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# Realization problem for singular positive continuous-time systems with delays

by

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**Abstract:** The positive realization problem for a class of singular continuous-time linear SISO systems with delays in state and in output is addressed. The notions of canonical forms of matrices and of dual systems are extended for singular linear systems with delays. Necessary and sufficient conditions for positivity of SISO singular continuous-time systems with delays and sufficient conditions for the existence of a positive singular realization are established. A procedure for computation of a positive singular realization of a given transfer function is proposed and illustrated by a numerical example.

**Keywords:** realization, positive, singular, continuous-time, system, existence, procedure.

## 1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in standard delay systems is given in Górecki et al. (1989) and in positive systems theory it is given in the monographs by Farina and Rinaldi (2000), and Kaczorek (2002). Recent developments in positive systems theory and some new results are given in Kaczorek (2003). Realizations problem of positive linear systems without time-delays has been considered in many papers and books, e.g. Benvenuti, Farina (2004), Farina, Rinaldi (2000), Kaczorek (2002).

Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in Busłowicz, Kaczorek (2004), Xie, Wang (2003).

The realization problem for positive multivariable discrete-time systems with one time-delay was formulated and solved in Kaczorek (2004a, b, 2005b, 2006a, b), Kaczorek, Busłowicz (2004), and the realization problem for positive continuous-time systems with delays was investigated in Kaczorek (2005a).

The main purpose of this paper is to present a method for computing a positive (singular) realization of a transfer function for a class of singular continuous-time linear systems with delays in state and in output. Sufficient conditions for the existence of a positive singular realization will be established and a procedure for computation of a positive singular realization of a given transfer function will be proposed.

To the best knowledge of the author the realization problem for singular continuous-time linear systems with delays in state vector and in output has not been considered yet.

## 2. Dual singular systems with delays

Let $R_+^{n\times m}$ be the set of $m \times n$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$. The $n \times n$ identity matrix will be denoted by $I_n$.

Consider the singular single-input single-output (SISO) continuous-time linear systems with $h$ delays in state and $q$ delays in output

$$E\dot{x}(t) = \sum_{i=0}^{h} A_i x(t-id) + Bu(t) \tag{1a}$$

$$y(t) = \sum_{j=0}^{q} C_j x(t-jd) \tag{1b}$$

where $x(t) \in R^n$, $u(t) \in R$, $y(t) \in R$ are the state vector, input and output, respectively and $E$, $A_i \in R^{n\times n}$, $i = 0, 1, ..., h$, $B \in R^n$, $C_j \in R^{1\times n}$, $j = 0, 1, ..., q$, $d > 0$ is the delay.

It is assumed that $\det E = 0$ and the characteristic polynomial is nonzero, i.e.

$$d(s,w) = \det\left[Es - A_0 - A_1 w - \cdots - A_h w^h\right] \neq 0\,,\; w = e^{-sd}. \tag{2}$$

It is also assumed that the initial conditions for (1a)

$$x_0(t)\,,\; t \in [-hd,\, 0] \tag{3}$$

belong to the set $X_0$ of admissible initial conditions.

Definition 1 *The singular system*

$$E^T\dot{x}(t) = \sum_{i=0}^{h} A_i^T x(t-id) + \sum_{j=0}^{q} C_j^T u(t-jd) \tag{4a}$$

$$y(t) = B^T x(t) \tag{4b}$$

*is called the dual system (with respect) to the system (1) where the upper index* $T$ *denotes the transpose and* $x(t)$, $u(t)$, $y(t)$ *and* $E$, $A_i$, $i = 0,1,...,h$, $B$ *and* $C_j$, $j = 0,1,...,q$ *are the same as for the system (1).*

The transfer function of the system (1) is given by

$$T(s,w) = (C_0 + C_1 w + \cdots + C_q w^q)\ [Es - A_0 - A_1 w - \cdots - A_h w^h]^{-1} B. \tag{5}$$

Lemma 1 *The characteristic polynomial*

$$\bar{d}(s,w) = \det\ [E^T s - A_0^T - A_1^T w - \cdots - A_h^T w^h] \tag{6}$$

*and the transfer function*

$$\bar{T}(s,w) = B^T\ [E^T s - A_0^T - A_1^T w - \cdots - A_h^T w^h]^{-1} (C_0^T + C_1^T w + \cdots + C_q^T w^q) \tag{7}$$

