to $\overline{x}$. Thus, by induction, we get from (38),

$$\|x_{k+1} - x_k\| \leq \varepsilon_k \varepsilon_{k-1} ... \varepsilon_1 \|x_1 - x_0\|. \tag{39}$$

Hence, we have for any $j > k$,

$$\|x_j - x_k\| \leq \|x_j - x_{j-1}\| + ... + \|x_{k+1} - x_k\|.$$

Then, by (39),

$$\|x_j - x_k\| \leq [(\varepsilon_{j-1}\varepsilon_{j-2}...\varepsilon_1) + ... + (\varepsilon_k \varepsilon_{k-1}...\varepsilon_1)] \|x_1 - x_0\|. \tag{40}$$

For the given $\delta \in (0, 1)$, we set $p > 0$ such that $\varepsilon_k \leq \delta$ for any $k > p$. We denote $\beta = \varepsilon_p ... \varepsilon_2 \varepsilon_1 \delta^p$. Then, for any $j > k > p$, (40) yields,

$$\|x_j - x_k\| \leq \beta\left(\delta^{j-1} + \delta^{j-2} + ... + \delta^k\right)\|x_1 - x_0\|.$$

Hence,

$$\|x_j - x_k\| \leq \frac{\beta\|x_1 - x_0\|}{1-\delta}\delta^k. \tag{41}$$

Let $\alpha = \frac{\beta\|x_1 - x_0\|}{1-\delta}$. Then, $\alpha < \infty$ and (41) gives

$$\|x_j - x_k\| \leq \alpha\delta^k.$$

From the inequality above it follows that $\|x_j - x_k\| \to 0$ as $j, k \to \infty$. Hence, the sequence $\{x_k\}$ is a Cauchy sequence, which converges to $\overline{x}$. Since x_j converges to $\overline{x}$ then by letting $k \to \infty$ in (41), we get the inequality

$$\|x_k - \overline{x}\| \leq \alpha\delta^k,$$

which, by Definition 6, provides that the sequence $\{x_k\}$ converges superlinearly to $\overline{x}$. ■

Proof. iii) Now, we assume that the matrix $\nabla^2 f_0$ satisfies the Lipschitz condition on D. Then

$$\left\|\nabla^2 f_0\left(\widetilde{x}_k\right) - \nabla^2 f_0\left(x_{k-1}\right)\right\| \leq L\left\|\widetilde{x}_k - x_{k-1}\right\|. \tag{42}$$

By definition, $\widetilde{x}_k = x_{k-1} + \theta_k\left(x_k - x_{k-1}\right)$, $\theta_k \in (0,1)$. Hence,

$$\widetilde{x}_k - x_{k-1} = \theta_k\left(x_k - x_{k-1}\right).$$

Using (42) together with (37), we get

$$\|x_{k+1} - x_k\| \leq \frac{2\theta_k L}{m}\|x_k - x_{k-1}\|^2. \tag{43}$$

Now, we show that the sequence $\{x_k\}$ generated by the algorithm converges quadratically to $\overline{x}$. We denote $\varepsilon = \frac{2L}{m}\|x_k - x_{k-1}\|$. Since $\|x_{k+1} - x_k\|$ converges to 0 then there exists p such that for any $k > p$ it follows that $\varepsilon < 1$. Thus, by induction, it follows from (43) that, for any $k \geq p$,

$$\|x_{k+1} - x_k\| < \frac{m\theta_k}{2L}\varepsilon^{2^{k-p}}. \tag{44}$$

Then, using (44), we obtain, for any $j > k \geq p$,

$$\|x_j - x_k\| \leq \frac{m\theta_k}{2L}\sum_{i=1}^{j-1}\varepsilon^{2^{i-p}}.$$

Hence,

$$\|x_j - x_k\| \leq \frac{m\theta_k}{2L}\varepsilon^{2^{k-p}}\sum_{i=0}^{\infty}\varepsilon^{2^i-1}. \tag{45}$$

