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Multiobservers and their application to output stabilization[1]

by

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Abstract: A construction of multiobserver is extended to arbitrary analytic control systems without any observability property. The dynamics of the multiobserver gives an estimation of the successive derivatives of the output and the output of the multiobserver is a set that approximates the whole class of states that are indistinguishable from the current state of the original system by the control applied to the system. The dynamics of the multiobserver is used to stabilize the system with the aid of output feedback. It is done under the assumption that the system is state-feedback stabilizable and the functions used in this feedback depend smoothly on the output function and its Lie derivatives.

Keywords: multiobserver, analytic system, output feedback stabilization.

1. Introduction

The idea of construction of high gain observers that estimate the whole class of indistinguishable states was introduced in Bartosiewicz and Wyrwas (1999). Such observers are called multiobservers. A multiobserver is constructed for an analytic system $\dot{x} = f(x,u)$, $y = h(x)$. We assume that there is an integer N, such that for any control u the following functions: $h(\cdot)$, $\mathcal{L}_f h(\cdot, u)$, $\ldots$, $\mathcal{L}_f^{N-1} h(\cdot, j^{N-2}u)$ distinguish states that are distinguishable by the control u. It is known (Wyrwas, 2001, 2004) that for systems without control such an integer exists and the finite number of functions from the observation space distinguish states that are distinguishable (if only the system is defined on the compact global semianalytic subset of $\mathbb{R}^n$).

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For analytic systems indistinguishability is an equivalence relation. If the system is not observable, there are states that are indistinguishable so a potential observer cannot follow the true state of the system. The most we can get is an approximation of the whole class of all states that are indistinguishable. Therefore the output of the multiobserver is a multivalued mapping (multifunction) whose values are subsets of $\mathbb{R}^n$. Such a multiobserver was constructed in Bartosiewicz, Wyrwas (1999, 2005) and Wyrwas (2001, 2004) for arbitrary locally observable systems. Under this assumption the values of the output map are discrete subsets of $\mathbb{R}^n$, which become finite when the state space is restricted to a compact subset of $\mathbb{R}^n$. In this paper we show that such a construction can be extended to arbitrary analytic systems without any observability property. Now the values of the output map may not be discrete sets, but they should be closed in order to be able to measure the distance between the state of the original system and the estimate coming from the multiobserver. Multiobservers work for all inputs applied to the systems. However, if we use the input that is not universal, then the output of the multiobserver is a subset of the state space that may contain states that are distinguishable by another control but the applied control does not distinguish them. Thus the output may be a big set, but the distance between this set and the true state goes to zero, when time is going to infinity.

We apply multiobservers to output stabilization following Jouan and Gauthier (1996) who used a standard high-gain observer. Our goal is to weaken their assumption as much as possible. We give up observability of the system as, this property is hard to check and there are many examples of systems that are not observable.

We do not use Ascending Chain Property (ACP) assumed in Jouan and Gauthier (1996), which is equivalent to the fact that the N^{th} derivative of the output, for some $N \in \mathbb{N}$, can be locally analytically expressed in terms of the output, its $N-1$ first derivatives together with the input and its $N-1$ first derivatives. Then, using a partition of unity one can show that there is a smooth function that globally describes the relation between output and input derivatives. Instead, we assume that there is $N \in \mathbb{N}$ such that, for every control u, if u distinguishes two states, then the $(N-2)$–jet of u also distinguishes these states and we show that one can express higher derivatives of the output as continuous functions of lower derivatives. One of such functions appears in the dynamics of the multiobserver. We additionally assume that this function is of class $\mathcal{C}^2$.