*of the dual system (4) is equal to the characteristic polynomial* $d(s,w)$ *and the transfer function* $T(s,w)$ *of the system (1), respectively, i.e.*

$$\bar{d}(s,w) = d(s,w) \tag{8a}$$

$$\bar{T}(s,w) = T(s,w). \tag{8b}$$

*Proof.* The equality (8a) follows from the well-known property that $\det X^T = \det X$ for any $X \in R^{n\times n}$. Taking into account that $\bar{T}^T(s,w) = \bar{T}(s,w)$ and $\left\{[E^T s - A_0^T - A_1^T w - \cdots - A_h^T w^h]^{-1}\right\}^T = [Es - A_0 - A_1 w - \cdots - A_h w^h]^{-1}$ and using (7) and (5) we obtain

$$\begin{aligned}
&\bar{T}(s,w) = \bar{T}^T(s,w) = \\
&= \left\{B^T [E^T s - A_0^T - A_1^T w - \cdots - A_h^T w^h]^{-1} \left(C_0^T + C_1^T w + \cdots + C_q^T w^q\right)\right\}^T = \\
&= (C_0 + C_1 w + \cdots + C_q w^q)\, [Es - A_0 - A_1 w - \cdots - A_h w^h]^{-1} B = T(s,w)
\end{aligned}$$

that is the equality (8b). ■

## 3. Positivity of the system with delays

Let us assume that the matrices of the system (1) have the following canonical forms

$$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n\times n}, \quad A_0 = \left[\begin{array}{c:c} 0 & I_{n-1} \\ \hdashline \multicolumn{2}{c}{a_0} \end{array}\right] \in R^{n\times n},$$

$$a_0 = \begin{bmatrix} a_{00} \cdots a_{0m-1} - 1 & 0 \cdots 0 \end{bmatrix},$$

$$A_i = \left[\begin{array}{c} 0 \\ \hdashline a_i \end{array}\right] \in R^{n\times n}, \quad a_i = \begin{bmatrix} a_{i0} \cdots a_{im-1} & 0 \cdots 0 \end{bmatrix}, \; i = 1, ..., h, \tag{9}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j = \begin{bmatrix} c_{j1} & c_{j2} \cdots c_{jn} \end{bmatrix} \in R^{1\times n}, \; j = 0, 1, ..., q\,.$$

LEMMA 2 *Let the matrices of the singular systems (1) have the forms (9). Then the singular system (1) is equivalent to the standard system*

$$\dot{x}(t) = \sum_{i=0}^{h} \bar{A}_i x(t - id) + Bu^{(p)}(t) \tag{10a}$$

$$y(t) = \sum_{j=0}^{q} C_j x(t - jd) \tag{10b}$$

*where*

$$\bar{A}_0 = \left[\begin{array}{c:c} 0 & I_{n-1} \\ \hdashline \multicolumn{2}{c}{\bar{a}_i} \end{array}\right], \; \bar{A}_j = \left[\begin{array}{c} 0 \\ \hdashline \bar{a}_i \end{array}\right], \; j = 1, ..., h,$$

$$\bar{a}_i = \begin{bmatrix} 0 \cdots 0 & a_{i0} \cdots a_{im-1} \end{bmatrix}, \; i = 0, 1, ..., h\,. \tag{11}$$

*Proof.* From (1a) with (9) we have

$$\dot{x}_k(t) = x_{k+1}(t) \quad \text{for} \quad k = 1, ..., n-1 \tag{12}$$

and

$$\begin{aligned} x_{m+1}(t) = {} & a_{00}x_1(t) + \cdots + a_{0m-1}x_m(t) + a_{10}x_1(t-d) + \cdots \\ & + a_{1m-1}x_m(t-d) + \cdots + a_{h0}x_1(t-hd) + \cdots \\ & + a_{hm-1}x_m(t-hd) + u(t)\,. \end{aligned} \tag{13}$$

