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Discrete-time control systems on homogeneous spaces: partition property [1]

by

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Abstract: If the system semigroup of a control system is a group, the system has the partition property, i.e., the reachable sets form a disjoint partition in the state space. The converse is not true in general. In this work we give sufficient conditions for the partition property for a family of discrete-time control systems on homogeneous spaces. We apply our results to Inverse Iteration systems on flag manifolds, which are closely related to numerical algorithms.

Keywords: discrete-time control systems, inverse iteration, partition property, system semigroup.

1. Introduction

Given a discrete-time control system, the reachable set of an initial point is defined as the set of points one may obtain in finitely many iteration steps using arbitrary controls. In applications it is necessary to understand the structure of the reachable sets, since they provide fundamental limitations on possible convergence behavior under feedback laws.

In general, the reachable sets of a discrete-time control system coincide with orbits of a certain semigroup action on the state space. Therefore, the reachable sets form a partition in the state space if this particular semigroup – the system semigroup – is a group. Unfortunately, in many important applications the system semigroup is not a group. We are interested in conditions, under which the state space is, nevertheless, a disjoint union of reachable sets. One sufficient condition for this is controllability of the system, i.e., the system semigroup acts transitively on the state space. For example San Martin and Mittenhuber showed necessary and sufficient conditions for transitive semigroup actions, see San Martin (1998) and Mittenhuber (2001) for more details. Unfortunately,

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in many applications – for example for systems with fixed points – we have naturally more than one reachable set and therefore no controllability.

For a family of discrete-time systems on homogeneous spaces, we will propose conditions for the partition property, which are weaker than transitivity of the semigroup action. This setting is motivated by a certain application concerning numerical algorithms. The idea is to interpret iterative algorithms as discrete-time dynamical systems. Interesting examples of such an approach can be found in the work by Ammar and Martin (1986), Batterson and Smillie (1989, 1990), and Shub and Vasquez (1987). In this context, topological and geometric structures naturally appear. Nevertheless, to prove our main result, we only use algebraic properties of the system. Therefore, we are able to state our result in a very general context.

The paper is organized as follows. In Section 2 we introduce the partition property in the general context of discrete-time control systems. For a family of such systems on homogeneous spaces we give a sufficient condition for the partition property in Section 3. In Section 4 we will apply our previous results on Inverse Iteration systems on flag manifolds, which are closely related to iterative numerical algorithms, such as QR algorithm. In particular, we show that a certain matrix semigroup - related to the system semigroup - is a group if and only if the Inverse Iteration system has partition property.

2. Partition property

Following the notation in Sontag (1998) we define a *discrete-time control system* as a triple $\Sigma = (M, U, f)$ containing of a *state space* M, a set of *control parameters* U and a *transition map* $f : M \times U \to M$. The system Σ describes the iteration

$$x_0 \in M, \qquad x_{t+1} := f(u_t, x_t), \quad u_t \in U. \tag{1}$$

The *reachable set* $\mathcal{R}(x)$ of a point x is the set of all states to which one may steer from x in finitely many iterations, using arbitrary controls in each step. For $T \in \mathbb{N}$ we define recursively $f_1 = f$ and $f_T : U^T \times H \to M$ by

$$f_T : (u_0, \dots u_{T-1}, x) \mapsto f(u_{T-1}, f_{T-1}(u_0, \dots, u_{T-2}, x)). \tag{2}$$

By S_Σ we denote the set of all maps one can generate in this way, i.e.,

$$S_\Sigma = \{F : M \to M \mid \exists\, T < \infty, \exists\, u \in U^T : F = f_T(u, \cdot)\}. \tag{3}$$

S_Σ is a semigroup, the so called *system semigroup* of Σ. The reachable set of a point $x \in M$ can be regarded as the orbit $\mathcal{R}(x) = S_\Sigma \cdot x$ of the semigroup action

$$S_\Sigma \times M \to M, \quad (s, m) \to s \cdot m := s(m). \tag{4}$$

Note that $x \in \mathcal{R}(y)$ implies $\mathcal{R}(x) \subset \mathcal{R}(y)$.