Assuming observability, ACP and the existence of a feedback, that stabilizes the original system and that can be expressed as a smooth function depending on output and its derivatives up to order $N-1$, Jouan and Gauthier showed how to use a constructed high-gain observer in output feedback stabilization. We follow here the main ideas of Jouan and Gauthier, but we give up observability and ACP. Similarly to Jouan and Gauthier we assume that there is a feedback that stabilizes the original system, but because we do not have ACP, so additionally we have to assume that feedbacks that stabilize the system and its extensions can be expressed as smooth functions that depend on the output and its derivatives up to order $N-1$ together with the input and its $N-2$ first derivatives. In fact, the values of these feedbacks are constant on the indistinguishability class. If we compose the extension of order $N-1$ of the original system with the dynamical part of the constructed multiobserver and use a feedback, that stabilizes this extension, we obtain a closed loop system of class $\mathcal{C}^2$. Then we can use the center manifold theorem to show local asymptotic stability of this closed loop system. This achieves the output feedback stabilization of the original system.

2. Analytic control systems and their properties

Let us consider an analytic control system Σ defined on $\mathcal{X} \subseteq \mathbb{R}^n$ of the following form

$$\dot{x} = f(x, u) \tag{1}$$

$$y = h(x). \tag{2}$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in U \subseteq \mathbb{R}$, $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}_+)$. The maps f and h are assumed to be analytic.

The following sequence

$$j^k u = \left(u, u', u'', \ldots, u^{(k)}\right)$$

will be called *k–jet of the control u*.

For a function ϕ depending on x and $j^k u$, let

$$(\mathcal{L}_f \phi)(x, j^{k+1}u) := \frac{\partial \phi(x, j^k u)}{\partial x} f(x, u) + \sum_{i=0}^{k} \frac{\partial \phi(x, j^k u)}{\partial u^{(i)}} u^{(i+1)},$$

where $\frac{\partial \phi(x,j^k u)}{\partial x} = \left(\frac{\partial \phi(x,j^k u)}{\partial x_1}, \ldots, \frac{\partial \phi(x,j^k u)}{\partial x_n}\right)$.

If we set the infinite jet of u as follows

$$j^\infty u = (u, u', \ldots, u^{(k)}, \ldots).$$

and we compute the successive derivatives of the output, we have the following family of analytic functions:

$$\begin{aligned}
\psi_i^0(x, j^\infty u) &= h_i(x), \\
\psi_i^1(x, j^\infty u) &= \tfrac{\partial h_i(x)}{\partial x} f(x, u), \\
\psi_i^2(x, j^\infty u) &= \tfrac{\partial \psi_i^1(x, j^\infty u)}{\partial x} f(x, u) + \tfrac{\partial \psi_i^1(x, j^\infty u)}{\partial u} u', \\
&\vdots \\
\psi_i^{k+1}(x, j^\infty u) &= \tfrac{\partial \psi_i^k(x, j^\infty u)}{\partial x} f(x, u) + \sum_{i=0}^{k-1} \tfrac{\partial \psi_i^k(x, j^\infty u)}{\partial u^{(i)}} u^{(i+1)}, \\
&\vdots
\end{aligned}$$

For the fixed $j^\infty u$, the space $\mathcal{K}_{j^\infty u}(\Sigma)$, defined as follows

$$\mathcal{K}_{j^\infty u}(\Sigma) := \operatorname{span}_{\mathbb{R}}\{\psi_i^k(\cdot, j^\infty u), i = 1, \ldots, p, k \geq 0\},$$

is a space of analytic functions defined on $\mathcal{X} \subseteq \mathbb{R}^n$ with values in $\mathbb{R}$.

Let $\chi_u(t, x_0)$ be the trajectory of the system Σ starting at x_0, corresponding to the control u and evaluated at time t. Assume that for all x_0 and u, the trajectory $\chi_u(t, x_0)$ is well defined on $[0, +\infty)$.

Definition 1 Let $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}_+)$. Two states $x_1, x_2 \in \mathcal{X}$ are called *indistinguishable by u* if

$$h(\chi_u(t, x_1)) = h(\chi_u(t, x_2)) \tag{3}$$

for every $t \geqslant 0$. Otherwise $x_1, x_2 \in \mathcal{X}$ are called *distinguishable by u*.