Differentiating (13) and using (12) we obtain

$$\begin{aligned} x_{m+2}(t) &= \dot{x}_{m+1}(t) = a_{00}\dot{x}_1(t) + \cdots + a_{0m-1}\dot{x}_m(t) + a_{10}\dot{x}_1(t-d) + \cdots \\ &\quad + a_{1m-1}\dot{x}_m(t-d) + \cdots + a_{h0}\dot{x}_1(t-hd) + \cdots + a_{hm-1}\dot{x}_m(t-hd) + \dot{u}(t) \\ &= a_{00}x_2(t) + \cdots + a_{0m-1}x_{m+1}(t) + a_{10}x_2(t-d) + \cdots \\ &\quad + a_{1m-1}x_{m+1}(t-d) + \cdots + a_{h0}x_2(t-hd) + \cdots \\ &\quad + a_{hm-1}x_{m+1}(t-hd) + \dot{u}(t) \end{aligned}$$

and after $p$ steps

$$\begin{aligned}\dot{x}_n(t) &= a_{00}x_{p+1}(t) + \cdots + a_{0m-1}x_n(t) + a_{10}x_{p+1}(t-d) + \cdots + \\ &\quad a_{1m-1}x_n(t-d) + \cdots + a_{h0}x_{p+1}(t-hd) + \cdots + \\ &\quad a_{hm-1}x_n(t-hd) + u^{(p)}(t) \\ u^{(p)}(t) &= \frac{d^p u(t)}{dt^p} \quad \text{and} \quad p = n-m\,. \end{aligned} \tag{14}$$

The equation (10a) follows immediately from (12) and (14). ■

Definition 2 *The system (1) is called (internally) positive if for any initial admissible conditions* $x_0(t) \in R_+^n$, $t \in [-hd, 0]$ *and all inputs* $u(t) \in R_+$, $u^{(p)}(t) \in R_+$, $t \geqslant 0$, $x(t) \in R_+^n$ *and* $y(t) \in R_+$ *for* $t \geqslant 0$.

Let $M$ be the set of Metzler matrices, i.e. the set of real matrices with nonnegative off diagonal entries.

It is worth emphasising that the positivity of the system imposes quite restrictive conditions on realizations of a given transfer function. For example, there exists always a realization $A = [-1]$, $B = [-1]$, $C = [1]$ of the transfer function $-1/(s+1)$ of the continuous-time standard linear system, but this realization is not positive, since for positive realization the matrices $B$ and $C$ of continuous-time systems should have nonnegative entries. Similarly, for the transfer function $1/(z+1)$ of discrete-time system there exists always a realization $A = [-1]$, $B = [1]$, $C = [1]$, but this realization is not positive, since for positive realization the matrix $A$ should have nonnegative entries.

Theorem 1 *The system (1) with matrices of the forms (9) is positive if and only if*

1. *the entries* $a_{ij}$ *of the matrices* $A_i$, $i = 0, 1, ..., h$ *are nonnegative except for* $a_{0m-1}$ *which can be arbitrary, i.e.*

$$a_{ij} \geqslant 0 \text{ for } i = 0, 1, ..., h;\ j = 0, 1, ..., m \quad \text{except for } a_{0m-1} \text{ which can be arbitrary} \tag{15}$$

2. *the entries* $c_{ij}$ *of the matrices* $C_i$, $i = 1, ..., q$ *are nonnegative, i.e.*

$$c_{ij} \geqslant 0 \text{ for } i = 0, 1, ..., q,\ j = 1, ..., n. \tag{16}$$

*Proof.* If the matrices of system (1) have the forms (9) then, by Lemma 2, the singular system (1) is equivalent to the standard system (10). It is well-known (Kaczorek, 2005a) that the standard system is positive if and only if the matrix $\bar{A}_0$ is a Metzler matrix ( $\bar{A} \in M$) and the remaining matrices of the system have nonnegative entries. From (11) for $i = 0$ it follows that $\bar{A} \in M$ if and only if the condition (15) is satisfied. The remaining matrices of the standard system have nonnegative entries if and only if the conditions (15) and (16) are satisfied. ■

REMARK 1 The singular dual system (4) with (9) and least one $c_{jn} \neq 0$ is not a positive system if the condition (16) is satisfied since from (4a) we have $x_{n-1}(t) = -\sum_{j=0}^{q} c_{jn} u(t - jd)$.

## 4. Formulation of the realization problem

DEFINITION 3 *Matrices*

$$A_0 \in M,\ E, A_i \in R_+^{n\times n},\ i = 0, 1, ..., h\ ,\ B \in R_+^n,\ C_j \in R_+^{1\times n},\ j = 0, 1, ..., q \quad (17)$$

*of the forms (9) are called a positive singular realization of a given transfer function $T(s,w)$ is they satisfy the equality (5). A realization is called minimal if the dimension $n\times n$ of $E$ and $A_i$, $i = 0, 1, ..., h$ is minimal among all realizations of $T(s,w)$.*

The positive singular realization problem can be stated as follows.