We define $\delta = \varepsilon^{2^{-p}}$ and $\alpha = \frac{m\theta_k}{2L}\sum_{i=0}^{\infty}\varepsilon^{2^i-1}$. Note that $\sum_{i=0}^{\infty}\varepsilon^{2^i-1}$ is a series of positive numbers for which

$$\frac{\varepsilon^{2^{i+1}-1}}{\varepsilon^{2^i-1}} = \varepsilon^{2^i} < 1.$$

Hence, the series $\sum_{i=0}^{\infty} \varepsilon^{2^i-1}$ in the definition of α is convergent. Thus, by (45), for any $j > p$,

$$\|x_j - x_k\| \leq \alpha\delta^{2^k}.$$

Since x_j converges to $\overline{x}$ then letting $k \to \infty$ in (41), we get the inequality

$$\|x_k - \overline{x}\| \leq \alpha\delta^{2^k}$$

which, by Definition 7, provides that the sequence $\{x_k\}$ converges quadratically to $\overline{x}$. ■

PROPOSITION 2 *Let the objective function f_0 be a second order function on the set of all feasible solutions D. Then, for each k,*

$$F_k(x_{k+1}) \leq F_k(x_k).$$

Proof. By definition,

$$F_k(x_{k+1}) = f_0(x_k) + (x_{k+1} - x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k) \tag{46}$$

and

$$F_k(x_k) = f_0(x_k). \tag{47}$$

Hence, by (46) and (47)

$$F_k(x_{k+1}) - F_k(x_k) = (x_{k+1} - x_k)^T \nabla f_0(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k). \tag{48}$$

By assumption, f_0 is a second order function on the set of all feasible solutions D. Then, by Definition 2,

$$f_0(x_{k+1}) - f_0(x_k) \geq (x_{k+1} - x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k).$$

Hence, by Theorem 6 i),

$$(x_{k+1} - x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x_{k+1} - x_k)^T \nabla^2 f_0(x_k)(x_{k+1} - x_k) \leq 0. \tag{49}$$

Hence, by (48) and (49), we get the inequality

$$F_k(x_{k+1}) - F_k(x_k) \leq 0.$$ ■

Theorem 7 *Let D be a convex compact subset of R^n and f_0 be a twice continuously differentiable function on D. Moreover, we assume that f_0 is a second order convex function on D. Then,*

$$\lim_{k\to\infty} f_0(x_k) = f_0(\overline{x}) = \min_{x\in D} f_0(x).$$

Proof. By assumption, D is a compact subset of R^n. As follows from the proof of Theorem 6, the iterative sequence $\{x_k\}$ generated by the algorithm is a Cauchy sequence. Then, it is convergent and its accumulation point $\overline{x}$ belongs to D. By assumption, f_0 is a second order convex function on D. Then it is continuous on D and, by Weierstrass' theorem, f_0 attains minimum at point $\overline{x}$. Hence, the sequence $\{f_0(x_k)\}$ converges to $f_0(\overline{x})$. ■

Now, we prove that the sequence $\{F_k(x_{k+1})\}$ is a monotonic decrease sequence. In other words, we show that the sequence of the values of objective functions at the optimal points in optimization problems $\left(P^2(x_k)\right)$ is monotonically decreasing.

Proposition 3 *Let the objective function f_0 be a second order function on the set of all feasible solutions D. Then, for any k,*

$$F_{k+1}(x_{k+2}) \leq F_k(x_{k+1}).$$

Proof. From the definition of the objective function in the second order modified objective function optimization problem, we have

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) = f_0(x_{k+1}) + (x_{k+2}-x_{k+1})^T \nabla f_0(x_{k+1}) + \frac{1}{2}(x_{k+2}-x_{k+1})^T \nabla^2 f_0(x_{k+1})(x_{k+2}-x_{k+1}) - \left[f_0(x_k) + (x_{k+1}-x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k)\right].$$

By assumption, f_0 is second order convex on D. Then, by Definition 2,

$$f_0(x_{k+2}) \geq f_0(x_{k+1}) + (x_{k+2}-x_{k+1})^T \nabla f_0(x_{k+1}) + \frac{1}{2}(x_{k+2}-x_{k+1})^T \nabla^2 f_0(x_{k+1})(x_{k+2}-x_{k+1}).$$

Hence,

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) \leq f_0(x_{k+2}) - \left[f_0(x_k) + (x_{k+1}-x_k)^T \nabla f_0(x_k) + \frac{1}{2}(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k)\right],$$

and, so

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) \leq f_0(x_{k+2}) - f_0(x_k) - \left[(x_{k+1}-x_k)^T \nabla f_0(x_k) + (x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k)\right] + \frac{1}{2}(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k).$$