The states are *distinguishable*, if they are distinguishable by some u, and they are *indistinguishable* ($x_1 \sim x_2$) if they are indistinguishable by all $u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}_+)$.

Define $\mathbb{K} := \bigcup_{u(\cdot) \in \mathcal{C}^\infty(\mathbb{R}_+)} \mathcal{K}_{j^\infty u}(\Sigma)$.

The following equivalence holds:

Proposition 1 *(Mozyrska, 2000; Wyrwas, 2004) States x_1 and x_2 are indistinguishable $\Leftrightarrow \psi(x_1) = \psi(x_2)$, for all functions $\psi \in \mathbb{K}$.*

Let $j^k u$ be the k-jet of u.

Definition 2 The states x_1 and x_2 are called *indistinguishable by $j^k u$* ($x_1 \sim_{j^k u} x_2$), if $h(x_1) = h(x_2), \mathcal{L}_f h(x_1, u) = \mathcal{L}_f h(x_2, u), \ldots, \mathcal{L}_f^{k+1} h(x_1, j^k u) = \mathcal{L}_f^{k+1} h(x_2, j^k u)$.

Remark 1 For the considered analytic systems the indistinguishability relation is an equivalence relation and the state space can be decomposed into nonempty and disjoint classes. The same holds for the indistinguishability by u and by $j^k u$, for a fixed input u.

We need the following assumption:

Assumption 1 $\exists N \in \mathbb{N}\ \forall u(\cdot) \in C^\infty$ *(x_1, x_2 are distinguishable by $u \Rightarrow x_1$, x_2 are distinguishable by $\boldsymbol{u} := j^{N-2} u$).*

Remark 2 If we consider an analytic system without control, defined on a compact global semianalytic subset $\mathcal{X}$ of $\mathbb{R}^n$, then there is a finite number of functions $\psi_1 := h$, $\psi_2 := L_f h, \ldots, \psi_k := L_f^{k-1} h$ from the observation space, such that every two states $x_1, x_2 \in \mathcal{X}$ are indistinguishable if and only if $\psi_i(x_1) = \psi_i(x_2)$, $i = 1, \ldots, k$. This fact was shown in Wyrwas (2004) with the aid of the Frisch theorem (Frisch, 1967), which asserts that the ring of analytic function germs at $\mathcal{X}$ is Noetherian. Using this theorem one can show that for an analytic control system defined on a compact global semianalytic subset of $\mathbb{R}^n$, for each control u there is a finite number of functions $h(\cdot), \mathcal{L}_f h(\cdot, u), \ldots, \mathcal{L}_f^{N-1} h(\cdot, j^{N-2}u)$ that distinguish states that are distinguishable by u. If a control $v : \mathbb{R}_+ \to U$ distinguishes all distinguishable states it is called *universal*. From Sussmann (1979) it is known that there is a universal control for any analytic control system and almost every control is universal because the set of universal C^∞ inputs is dense and open.

If all controls are universal then Assumption 1 is equivalent to the following condition: $\exists N \in \mathbb{N}\ \forall u(\cdot) \in C^\infty \left\{ h(\cdot), \mathcal{L}_f h(\cdot, u), \ldots, \mathcal{L}_f^{N-1} h(\cdot, j^{N-2}u) \right\}$ distinguish distinguishable states.

If we take an input that is not universal, then it may not distinguish some points that could be distinguished by another control. In this case we obtain different indistinguishability classes parameterized by controls u.

Assume that the set $\mathcal{X}$ is a compact subset of $\mathbb{R}^n$ and Assumption 1 holds. Let $\mathrm{S}\Phi_N : \mathcal{X} \times \mathbb{R}^{N-1} \to \mathbb{R}^{N \cdot p} \times \mathbb{R}^{N-1}$ be the map defined as follows:

$$\mathrm{S}\Phi_N(x, j^{N-2}u) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x, u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \\ j^{N-2}u \end{pmatrix}. \tag{4}$$

The map $\mathrm{S}\Phi_N$ is related to the indistinguishability relation and from Assumption 1 there is $N \in \mathbb{N}$ such that components of $\mathrm{S}\Phi_N(\cdot, j^{N-2}u)$ distinguish states that are distinguishable by the input u.