Given a transfer function of the form

$$T(s,w) = \frac{b_{m-1}(w)s^{n-1} + \cdots + b_1(w)s + b_0(w)}{s^m - a_{m-1}(w)s^{m-1} - \cdots - a_1(w)s + a_0(w)} = \frac{b(s,w)}{a(s,w)} \quad (18)$$

where

$$a_k(w) = a_{hk}w^h + \cdots + a_{1k}w + a_{0k} \quad k = 0, 1, ..., m-1\ , \quad (19a)$$
$$b_j(w) = b_{jq}w^q + \cdots + b_{j1}w + b_{j0} \quad j = 0, 1, ..., n-1\ , \quad (19b)$$

find a positive singular realization (17) of (18).

In this paper sufficient conditions for solvability of the problem will be established and a procedure for computation of a positive singular realization (17) will be proposed.

## 5. Solution of the realization problem

Solution of the problem is based on the following two lemmas:

LEMMA 3 *If*

$$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n\times n},\ \ A_0 = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline \multicolumn{2}{c}{a_0} \end{array}\right] \in R^{n\times n},$$
$$a_0 = \begin{bmatrix} a_{00} \cdots a_{0m-1} - 1 & 0 \cdots 0 \end{bmatrix},$$
$$A_i = \left[\begin{array}{c} 0 \\ \hline a_i \end{array}\right] \in R^{n\times n},\ \ a_i = \begin{bmatrix} a_{i0} \cdots a_{im-1} & 0 \cdots 0 \end{bmatrix},\ i = 1, ..., h, \quad (20)$$

*then*

$$\det[Es - A_0 - A_1 w - \cdots - A_h w^h] = s^m - a_{m-1}(w)s^{m-1} - \cdots \\ -a_1(w)s - a_0(w) \tag{21}$$

*and the coefficients $a_k(w)$ are defined by (19a).*

*Proof.* Expansion of the determinant with respect to $n$-th row yields

$$\det[Es - A_0 - A_1 w - \cdots - A_h w^h] =$$

$$= \begin{vmatrix} s & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & s & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & s & -1 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & s & -1 \\ -a_0(w) & -a_1(w) & -a_2(w) & \cdots & -a_{m-1}(w) & 1 & \cdots & 0 & 0 \end{vmatrix} =$$

$$= s^m - a_{m-1}(w)s^{m-1} - \ldots - a_1(w)s - a_0(w)\,. \qquad \blacksquare$$

LEMMA 4 *If the matrices $E$ and $A_i$, $i = 0, 1, ..., h$ have the forms (20) then the* $n$th *column $C_n(s)$ of the adjoint matrix*

$$Adj[Es - A_0 - A_1 w - \cdots - A_h w^h] \tag{22}$$

*has the form*

$$C_n(s) = [\ 1 \quad s \quad \ldots \quad s^{n-1}\ ]^T. \tag{23}$$

*Proof.* Taking into account that

$$[Es - A_0 - A_1 w - \cdots - A_h w^h](\mathrm{Adj}\,[Es - A_0 - A_1 w - \cdots A_h w^h]) = I_n a(s, w)$$

it is easy to verify that

$$[Es - A_0 - A_1 w - \cdots - A_h w^h]C_n(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} a(s, w). \qquad \blacksquare$$

Knowing the coefficients $a_{ij}$, $i = 0, 1, \ldots, h$; $j = 0, 1, , ..., m - 1$ of the denominator $a(s, w)$ of the transfer function (18) we may find the matrices (20).

Taking into account (21), (23) and (18) we may write the transfer function (5) for matrices (9) in the form

$$T(s, w) = C(w)\left[Es - A_0 - A_1 w - \cdots - A_h w^h\right]^{-1} B \\ = \frac{C(w)}{a(s, w)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} = \frac{b(s, w)}{a(s, w)} \tag{24}$$

where

$$C(w) = C_q w^q + \cdots + C_1 w + C_0 \\ = [c_{q1}w^q + \cdots + c_{11}w + c_{01}, ..., c_{qn}w^q + \cdots + c_{1n}w + c_{0n}] . \quad (25)$$