We say that a system Σ has the *partition property* if the reachable sets form a *partition* in the state space, i.e., for every $x \in M$ there exists $y \in M$, such that $x \in \mathcal{R}(y)$ and for all $x, y \in M$ it is either $\mathcal{R}(x) = \mathcal{R}(y)$ or $\mathcal{R}(x) \cap \mathcal{R}(y) = \emptyset$. Note that the partition property implies $x \in \mathcal{R}(x)$ for all $x \in M$, even if the identity is not an element of the system semigroup.

Since the reachable sets are orbits of a semigroup action, a system Σ has the partition property whenever S_Σ is a group. The following example shows that the partition property may hold even when the system semigroup is not a group.

Example 1 Let $M = \mathbb{R}$, $U = \mathbb{R}^+$ and

$$f(x,u) = \begin{cases} ux & x \geq 0 \\ 2ux & x < 0. \end{cases} \tag{5}$$

Note that $f(\cdot, u)$ is bijective for every $u \in U$. Every element of S_Σ has the form

$$F(x) = \begin{cases} ux & x \leq 0 \\ 2^k ux & x < 0 \end{cases} \tag{6}$$

with $u \in U$ and $k \in \mathbb{N}$. In particular, identity does not belong to S_Σ and therefore S_Σ is not a group.

On the other hand, Σ has the partition property, since $\mathcal{R}(x) = \mathbb{R}^+$ for every $x > 0$, $\mathcal{R}(0) = \{0\}$ and $\mathcal{R}(x) = \mathbb{R}^-$ for $x < 0$.

3. Control systems on homogeneous spaces

This work was motivated by the analysis of the shifted eigenvector algorithms such as Inverse Iteration and QR algorithm. Both methods can be formulated as discrete-time control systems on related homogeneous spaces. It turns out that the reachable sets of both systems are semigroup orbits of the same semigroup. Moreover, for a given matrix the Inverse Iteration system has the partition property if and only if the QR algorithm has the partition property. The proof of these facts is purely algebraic and can be extended to a very general setting.

Let G be a group and H be a subgroup of G. The set of cosets $G/H := \{gH \,|\, g \in G\}$ is called homogeneous space. Every element $x \in G/H$ can be represented in the form $x = gH$. Note that $gH = \tilde{g}H$ if and only if $g^{-1}\tilde{g} \in H$. Canonically, we define for any subsemigroup $\tilde{S}$ of G a product

$$\tilde{S} \times G/H \to G/H, \quad s \cdot gH = sgH. \tag{7}$$

Now, let U be a set of control parameters and $\Phi : U \to G$ be a map, which induces the transition map $f : G/H \times U \to G/H$ by $f(x,u) = \Phi(u) \cdot x$. In the following we analyze the system $\Sigma = (G/H, U, \Phi)$, respectively the iteration

$$x_0 \in G/H, \qquad x_{t+1} := \Phi(u_t) \cdot x_t, \quad u_t \in U. \tag{8}$$

In this setting the reachable sets are orbits of the semigroup

$$\tilde{S} := \left\{ \prod_{t=0}^{T} \Phi(u_t) \,|\, T \in \mathbb{N}_0, u_t \in U \right\}, \tag{9}$$

i.e., $\mathcal{R}(x) = \{\Phi \cdot x \,|\, \Phi \in \tilde{S}\}$. Typically, $\tilde{S}$ is easier to handle than the system semigroup S_Σ since $\tilde{S}$ is a subsemigroup of G.

Obviously, system Σ has the partition property if $\tilde{S}$ is a group. In the following we show a weaker condition. For that purpose we introduce the *normal core* of a homogeneous space G/H defined as

$$C := \bigcap_{g \in G} gHg^{-1}. \tag{10}$$

Note that C is the largest normal subgroup of G contained in H. In particular, it is $gC = Cg$ for all $g \in G$.

LEMMA 1 *Let $\Sigma = (G/H, U, \Phi)$ be a system of Type (8).*
a) $C\tilde{S} = \{cs \,|\, c \in C, s \in \tilde{S}\}$ is a subsemigroup of G.
b) If $\tilde{S}$ is a group then $C\tilde{S}$ is a group.
c) For every point $x \in G/H$ there holds $\mathcal{R}(x) = C\tilde{S} \cdot x$.