Let $\mathcal{U}_{k,B}$ be the set of all the maps $u : \mathbb{R}_+ \to \mathbb{R}$ that are k times continuously differentiable and satisfy the following inequality:

$$\left\| \frac{d^i}{dt^i} u(t) \right\| \leqslant B, \tag{5}$$

for all $t \geqslant 0$, where $0 \leqslant i \leqslant k$.
Assume that controls applied to the system Σ belong to the set $\mathcal{U}_{N-1,B}$. Then

$$(u(t), u'(t), \ldots, u^{(N-1)}(t)) \in \mathbb{U}_N \subset \mathbb{R}^N,$$

where $\mathbb{U}_N$ is a compact subset.

For the considered analytic systems we have the following property:

Proposition 2 *Let Assumption 1 hold. If the system Σ is analytic and it is defined on a compact set $\mathcal{X}$, then there is a continuous map $\varphi : \mathbb{R}^{Np} \times \mathbb{U}_N \to \mathbb{R}^p$ such that*

$$\mathcal{L}_f^N h = \varphi(h, \ldots, \mathcal{L}_f^{N-1} h, u, u', \ldots, u^{(N-1)}). \tag{6}$$

Proof. Let u be a control applied for the analytic control system Σ and $\boldsymbol{u} = j^{N-2}u \in \mathbb{U}_{N-1}$. From Assumption 1 there exists $N \in \mathbb{N}$ such that the following functions

$$\psi_i(x) := h_i(x), \psi_{i,u}(x) := \mathcal{L}_f h_i(x,u), \ldots, \psi_{i,j^{N-2}u}(x) := \mathcal{L}_f^{N-1} h_i(x, j^{N-2}u),$$

distinguish states that are distinguishable by u. It means that states x_1 and x_2 are indistinguishable by u if and only if we have

$$\begin{aligned} \psi_i(x_1) &= \psi_i(x_2), \\ \psi_{i,u}(x_1) &= \psi_{i,u}(x_2), \\ &\vdots \\ \psi_{i,j^{N-2}u}(x_1) &= \psi_{i,j^{N-2}u}(x_2), \end{aligned}$$

when $i = 1, \ldots, p$.

Then the function $x \mapsto \mathcal{L}_f^N h_i(x, j^{N-1}u)$, $i = 1, \ldots, p$, does not distinguish states from the set

$$[x]_{\boldsymbol{u}} = \{\tilde{x} \in \mathcal{X} : \ \Phi_{N,j^{N-2}u}(x) = \Phi_{N,j^{N-2}u}(\widetilde{x})\},$$

where $\Phi_{N,j^{N-2}u} : \mathcal{X} \to \mathbb{R}^{Np}$ is defined by

$$\Phi_{N,j^{N-2}u}(x) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x,u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \end{pmatrix}. \tag{7}$$

Denote by $\widetilde{\mathcal{X}}_{\boldsymbol{u}}$ the quotient space of $\mathcal{X}$ with respect to the indistinguishability relation by $\boldsymbol{u}$ ($\widetilde{\mathcal{X}}_{\boldsymbol{u}} := \mathcal{X}/_{\sim_{\boldsymbol{u}}}$).

Let $\mathcal{Z} := \bigcup_{\boldsymbol{u} \in \mathbb{U}_{N-1}} \widetilde{\mathcal{X}}_{\boldsymbol{u}} \times \{\boldsymbol{u}\}$ and $\pi : \mathcal{X} \times \mathbb{U}_{N-1} \to \mathcal{Z}$, $\pi(x, \boldsymbol{u}) := ([x]_{\boldsymbol{u}}, \boldsymbol{u}) \in \mathcal{Z}$. A natural topology on the space $\mathcal{Z}$ is introduced in the following manner: we call a set $\widetilde{V} \subset \mathcal{Z}$ open in $\mathcal{Z}$ if and only if the set $\pi^{-1}(\widetilde{V})$ is open in $\mathcal{X} \times \mathbb{U}_{N-1}$. It will be called the *quotient* topology. Then π is continuous and $\mathcal{Z}$ is compact.