Since x_{k+1} is an optimal point in the second order modified objective optimization problem $\left(P^2(x_k)\right)$ and f_0 is second order convex on D then by Lemma 1, we get

$$(x_{k+1}-x_k)^T \nabla f_0(x_k) + (x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k) \leq 0.$$

Hence, using the two last inequalities, it follows that

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) \leq f_0(x_{k+2}) - f_0(x_k) \\ +\tfrac{1}{2}(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k). \tag{50}$$

Since f_0 is second order convex on D then by Lemma 2,

$$(x_{k+1}-x_k)^T \nabla^2 f_0(x_k)(x_{k+1}-x_k) \geq 0. \tag{51}$$

Then, by (50) and (51), we obtain

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) \leq f_0(x_{k+2}) - f_0(x_k).$$

Thus, by Theorem 6 i), it follows that the following inequality

$$F_{k+1}(x_{k+2}) - F_k(x_{k+1}) \leq 0$$

is satisfied. This means that the conclusion of this proposition is established. ■

THEOREM 8 *Let D be a convex compact subset of R^n and f_0 be a twice continuously differentiable function on D. Moreover, we assume that f_0 is a second order convex function on D. Then,*

$$\lim_{k\to\infty} F_k(x_{k+1}) = F(\overline{x}) = \min_{x\in D} F(x).$$

6. Preliminary computational results

In order to gain some understanding of the practical performance of the proposed method, we have implemented the iterative algorithm from the preceding section.

We describe the implementation details for the nonlinear mathematical programming problem from Example 1. We have used various starting points, where these starting points are feasible for the optimization problem considered in Example 1.

We write the auxiliary optimization problem $(P^2(x_k))$ for the considered O.R. problem from Section 4 in the following form:

$$\begin{array}{c} F_k(x) = a_k\left(x^2 + b_k x\right) + c_k \to \min \\ f_1(x) \leq 0. \end{array} \qquad (\mathrm{P}^2(x_k))$$

In the following tables we report the computational results obtained with the above implementation by using two values of x_0 in the iterative sequence generated by the algorithm presented in the preceding section.

x_k	$f_0(x_k)$	$F_{k-1}(x_k)$	a_k	b_k	c_k
0.000000000000	200.0000		0.100000	-50.000000	200.0000
25.000000000000	124.1836	137,5000	0.045489	-83.333333	190.5230
41.666666666667	109.4587	111,5476	0.025351	-97.619047	168.5621
48.809523809524	108.0566	108,1653	0.019285	-99.944629	156.1914
49.972314507198	108.0301	108,0306	0.018414	-99.9999693	154.0660
49.999984678753	108.0301	108,0301	0.018393	-99.9999999	154.0151
49.999999999995	108.0301	108,0301	0.018393	-100.0000000	154.0151
50.000000000000	108.0301	108,0301	0.018393	-100.0000000	154.0151

x_k	$f_0(x_k)$	$F_{k-1}(x_k)$	a_k	b_k	c_k
39.000000000000	110.6108		0.027962	-96.032786	172.8078
48.016393442623	108.1045	108.3376	0.019897	-99.848617	157.6256
49.924308926697	108.0302	108.0320	0.018449	-99.999771	154.1542
49.999885590424	108.0301	108.0301	0.018394	-99.999999	154.0153
49.999999999738	108.0301	108.0301	0.018393	-100.000000	154.0151
50.000000000000	108.0301	108.0301	0.018393	-100.000000	154.0151

From the results reported in the tables, we see that the implemented iterative method has been able to find a very good approximation of the optimal solution within an acceptable solution set. For different start points x_0, the method is convergent to an unique solution $\overline{x} = 50$ of the considered optimization problem. The iterative method used to solve of this optimization problem generates feasible iterates that mark an improvement of the objective value at each iteration.

Further, the proposed algorithm has several advantages. First of all it features sure convergence properties and has fast (quadratic) rate of convergence. It also requires a feasible starting point but then generates a sequence of feasible points. Thus, the method is globally convergent if a starting point is chosen as feasible in the optimization problem.

Because of the remarks above and because of the uncertainty of the convergence properties, due mainly to the use of a suitable method for solving auxiliary optimization problems $(P^2(x_k))$, research on the algorithm will continue. However, the discussion and the numerical results presented suggest that the method as is stands will solve many constrained minimization problems with twice differentiable functions, more precisely, with second order convex functions.