Proof. All statements follow from the fact that C is a normal subgroup of G.
a) For all $c_1 s_1, c_2 s_2 \in C\tilde{S}$ there exist $\tilde{c}_2 \in C$, such that $c_1 s_1 c_2 s_2 = c_1 \tilde{c}_2 s_1 s_2 \in C\tilde{S}$. Therefore $C\tilde{S}$ is a semigroup.
b) If $\tilde{S}$ is a group, then $C\tilde{S}$ is a group, since $C\tilde{S} = \tilde{S}C$.
c) Since C is a subgroup of H it is $C \cdot gH = gH$ for all $g \in G$ and therefore $\mathcal{R}(gH) = \tilde{S} \cdot gH = \tilde{S}CgH = C\tilde{S} \cdot gH$.

■

Note that $C\tilde{S}$ is a group if and only if the system semigroup S_Σ is a group. Nevertheless, for our applications it is more convenient to deal with $C\tilde{S}$ instead of S_Σ.

Lemma (1) shows, that in order to analyze the reachable sets of a system of Type (8) we can use more algebraic structure if we regard the reachable sets as orbits of $C\tilde{S}$ instead of $\tilde{S}$. It follows that the partition property is already given, if $C\tilde{S}$ is a group. In Section 4 we will give an example where $C\tilde{S}$ is a group but $\tilde{S}$ is not.

In the following we give a sufficient condition, under which the partition property implies that $C\tilde{S}$ is a group. Recall that a subset $\tilde{S} \subset G$ of a group G generates the subgroup

$$\langle \tilde{S} \rangle := \left\{ \prod_{i=1}^{N} s_i \,|\, N < \infty, s_i \in \tilde{S} \cup \tilde{S}^{-1} \right\}. \tag{11}$$

Note that $\langle \tilde{S} \rangle$ is the smallest subgroup of G which contains $\tilde{S}$. Moreover, $C\tilde{S}$ is a group if and only if $C\tilde{S} = \langle C\tilde{S} \rangle$.

Theorem 1 *Let $\Sigma = (G/H, U, \Phi)$ be a system of Type (8) where $H \cap \langle \tilde{S} \rangle$ is a subgroup of C. The following statements are equivalent:*
(i) $C\tilde{S}$ is a group.
(ii) For all $x \in G/H$ the equation $\mathcal{R}(x) = \langle \tilde{S} \rangle \cdot x$ holds.
(iii) System Σ has the partition property.

Proof. The implications (i)⇒(iii) and (ii)⇒(iii) are obvious.
(iii)⇒(ii) For any $\tilde{s} \in \tilde{S}^{-1}$ and $x \in G/H$ it is $x = \tilde{s}^{-1}\tilde{s} \cdot x \in \mathcal{R}(\tilde{s} \cdot x)$. It follows that $\mathcal{R}(x) = \mathcal{R}(\tilde{s} \cdot x)$ and therefore $\tilde{s} \cdot x \in \tilde{S} \cdot x$. We conclude that for an arbitrary $\delta \in \langle \tilde{S} \rangle$, i.e., $\delta = \prod_{t=1}^{N} \tilde{s}_t$ with $\tilde{s}_t \in \tilde{S} \cup \tilde{S}^{-1}$ there exist $s_1, \ldots, s_N$ such that $\delta \cdot x = \prod_{t=1}^{N} s_t \cdot x \in \tilde{S} \cdot x$ and therefore $\mathcal{R}(x) = \tilde{S} \cdot x = \langle \tilde{S} \rangle \cdot x$.
(iii)⇒(i) If $C\tilde{S}$ is not a group, then there exists $a \in \langle C\tilde{S} \rangle \setminus C\tilde{S}$. Since C is a normal subgroup of G we can factorize $a = \tilde{c}\delta$ with $\tilde{c} \in C$ and $\delta \in \langle \tilde{S} \rangle$. Supposing that the reachable sets form a partition in G/H, then for every $gH \in G/H$ — and in particular for $\delta^{-1}H$ — there exists $cs \in C\tilde{S}$ such that

$$a \cdot \delta^{-1}H = cs \cdot \delta^{-1}H. \tag{12}$$

Since C is a normal subgroup of G, with $C \subset H$, it follows that $s\delta^{-1} \in H$. Moreover, since $s, \delta \in \langle \tilde{S} \rangle$ and $H \cap \langle \tilde{S} \rangle \subset C$ we have $\check{c}s = \delta$ with $\check{c} \in C$. Therefore, $a = \tilde{c}\check{c}s$, which is a contradiction to $a \notin C\tilde{S}$. Hence, $C\tilde{S}$ is a group. ∎

4. Partition property of Inverse Iteration

In the following we want to apply our results of the previous section to shifted Inverse Iteration on flag manifolds.

