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Nonessential objective functions in linear multiobjective optimization problems*

by

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Abstract: In multiobjective (vector) optimization problems, among the given objective functions there exist some, which do not influence the set of efficient solutions. These objective functions are said to be nonessential. In this paper we present a new method to decide if a given linear objective function is nonessential or not.

Keywords: multiple criteria decision making, multiobjective (vector) optimization, efficient solutions, nonessential objectives.

1. Introduction and notations

In this paper we study the following multiobjective (vector) optimization problem

$$P = (X, \mathbf{F}^n, \Re) \tag{1}$$

where:
1) X is a feasible set;
2) $\mathbf{F}^n = [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})]^T : X \to \mathbb{R}^n$ $(n > 1)$ is a vector-valued function, each f_i is called an objective function;
3) $\Re$ is a binary relation on $\mathbb{R}^n$
$\mathbf{y}^1 = [y_1^1, ..., y_n^1]^T, \mathbf{y}^2 = [y_1^2, ..., y_n^2]^T$
$(\mathbf{y}^1, \mathbf{y}^2) \in \Re \Leftrightarrow \mathbf{y}^1 \geq \mathbf{y}^2 \Leftrightarrow \forall i \in \{1, ..., n\}\ y_i^1 \geq y_i^2 \wedge \exists i \in \{1, ..., n\}\ y_i^1 > y_i^2$.

The solution of the problem (1) is to find all solutions that are efficient in the sense of the following definition:

Definition 1 *A vector $\boldsymbol{x}^0 \in X$ is said to be an efficient (Pareto - optimal) solution of the problem (1) iff there exists no $\boldsymbol{x} \in X$ such that $\boldsymbol{F}^n(\boldsymbol{x}) \geq \boldsymbol{F}^n(\boldsymbol{x}^0)$. The set of efficient solutions of the problem (1) is denoted by X_E^n.*

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We form a new vector optimization problem $\tilde{P}$ from P by adding an objective function f_{n+1} to the problem P. If the set of efficient solutions of the problem $\tilde{P}$ equals that of P, then the objective function f_{n+1} is called nonessential. Information about nonessential objectives helps a decision maker to know better and understand the problem and this might be a good starting point for a further investigation or revision of the model. Dropping of nonessential functions leads to new problems with a smaller number of objectives, thus easier to solve. For this reason the issue of nonessential objectives must be considered in works dealing with Multiple Criteria Decision Making (MCDM). However, research in this field is indeed scarce. The seminal papers by Gal and Leberling (1977) and Gal (1980) defined and investigated nonessential objectives in linear problems. With respect to these papers Gal and Hanne (1999, 2006) considered the consequences of dropping nonessential objectives functions for application of MCDM methods. The concept of nonessential objectives was generalized by the present author. In Malinowska (2002a,b) convex vector optimization problems are considered. Furthermore, new definitions of weak nonessential and proper nonessential functions are introduced and investigated.

The aim of this paper is to give an algorithm to determine if a given objective function of a certain linear problem is essential or not. In comparison with the Gal - Leberling method, which is based on the sufficient condition for the function to be nonessential, our method is based on the necessary and sufficient conditions. The outline of the paper is as follows: In Section 2 we develop the theory of nonessential objectives. Section 3 presents how to verify that the set of efficient solutions of the problem $\tilde{P}$ is contained in that of P. Section 4 is devoted to the main result of the paper: the method to determine if a given objective function of a linear problem is essential or nonessential. In Section 5 we provide examples to illustrate our method. Finally some conclusions are given.

2. Nonessential objective functions

Let X_E^{n+1} denote the set of solutions of the problem

$$\tilde{P} = (X, \mathbf{F}^{n+1}, \Re) \tag{2}$$

where $\mathbf{F}^{n+1} = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}), f_{n+1}(\mathbf{x})]^T$ and $\Re$ is a binary relation on $\mathbb{R}^{n+1}$. With this notation we introduce the definition of nonessential objective function.

Definition 2 *The objective function f_{n+1} is said to be nonessential in $\tilde{P}$ iff $X_E^n = X_E^{n+1}$.*
An objective function which is not nonessential is called essential.

Let X_{n+1} denote the set of solutions of the single objective optimization problem

$$Max\{f_{n+1} : x \in X\}. \tag{3}$$

In other words

$$X^{n+1} = \{\mathbf{x}^0 \in X : \forall \mathbf{x} \in X \; f_{n+1}(\mathbf{x}^0) \geq f_{n+1}(\mathbf{x})\}. \tag{4}$$

From now on we make the following assumptions:
A1. X is a nonempty, compact and convex set;
A2. Each f_i is a continuous and convex function.
Below we provide the conditions under which the objective is nonessential.