Let $\mathrm{S}\widehat{\Phi}_N : \mathcal{Z} \to \mathbb{R}^{Np} \times \mathbb{U}_{N-1}$ be the map defined as follows:

$$\mathrm{S}\widehat{\Phi}_N\left([x]_{\boldsymbol{u}}, \boldsymbol{u}\right) := \mathrm{S}\Phi_N\left(x, \boldsymbol{u}\right). \tag{8}$$

It is continuous and injective, so (from Lemma 4.1 of Jouan and Gauthier, 1996) $\mathrm{S}\widehat{\Phi}_N$ is a homeomorphism onto its image, which is closed. It is easy to see that

$$\mathrm{S}\widehat{\Phi}_N\left(\mathcal{Z}\right) = \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1}).$$

Consider the map $\mathrm{S}\widetilde{\Phi}_N : \mathcal{Z} \times [-B, B] \to \mathbb{R}^{Np} \times \mathbb{R}^N$

$$\mathrm{S}\widetilde{\Phi}_N\left([x]_{\boldsymbol{u}}, u, u', \ldots, u^{(N-1)}\right) := \left(\mathrm{S}\Phi_N(x, \boldsymbol{u}), u^{(N-1)}\right). \tag{9}$$

The map $S\widetilde{\Phi}_N$ is similar to $S\widehat{\Phi}_N$ and, as before, it is a homeomorphism onto its image, which is closed. Then $\Psi := S\widetilde{\Phi}_N^{-1}$ is a continuous map from $S\widetilde{\Phi}_N(\mathcal{Z} \times [-B, B])$ to $\mathcal{Z} \times [-B, B]$. It satisfies

$$\Psi\left(h(x), \mathcal{L}_f h(x,u), \ldots, \mathcal{L}_f^{N-1} h(x, j^{N-2}u), j^{N-1}u\right) = \left([x]_{\boldsymbol{u}}, j^{N-1}u\right). \quad (10)$$

For every state $x \in \mathcal{X}$ and every jet $j^{N-1}u \in \mathbb{U}_N \subset \mathbb{R}^N$ we have

$$\begin{aligned} &\mathcal{L}_f^N h\left(x, u, u', \ldots, u^{(N-2)}, u^{(N-1)}\right) = \mathcal{L}_f^N h\left([x]_{\boldsymbol{u}}, j^{N-1}u\right) = \\ &= \mathcal{L}_f^N h\left(\Psi\left(Y, j^{N-1}u\right)\right) = \left(\mathcal{L}_f^N h \circ \Psi\right)\left(Y, j^{N-1}u\right) \end{aligned}$$

where $Y := \Phi_{N, j^{N-2}u}(x)$. The function $\mathcal{L}_f^N h \circ \Psi$ is continuous and it is defined on a compact set $S\widetilde{\Phi}_N\left(\mathcal{Z} \times [-B, B]\right)$. Then there is a continuous extension of $\mathcal{L}_f^N h \circ \Psi$ on the entire $\mathbb{R}^{Np} \times \mathbb{U}_N$ (from the Tietze's theorem).