Let $\mathbb{F}$ be an arbitrary field. With $\mathbb{F}^*$ we denote the multiplicative group of $\mathbb{F}$. A flag $\mathcal{V}$ is an increasing sequence of $\mathbb{F}$-linear subspaces

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_k \subset \mathbb{F}^n.$$

The *type* of the flag $\mathcal{V} = (V_1, \ldots, V_k)$ is defined by the k-tuple $d := (d_1, \ldots, d_k)$ of dimensions $d_i = \dim_{\mathbb{F}} V_i, i = 1, \ldots, k$. For any such sequence of integers $d = (d_1, \ldots, d_k), 1 \leq d_1 < \cdots < d_k \leq n$, we denote the set of all flags of type d with $\mathrm{Flag}(d, \mathbb{F}^n)$. Note that every $A \in \mathrm{GL}_n(\mathbb{F})$ acts on $\mathrm{Flag}(d, \mathbb{F}^n)$ via $A(V_1, \ldots, V_k) = (AV_1, \ldots, AV_k)$.

Via the bijection $T_{\mathcal{V}} : gH_{\mathcal{V}} \mapsto (gV_1, \ldots, gV_k)$ we can identify $\mathrm{Flag}(d, \mathbb{F}^n)$ with the homogeneous space $\mathrm{Flag}(d, \mathbb{F}^n) := \mathrm{GL}_n(\mathbb{F})/H_{\mathcal{V}}$, where $\mathcal{V} = (V_1, \ldots, V_k)$ is any fixed reference flag of type d and $H_{\mathcal{V}} := \{g \in \mathrm{GL}_n(\mathbb{F}) \,|\, gV_i = V_i\}$ is the stabilizer group of $\mathcal{V}$.

For a given matrix $A \in \mathbb{F}^{n \times n}$ we define $U = \mathbb{F} \setminus \mathrm{Spec}(A)$ and the map $\Phi : U \to \mathrm{GL}_n(\mathbb{F})$, $\Phi(u) = (A - uI)^{-1}$. Note that for every $x \in \mathrm{GL}_n(\mathbb{F})/H_{\mathcal{V}}$ the equation $T_{\mathcal{V}}(\Phi(u) \cdot x) = \Phi(u)T_{\mathcal{V}}(x)$ holds.

The Inverse Iteration system is given by $\Sigma_{A,d} = (\mathrm{Flag}(d, \mathbb{F}^n), U, \Phi)$. Obviously,

$$\tilde{S}_{II} = \left\{ \prod_{t=1}^{N} (A - u_t I)^{-1} \,|\, N < \infty, u_t \in \mathbb{F} \setminus \mathrm{Spec}(A) \right\}. \tag{13}$$

Note that $\mathrm{Flag}(d, \mathbb{F}^n)$ is the projective space $\mathbb{PF}^{n-1}$ for $d = (1)$. In this case $\Sigma_{A,(1)}$ describes the well known Inverse Iteration algorithm. The fixed points of $\Sigma_{A,(1)}$ coincide with the eigenspaces of A.

Another important special case is $d_c = (1, 2, \ldots, n-1)$, yielding the *complete flag manifold*. The fixed points of Σ_{A,d_c} coincide with the eigenflags $\mathcal{E} = (V_1, \ldots, V_{n-1})$ of A, i.e., $AV_i = V_i$ for $i = 1, \ldots, n-1$. It is known that the dynamics of Inverse Iteration on complete flag manifolds is closely related to the QR algorithm. We refer to Ammar, Martin (1986), Shub, Vasquez (1987), for more details.