Comparison of the coefficients at the same power of $s$ and $w$ of the numerator of the equality (24) yields

$$c_{qn}w^q + \cdots + c_{1n}w + c_{0n} = b_{n-1q}w^q + \cdots + b_{n-1,1n}w + b_{n-1,0}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{q1}w^q + \cdots + c_{11}w + c_{01} = b_{0q}w^q + \cdots + b_{01}w + b_{00}$$

and

$$b_{00} = c_{01},\ \ b_{01} = c_{11},\ \ldots\ ,\ b_{0q} = c_{q1}, ..., b_{n-1,0} = c_{0n}, \\ b_{n-1,1} = c_{1n}, ..., b_{n-1,q} = c_{qn}. \quad (26)$$

Knowing the coefficients $b_{jk}$, $j = 0, 1, ..., n-1$; $k = 0, 1, ..., q$ of the numerator $b(s,w)$ of (18) and using (26) we may find the entries of the matrices $C_j$, $j = 0, 1, ..., q$.

THEOREM 2 *There exists a positive singular realization of the form (9) of the transfer function (18) if*

1. *the coefficients $a_{ij}$, $i = 0, 1, ..., m-1$; $j = 0, 1, ..., h$ of the denominator $a(s,w)$ are nonnegative except for $a_{0m-1}$, which can be arbitrary*
2. *the coefficients $b_{ij}$, $i = 0, 1, ..., n-1$; $j = 0, 1, ..., q$ of the numerator $b(s,w)$ are nonnegative*

*Proof.* If the condition i) is satisfied then $A_0 \in M$ and $A_i$, $i = 1, ..., h$ have nonnegative entries. From (26) it follows that if condition ii) is met then the matrices $C_j$, $j = 0, 1, ..., q$ have nonnegative entries. Therefore, if both conditions are satisfied, then by Theorem 1 the realization is a positive one. ■

If the conditions of Theorem 2 are satisfied then the positive singular realization (17) can be found by the use of the following procedure.

**Procedure 1.**

Step 1. For given transfer function (18) find

$$n = 1 + \deg_s b(s,w)\,;\ q = \deg_w b(s,w);\ h = \deg_w a(s,w) \\ \text{and } m = \deg_s a(s,w) \quad (27)$$

($\deg_s(\cdot)$ denotes the degree of $(\cdot)$ with respect to $s$).

Step 2. Knowing the coefficients $a_{ij}$ of the polynomial $a(s,w)$ find the matrices (20) satisfying the condition (21).

Step 3. Knowing the coefficients $b_{ij}$ of the polynomial $b(s,w)$ and using (26) find the matrices $C_j$, $j = 0, 1, ..., q$ and the matrix $B$.

EXAMPLE 1 Compute a positive singular realization (17) of the transfer function

$$T(s,w) = \frac{w^2s^3 + (w^3+2w)s^2 + (w+1)s + 2w^2 + w + 1}{s^2 - (2w+1)s - (w^2+2w+1)}\,. \tag{28}$$

It is easy to check that the transfer function (28) satisfies the conditions of Theorem 2. Using the Procedure we obtain:

Step 1. In this case

$$\begin{aligned} b(s,w) &= w^2s^3 + (w^3+2w)s^2 + (w+1)s + 2w^2 + w + 1 \\ a(s,w) &= s^2 - (2w+1)s - (w^2+2w+1) \end{aligned}$$

and using (27) we obtain $n = 1 + \deg_s b(s,w) = 4$; $q = \deg_w b(s,w) = 3$; $h = \deg_w a(s,w) = 2$; $m = \deg_s a(s,w) = 2$.

Step 2. Taking into account that $a_0(w) = w^2 + 2w + 1$, $a_1(w) = 2w + 1$ and using (20) we obtain

$$E = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},\ A_0 = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&-1&0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\2&2&0&0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}. \tag{29}$$

Step 3. Using (26) we obtain

$$\begin{aligned} C_0 &= [c_{01}\, c_{02}\, c_{03}\, c_{04}] = [1,1,0,0],\ C_1 = [c_{11}\, c_{12}\, c_{13}\, c_{14}] = [1,1,2,0] \\ C_2 &= [c_{21}\, c_{22}\, c_{23}\, c_{24}] = [2,0,0,1],\ C_3 = [c_{31}\, c_{32}\, c_{33}\, c_{34}] = [0,0,1,0] \end{aligned} \tag{30}$$

and the matrix $B$ has the form $B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

The desired positive singular realization of (28) is given by (29) and (30).