7. Conclusions

In the paper, we present the second order modified objective function method for solving constrained optimization problems with second order convex functions. Further, we develop an algorithm for finding an optimal solution to the original mathematical programming problem by using the proposed approach. Since the functions constituting the original programming problem are assumed to be second order convex then the presented algorithm requires calculations with the Hessian matrix. As follows from the construction of the introduced method, the original programming problem is equivalent to an associated second order modified objective function optimization problem. However, to solve the second order objective function optimization problem the presented iterative algorithm requires to solve the sequence of optimization problems of such type.

To prove most of results in the paper, the objective function f_0 in the original mathematical programming problem (P) is assumed to be a (strictly) second order convex function on the set of all feasible solutions D. However, to establish the equivalence between the original mathematical programming problem (P) and its second order modified objective function optimization problem $(P^2(\overline{x}))$, it is sufficient to assume only that the objective function f_0 is second order convex at the point $\overline{x}$.

It is known that efficient and finite methods for solving subproblems $(P^2(x_k))$ exist. For example, if x_k is known then the next point x_{k+1} can be determined by the principal pivoting method, Cottle and Dantzing (1968), or the finite method, Stoer (1971). Further, finite methods for solving a quadratic optimization problem have been considered also by Fletcher (2000), Sukharev, Timokhov and Fedorov (1986).

In this paper, we give an example of O.R. problem which can be solved by using the second order modified objective function method. Of course, there exist O.R. problems with more complicated objective function, which can be solved by the introduced method. As follows from the example of the O.R. problem given in the paper, we obtain, using the presented method, a simpler optimization problem to solve. Thus, the computational procedures for solving a quadratic optimization problem can be applied. In this way, O.R. problems with a strongly nonlinear objective function can be solved by using computational algorithms for solving quadratic optimization problems (moreover, we are in a position to solve immediately some second order modified objective function optimization problems).

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References

ANTCZAK, T. (2004) An η-approximation approach for nonlinear mathematical programming problems involving invex functions. *Numerical Functional Analysis and Optimization* **25** (5&6), 423-438.

BAZARAA, M.S., SHERALI, H.D. and SHETTY, C.M. (1991) *Nonlinear Programming: Theory and Algorithms.* John Wiley and Sons, New York.

BECTOR, C.R. and BECTOR, B.K. (1986) On various duality theorems for second order duality in nonlinear programming. *Cahiers Centre Etudes Rech. Opér.* **28**, 283-292.

BECTOR, C.R. and CHANDRA, S. (1985) Generalized bonvex functions and second order duality in mathematical programming. *Res. Rep.* 85-2, Department of Actuarial and Management Sciences, University of Manitoba, Winnipeg, Manitoba, Canada.

BEN-TAL, A. (1980) Second-order and related extremality conditions in nonlinear programming. *Journal of Optimization Theory and Applications* **31**, 143-165.

COTTLE, R.W. and DANTZING, G.B. (1968) Complementary pivot theory of mathematical programming. *Journal of Linear Algebra and Applications* **1**, 103-125.

DANTZING, G.B. (1963) *Linear Programming and Extensions.* Princeton University Press, Princeton, N.J.

FIACCO, A.V. and MCCORMICK, G.P. (1968) *Nonlinear Programming: Sequential Minimization Techniques.* John Wiley & Sons.

FLETCHER, R. (2000) *Practical methods ofoptimization*, Second Edition. John Wiley & Sons, Ltd.

MOND, B. and WEIR, T. (1981) Generalized convexity and higher order duality. *Pure Math. Res. Rep.* 81-16, Math. Dept., La Trobe University, Australia.

ORTEGA, J.M. and RHEINBOLDT, W.C. (1970) *Iterative Solution of Nonlinear Equations of Several Variables.* Academic Press, New York

PANG, J.S. (1984) Necessary and sufficient conditions for the convergence of iterative methods for the linear complementarity problem. *Journal of Optimization Theory and Applications* **42**, 1-17.

STOER, J. (1971) On the numerical solution of constrained least-squares problems. *SIAM Journal of Numerical Analysis*, **8**, 382-411.

SUKHAREV, A.G., TIMOKHOV, A.W. and FEDOROV, W.W. (1986) *Course of optimization methods* (in Russian). “Nauka”, Moscow.

ZANGWILL, W.I. (1969) *Nonlinear Programming.* Prentice Hall, Englewood Cliffs.