In applications as above, one is usually interested in the behavior of *cyclic* matrices, i.e., matrices $A \in \mathbb{F}^{n\times n}$, such that there exists a *cyclic vector* $v \in \mathbb{F}^n$ such that $\mathrm{span}(v, Av, \ldots, A^{n-1}v) = \mathbb{F}^n$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ the set of cyclic matrices is open and dense in $\mathbb{F}^{n\times n}$. Therefore, cyclicity is a generic assumption on a matrix.

The following lemma shows that the conditions of Theorem 1 are satisfied.

LEMMA 2 *Let $A \in \mathbb{F}^{n\times n}$ be cyclic and $\mathrm{Flag}(d, \mathbb{F}^n)$ a flag manifold of type d, such that $d_1 = 1$. The Inverse Iteration system $\Sigma_{A,d} = (\mathrm{Flag}(d, \mathbb{F}^n), U, \Phi)$ satisfies the following conditions:*

a) $C = \mathbb{F}^* I$.

b) $H_{\mathcal{V}} \cap \langle \tilde{S}_{II} \rangle \subset C$ for a reference flag $\mathcal{V}$ of type d.

Proof. As above we identify the set $\mathrm{Flag}(d, \mathbb{F}^n)$ with the homogeneous space $\mathrm{GL}_n(\mathbb{F})/H_{\mathcal{V}}$. For that purpose we choose a reference flag $\mathcal{V} = (V_1, \ldots, V_k)$ of type d, such that a nonzero vector $v \in V_1$ is a a cyclic vector.

a) For any $c \in C$ and for all $g \in \mathrm{GL}_n(\mathbb{F})$ we have $cgV_1 = gV_1$. Therefore, every one dimensional vector space is an eigenspace of c. It follows that $c \in \mathbb{F}^* I$.

b) Obviously, $\langle \tilde{S}_{II} \rangle$ is a subgroup of the centralizer $Z(A) := \{B \in \mathrm{GL}_n(\mathbb{F}) \,|\, AB = BA\}$. Moreover, since A is cyclic, there is

$$Z(A) = \{p(A) \,|\, p \in \mathbb{F}[t], p \text{ coprim } \chi_A, \deg(p) \leq n-1\} \tag{14}$$

where χ_A is the characteristic polynomial of A (see Fuhrmann, 1996, Proposition 6.1.2). Therefore, if $X = \sum_{j=0}^{n-1} \alpha_j A^j \in H_{\mathcal{V}} \cap \langle \tilde{S}_{II} \rangle$ and $v \in V$ cyclic, then $Xv = \lambda v$ and $(\alpha_0 - \lambda)v + \sum_{j=1}^{n-1} \alpha_i A^i v = 0$. Since $v, Av, \ldots, A^{k-1}v$ is a basis of $\mathbb{F}^n$ we conclude that $X = \lambda I$. ∎

Applying Theorem 1 we have the following result:

THEOREM 2 *Let A be cyclic and* $\mathrm{Flag}(d, \mathbb{F}^n)$ *a flag manifold with* $d_1 = 1$. *Then the following statements are equivalent:*
(i) $\mathbb{F}^*\tilde{S}_{II}$ *is a group.*
(ii) System $\Sigma_{d,A}$ *has the partition property.*

The following example shows that $\mathbb{R}^*\tilde{S}_{II}$ may be a group even if $\tilde{S}_{II}$ is not.

EXAMPLE 2 Let $\mathbb{F} = \mathbb{R}$ and

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

We show that $\tilde{S}_{II}$ is not a group. Let

$$B := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \in \tilde{S}_{II} = \left\{ \prod_{t=1}^{N} \begin{pmatrix} -u_t & -1 \\ 1 & -u_t \end{pmatrix}^{-1} \middle| N < \infty, u_t \in \mathbb{R} \right\}. \tag{16}$$

Assume $B^{-1} \in \tilde{S}_{II}$, i.e., there exist shift parameters $u_1, \ldots, u_N \in \mathbb{R}$ such that $B^{-1} = \prod_{t=1}^{N}(A - u_t I)^{-1}$. Then

$$\det(B^{-1}) = \det \left(\prod_{t=1}^{N} \begin{pmatrix} -u_t & -1 \\ 1 & -u_t \end{pmatrix}^{-1} \right) = \prod_{t=1}^{N} \frac{1}{u_t^2 + 1} \leq 1, \tag{17}$$

which is a contradiction to $\det(B) = \frac{1}{2}$. Hence, $\tilde{S}_{II}$ is not a group.