REMARK 2 By Lemma 1 the matrices

$$E = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix},\ A_0 = \begin{bmatrix} 0&0&0&1\\1&0&0&1\\0&1&0&-1\\0&0&1&0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0&0&0&2\\0&0&0&2\\0&0&0&0\\0&0&0&0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 1\\1\\0\\0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1\\1\\2\\0 \end{bmatrix},\ B_2 = \begin{bmatrix} 2\\0\\0\\1 \end{bmatrix},\ B_3 = \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix},\ C = \begin{bmatrix} 0&0&0&1 \end{bmatrix}.$$

are also a singular realization of the transfer function (28) but this realization is not a positive one (see Remark 1).

## 6. Concluding remarks

The positive realization problem for a class of singular continuous-time linear SISO systems with delays in state and in output has been considered. The notions of canonical forms of matrices and of dual systems have been extended for singular linear continuous-time systems with delays. Conditions for positivity and for the existence of positive singular realizations have been established for singular linear continuous-time SISO system with delays in state and in output. A procedure for computation of a positive singular realization of a given transfer function has been proposed and illustrated by a numerical example.

The proposed method solves only the realization problem for a class of singular positive continuous-time systems with delays when the matrices of the system (1) have the canonical form (9). This class is quite large, since for standard (non positive) singular systems there exists a realization with matrices in the canonical forms (9) for any given improper transfer function.

The considerations can be extended for multi-input multi-output singular continuous-time linear systems with delays in state and in output. An extension of these considerations for 2D singular linear systems with delays is an open problem.

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## References

Arnold, W.F. and Laub, A.J. (1984) Generalized eigenproblem algorithms and software for algebraic Riccati equations. *Proc. IEEE* **72**(12), 1746–1754.

Benvenuti, L. and Farina, L. (2004) A tutorial on the positive realization problem. *IEEE Trans. Autom. Control* **49** (5), 651-664.

Busłowicz, M. and Kaczorek, T. (2004) Reachability and minimum energy control of positive linear discrete-time systems with one delay. *12$^{th}$ Mediterranean Conference on Control and Automation*, June 6-9, 2004, Kusadasi, Izmir, Turkey.

Farina, L. and Rinaldi, S. (2000) *Positive Linear Systems; Theory and Applications.* J. Wiley, New York.

Górecki, H., Fuksa, S., Grabowski, P. and Korytowski, A. (1989) *Analysis and Synthesis of Time Delay Systems.* PWN and J. Wiley, Warszawa.

Kaczorek, T. (2002) *Positive 1D and 2D Systems.* Springer-Verlag, London.

Kaczorek, T. (2003) Some recent developments in positive systems. *Proc. 7$^{th}$ Conference of Dynamical Systems Theory and Applications*, Łódź 2003, 25-35.

Kaczorek, T. (2004a) Realization problem for positive multivariable linear systems with time-delay. *Proc. Intern. Workshop Computational Prob-*

*lems of Electrical Engineering*. Zakopane 2004, 186-192.

Kaczorek, T. (2004b) Realization problem for positive discrete-time systems with delay. *System Science* **30** (4), 17-30.

Kaczorek, T. (2005a) Realization problem for a class of positive continuous-time systems with delays. *Int. J. Appl. Math. Comput. Sci.* **15** (4), 447-453.

Kaczorek, T. (2005b) Realization problem for positive linear systems with time delay. *Mathematical Problems in Engineering* **2005** (4), 455-463.

Kaczorek, T. (2006a) Minimal positive realizations for discrete-time systems with delays. *The International Journal for Computation and Mathematics in Electrical and Electronic Engineering COMPEL* **25** (4), 812-826.

Kaczorek, T. (2006b) Computation of realizations of discrete-time cone-systems. *Bull. Pol. Acad. Sci. Techn.* **53** (3), 347-350.

Kaczorek, T. and Busłowicz, M. (2004) Minimal realization for positive multivariable linear systems with delay. *Int. J. Appl. Math. Comput. Sci.* **14** (2), 181-187.

Xie, G. and Wang, L. (2003) Reachability and controllability of positive linear discrete-time systems with time-delays. In: L. Benvenuti, A. De Santis and L. Farina, eds., *Positive Systems, LNCIS* **294**, Springer-Verlag, Berlin, 377-384.