THEOREM 1 *Let the problem $\tilde{P}$ satisfy the assumptions A1, A2. The objective function f_{n+1} is nonessential in $\tilde{P}$ if and only if the following three conditions hold:*
a) $X_E^n \subset X_E^{n+1}$;
b) $X_E^n \cap X_{n+1} \neq \emptyset$;
c) $\forall \boldsymbol{x} \in X \backslash X_E^n \; \exists \boldsymbol{x}' \in \mathbb{R}^k \; \boldsymbol{F}^{n+1}(\boldsymbol{x}') \geq \boldsymbol{F}^{n+1}(\boldsymbol{x})$.

For the proof we refer the reader to Malinowska (2002a,b).

Now assume that $\tilde{P}$ is a linear vector optimization problem,that is

$$X = \{\mathbf{x} \in \mathbb{R}^k : A\mathbf{x} = \mathbf{b}, \forall i \in \{1, ..., k\} \; x_i \geq 0\}, A \in \mathbb{R}^{m\times k}, b \in \mathbb{R}^m$$
$$\mathbf{F}^{n+1}(\mathbf{x}) = \tilde{C}\mathbf{x} = [(\mathbf{c}^1)^T\mathbf{x}, ..., (\mathbf{c}^{n+1})^T\mathbf{x}]^T, \mathbf{c}^i \in \mathbb{R}^k (i = 1, ..., n)$$

The following theorem was proved by Gal and Leberling (1977).

THEOREM 2 *Let the problem $\tilde{P}$ be linear. The objective function f_{n+1} is nonessential in $\tilde{P}$ if the following holds:*

$$\boldsymbol{c}^{n+1} = \sum_{i=1}^{n} \alpha_i \boldsymbol{c}^i, \alpha_i \geq 0 \; (i = 1, ..., n). \tag{5}$$

3. The inclusion $X_E^{n+1} \subset X_E^n$ in linear vector optimization problems

In the linear case, in order to verify that the inclusion $X_E^{n+1} \subset X_E^n$ holds, we need only to solve a finite number of single objective linear optimization problems. If we want to test the condition b), first we solve the problem (3) and next we check if there is a vertex of X_{n+1} which belongs to X_E^n (for the method we refer the reader to Benson, 1978). In order to verify the condition c) we may test the equality of sets $\tilde{U} = \{\mathbf{u} \in \mathbb{R}^k : \tilde{C}\mathbf{u} \geq \mathbf{0}\} = \emptyset$. Because the following remark holds.

REMARK 1 If $\tilde{U} = \{\mathbf{u} \in \mathbb{R}^k : \tilde{C}\mathbf{u} \geq \mathbf{0}\} \neq \emptyset$, then the condition c) of Theorem 1 holds.

Proof. Let $\tilde{U} \neq \emptyset$. Assume, on the contrary, that

$$\exists \mathbf{x} \in X \backslash X_E^n \ \forall \mathbf{x}' \in \mathbb{R}^k \ \neg(\tilde{C}\mathbf{x}' \geq \tilde{C}\mathbf{x}). \tag{6}$$

Since $\tilde{U} \neq \emptyset$, there exists $\mathbf{u} \in \mathbb{R}^k$ such that $\tilde{C}\mathbf{u} \geq \mathbf{0}$. Applying $\mathbf{x}' = \mathbf{u} + \mathbf{x}$ to (6) we obtain

$$\neg(\tilde{C}(\mathbf{u} + \mathbf{x}) \geq \tilde{C}(\mathbf{x})),$$
$$\neg(\tilde{C}\mathbf{u} \geq \mathbf{0}).$$

This is a contradiction by free choice of $\mathbf{x}'$. ■

Below, we present the theorem which will be useful in next section.

THEOREM 3 *(Galas, Nykowski, Zółkiewski, 1987) A sufficient condition for* $X_E^{n+1} = X$ *is* $\tilde{U} = \emptyset$. *If* $intX \neq \emptyset$, *this condition is also necessary (*$intX$ *stands for the interior of the set* X*).*

4. Main result

The theory presented in the previous section enables us to work out the method: how to test whether an objective function of a linear problem is essential or not. Generally, the method consists of seven steps.