On the other hand, the inverse of $(A - uI)^{-1} \in \tilde{S}_{II}$ is given by

$$A - uI = (u^2 + 1)A^{-1}A^{-1}(A + uI)^{-1} \in \mathbb{R}^*\tilde{S}_{II}. \tag{18}$$

Therefore, $\mathbb{R}^*\tilde{S}_{II}$ is a group.

Let $\Sigma_{A,d}$ be the Inverse Iteration system on a flag manifold $\mathrm{Flag}(d, \mathbb{F}^n)$ and $(A - uI)^{-1}$ be an arbitrary element of $\tilde{S}_{II}$. In order to find out if $\mathbb{F}^*\tilde{S}_{II}$ is a group one has to find elements $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F} \setminus \mathrm{Spec}(A)$ and $r \in \mathbb{F}^*$ such that

$$r \prod_{t=1}^{n-1} (A - \alpha_t I)^{-1}(A - uI)^{-1} = I. \tag{19}$$

In the case $\mathbb{F} = \mathbb{C}$ this can always be done. Let χ_A be the characteristic polynomial of A. Since $\mathbb{C}[z]$ is an Euclidean ring and since u is not a zero of χ_A we have the identity $\chi_A = (t - u)k - r$ with $k \in \mathbb{C}[z]$ and $r \in \mathbb{C} \setminus \{0\}$. The linear factorization of $\chi_A + r$ gives us $\chi_A + r = (t - u)\prod_{t=1}^{n-1}(t - \alpha_t)$ for $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C} \setminus \mathrm{Spec}(A)$. Therefore, there holds

$$I = \frac{1}{r}(A - uI) \prod_{t=1}^{n-1} (A - \alpha_i I). \tag{20}$$

We conclude that in the case $\mathbb{F} = \mathbb{C}$ the Inverse Iteration system $\Sigma_{A,d}$ always has the partition property.

In the case $\mathbb{F} = \mathbb{R}$ the situation is much more complicated. The following example shows that we cannot expect to have the partition property for all matrices $A \in \mathbb{R}^{n\times n}$.

Example 3 Let $\Sigma_{A,d} = (\mathbb{P}\mathbb{R}^2, \mathbb{R}\setminus \mathrm{Spec}(A), \Phi_{II})$ be the Inverse Iteration system for the cyclic matrix

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

We show that the identity is not an element of the semigroup $\mathbb{R}^*\tilde{S}_{II}$. By Theorem 2 it follows that $\Sigma_{A,d}$ fails to have the partition property.

Suppose there exist $r \in \mathbb{R}^*$ and control parameters $u_1, \ldots, u_N \in U$ such that $I = r\prod_{t=0}^{N}(A - u_tI)^{-1}$. The block structure of A yields the equations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = r\prod_{t=1}^{N}\begin{pmatrix} -u_t & -1 \\ 1 & -u_t \end{pmatrix} \qquad \text{and} \qquad 1 = r\prod_{t=0}^{N}(-u_t). \tag{22}$$

Comparison of the determinants shows that

$$r^2\prod_{t=0}^{N}(u_t^2+1) = r^2\prod_{t=0}^{N}u_t^2\ , \tag{23}$$

which is a contradiction to $r \neq 0$.

5. Conclusion and remarks

We have shown a sufficient condition for partition property for a family of discrete-time control systems on homogeneous space. As an application we obtain a necessary and sufficient condition, for which Inverse Iteration on flag manifolds has the partition property. In particular, for a given matrix, Inverse Iteration on projective space has the partition property if and only if Inverse Iteration on complete flag manifolds – and therefore the QR algorithm – has the partition property.

To analyze Inverse Iteration concerning its controllability properties and adherence structure of the reachable sets – as it is done in Helmke and Fuhrmann (2000), Helmke and Wirth (2001), Helmke and Jordan (2002) – it is appropriate to use the specific topological, geometric and algebraic structure of the system. Nevertheless, for the proof of our results stated in the paper we only need purely algebraic properties. It might be possible to apply and adapt those results to other applications of control theory with discrete state spaces, such as coding theory or cryptography.

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