Step 1. Solve the problem: Is there $\tilde{U} = \emptyset$?
If the answer is "Yes", then $X_E^{n+1} = X$ (by Theorem 3), go to Step 2. If the answer is "No", then the condition c) of Theorem 1 holds (by Remark 1), go to Step 5.
Step 2. Solve the problem: Is there $U = \{\mathbf{u} \in R^k : C\mathbf{u} = [(\mathbf{c}^1)^T\mathbf{u}, ..., (\mathbf{c}^n)^T\mathbf{u}]^T \geq \mathbf{0}\} = \emptyset$?
If the answer is "Yes", then $X_E^n = X$ and the objective function f_{n+1} is nonessential. Otherwise go to Step 3.
Step 3. Solve the problem: Is there $intX \neq \emptyset$?
For the method we refer the reader, for instance, to Galas, Nykowski, Zółkiewski (1987). If the answer is "Yes", then $X_E^n \neq X$ (by Theorem 3) and the objective function f_{n+1} is essential. If the answer is "No", then go to Step 4.
Step 4. Solve the problem: Is there $X_E^n = X$?
(This can be done, for instance, by checking the efficiency of some feasible vectors.) If the answer is "Yes", then the objective function f_{n+1} is nonessential. Otherwise f_{n+1} is essential.
Step 5. Determine the set X_{n+1} and go to Step 6.
Step 6. Solve the problem: Is there $X_E^n \cap X_{n+1} \neq \emptyset$?
If the answer is "Yes", then $X_E^{n+1} \subset X_E^n$, go to Step 7. If the answer is "No", then the objective function f_{n+1} is essential.

Step 7. Solve the problem: Is there $X_E^n \subset X_E^{n+1}$?
If the answer is "Yes", then the objective function f_{n+1} is nonessential. Otherwise f_{n+1} is essential.

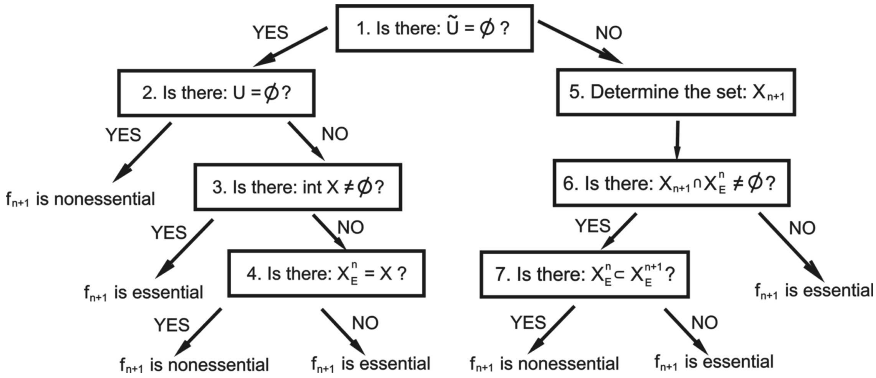

Figure 1. The scheme of the method.

The advantage of using the above method lies in the fact that the basis of the method is provided by the necessary and sufficient condition for the objective function to be nonessential. It is worth pointing out that in Steps 1 - 6 we must solve only single objective linear problems (with one exception in Step 4, if it is necessary).

The disadvantage of the method is that we cannot use it in the case of lack of any knowledge of the set X. The next problem is Step 7. It is not always easy to verify that $X_E^n \subset X_E^{n+1}$. The inclusion holds, for example, if the vector - valued function $\mathbf{F}^n$ is one-to-one on the set X_E^n (Gutenbaum, Inkielman, 1998) or the objective f_{n+1} is a linear combination of the other objectives (Malinowska, 2002b).

5. Illustrative examples

To illustrate the proposed method, we now consider some simple problems.

Example 1 The set X, the vector-valued function $F^{\mathbf{n}}$ and the objective function f_{n+1} are given by

$$X = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + 4x_2 \leq 24, -x_1 - x_2 \leq -6, x_2 \leq -1, 3x_1 + 2x_2 \leq 32\}$$
$$\mathbf{F}^2(\mathbf{x}) = [x_2, 4x_1 + x_2]^T$$
$$f_3 = -x_1 + 4x_2.$$

Step 1. Since in the problem

$$Max\{v_1+v_2+v_3 : -C(\mathbf{x}-\mathbf{1}a)+\mathbf{v}=\mathbf{0}, a\geq 0, \forall i\in\{1,2\}\ x_i\geq 0, \forall i\in\{1,2,3\}\ v_i\geq 0\}$$

there is $(v_1+v_2+v_3)_{max}=\infty$, the condition c) of Theorem 1 holds.
Step 5. The solution of the problem

$$Max\{-x_1+4x_2 : x\in X\}$$

is $X_3=\{[0,6]^T\}$.
Step 6. Since the solution of the problem

$$Max\{v_1+v_2 : \mathbf{x}\in X, C\mathbf{x}-\mathbf{v}=C[0,6]^T, \forall i\in\{1,2\}\ x_i\geq 0, \forall i\in\{1,2\}\ v_i\geq 0\}$$

is $\mathbf{v}=0, \mathbf{x}=[0,6]^T$, $\mathbf{x}=[0,6]^T\in X_E^2$. Hence $X_E^3\subset X_E^2$.
Step 7. Since $\det\begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}=\det C\neq 0$, $\mathbf{F}^2$ is one-to-one and $X_E^2\subset X_E^3$.
Hence the objective function f_3 is nonessential.

Let us notice that the condition (5) does not hold. The solution of the system of equations $\mathbf{c}^3=\alpha_1\mathbf{c}^1+\alpha_2\mathbf{c}^2$ is $\alpha_1=17/4, \alpha_2=-1/4$.

In this problem the objective function f_1 is also nonessential. It is easy to check it by using the condition (5) or our method.

Example 2 Let the set X satisfy the assumption A1. The vector-valued function $\mathbf{F}^n$ and the objective function f_{n+1} are given by

a)

$$\mathbf{F}^3(\mathbf{x})=[x_1+3x_2, 3x_1, -3x_1-x_2]^T$$
$$f_4=2x_1+x_2.$$

Step 1. Since the solution of the problem

$$Max\{v_1+v_2+v_3+v_4 : -C(\mathbf{x}-\mathbf{1}a)+\mathbf{v}=\mathbf{0}, a\geq 0, \forall i\in\{1,2\}\ x_i\geq 0, \forall i\in\{1,2,3,4\}\ v_i\geq 0\}$$

is $\mathbf{v}=\mathbf{0}, \mathbf{x}=\mathbf{0}, a=0$, $X_E^4=X$.
Step 2. Since the solution of the problem

$$Max\{v_1+v_2+v_3 : -C(\mathbf{x}-\mathbf{1}a)+\mathbf{v}=\mathbf{0}, a\geq 0, \forall i\in\{1,2\}\ x_i\geq 0, \forall i\in\{1,2,3\}\ v_i\geq 0\}$$

is $\mathbf{v}=\mathbf{0}, \mathbf{x}=\mathbf{0}, a=0$, $X_E^3=X$.
Hence the objective f_4 is nonessential.

b)

$$\mathbf{F}^3(\mathbf{x})=[x_1+3x_2, 2x_1+x_2, -3x_1-x_2]^T$$
$$f_4=3x_1.$$

Step 1. Like in a).
Step 2. There is $(v_1 + v_2 + v_3)_{max} = \infty$ in the problem

$$Max\{v_1 + v_2 + v_3 : -C(\mathbf{x} - \mathbf{1}a) + \mathbf{v} = \mathbf{0}, a \geq 0, \forall i \in \{1,2\}\ x_i \geq 0, \\ \forall i \in \{1,2,3\}\ v_i \geq 0\}.$$

Step 3. If we know that $intX \neq \emptyset$, then $X_E^3 \neq X$ and the objective function f_4 is essential. Otherwise, we do not know whether f_4 is essential or not.

It is easy to check that the condition (5) does not hold. Therefore by using the method from Gal, Leberling (1977) we also cannot verify whether f_4 is essential or not.

Example 3 Let the set X satisfy the assumption A1. The vector-valued function $\mathbf{F}^n$ and the objective function f_{n+1} are given by

$$\mathbf{F}^2(\mathbf{x}) = [x_1 - 3x_2 - x_3, 2x_2 + 3x_3]^T$$
$$f_3 = 2x_1 - 4x_2 + x_3.$$

According to the method from Gal, Leberling (1977) the objective function f_3 is nonessential, since it is a linear combination of the other two objective functions (with $\alpha_1 = 2$ and $\alpha_2 = 1$). Now we use our method.
Step 1. In the problem

$$Max\{v_1 + v_2 + v_3 : -C(\mathbf{x} - \mathbf{1}a) + \mathbf{v} = \mathbf{0}, a \geq 0, \forall i \in \{1,2,3\}\ x_i \geq 0, \\ \forall i \in \{1,2,3\}\ v_i \geq 0\}$$

there is $(v_1 + v_2 + v_3)_{max} = \infty$. Since we do not know the set X, we cannot go to Step 5. Hence in this problem we cannot use our method.

6. Conclusions

As mentioned in the introduction, there are theoretical and practical reasons for developing a method to test if a given objective function is nonessential. One such method, for linear problems, was given by Gal and Leberling in 1977. In this paper we present a new method. From the examples in Section 5, it is clear that the effectiveness of both methods depends on the particular problem under consideration.

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