Let $\varphi : \mathbb{R}^{Np} \times \mathbb{U}_N \to \mathbb{R}^p$ be the continuous extension of the function $\mathcal{L}_f^N h \circ \Psi$. Then $(Y, j^{N-1}u) \in \mathcal{Z} \times [-B, B]$ and we have $\varphi\left(Y, j^{N-1}u\right) = \left(\mathcal{L}_f^N h \circ \Psi\right)(Y, j^{N-1}u)$, so

$$\mathcal{L}_f^N h(x, j^{N-1}u) = \varphi(h(x), \ldots, \mathcal{L}_f^{N-1} h(x, j^{N-2}u), j^{N-1}u). \qquad \blacksquare$$

3. The construction of multiobservers

DEFINITION 3 The system $\mathcal{M}$ defined on $\mathbb{R}^{Np}$ of the following form

$$\dot{z} = F(z, u, \dot{u}, \ldots, u^{(N-1)}, y) \quad (11)$$

$$\hat{x} = g(z, u, \dot{u}, \ldots, u^{(N-2)}) \quad (12)$$

is called *a multiobserver* of the control system Σ, if

(i) the input y is the output of Σ,

(ii) g is a multivalued function (multifunction) whose values are closed subsets of $\mathbb{R}^n$,

(iii) $\lim_{t \to +\infty} d(x(t); \hat{x}(t)) = 0$, for every trajectory $z(\cdot)$ of the system (11),

where $d(x_0; \mathbb{A}) := \inf_{x \in \mathbb{A}} \|x_0 - x\|$ is the distance of the point x_0 to the set $\mathbb{A}$, and $\|\cdot\|$ is the Euclidean norm.

REMARK 3 Definition 3 is an extension of the definition of multiobserver given in Bartosiewicz, Wyrwas (1999, 2005) and Wyrwas (2001, 2004), where it was assumed that the values of the output of a multiobserver were finite subsets of $\mathbb{R}^n$ as the multiobserver was constructed only for locally observable systems on a compact subset of $\mathbb{R}^n$.

Let u be a smooth control applied to the system Σ. Assume we know the output y. Then we consider the system $\mathcal{M}_{\Sigma,\theta,\mathcal{X}}$ defined on $\mathbb{R}^{Np}$ with inputs u, u', ..., $u^{(N-1)}$ and y as follows:

$$\dot{z} = (\mathrm{A} - \mathrm{K}_\theta \mathrm{C})z + \mathrm{K}_\theta y + \mathrm{b}\varphi\left(z, u, u', \ldots, u^{(N-1)}\right), \tag{13}$$

where $z \in \mathbb{R}^{Np}$, $\mathrm{A} = \begin{pmatrix} 0 & \mathrm{I}_p & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \mathrm{I}_p & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \mathrm{I}_p \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{Np\times Np}$, $\mathrm{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I}_p \end{pmatrix}_{Np\times p}$,

$\mathrm{C} = \begin{pmatrix} \mathrm{I}_p & 0 & \ldots & 0 \end{pmatrix}_{p\times Np}$, $\Delta_\theta = \begin{pmatrix} \theta \mathrm{I}_p & 0 & 0 & \ldots & 0 \\ 0 & \theta^2 \mathrm{I}_p & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \theta^N \mathrm{I}_p \end{pmatrix}_{Np\times Np}$, $\theta \in \mathbb{R}_+$

and $\theta \gg 1$, $\mathrm{K}_\theta = \Delta_\theta \mathrm{K}$. $\mathrm{K} \in \mathbb{R}^{Np\times p}$ is a matrix such that $\mathrm{A} - \mathrm{KC}$ is a Hurwitz matrix, and the function φ comes from Proposition 2. $b \gg a$ denotes that $\frac{a}{b}$ is small enough.

Let $\Phi_N : \mathcal{X} \times \mathbb{U}_{N-1} \to \mathbb{R}^{Np}$ be a map defined as follows:

$$\Phi_N(x, j^{N-2}u) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x, u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \end{pmatrix}. \tag{14}$$

THEOREM 1 *Assume that φ is a Lipschitz function with Lipschitz constant $L_{\mathcal{X},B}$ and inputs u belong to $\mathcal{U}_{N-1,B}$. For every $z(0) \in \mathbb{R}^{Np}$, $x(0) \in \mathcal{X}$ and $u(\cdot) \in \mathcal{U}_{N-1,B}$ the solution of the system (13) satisfies the following inequality*

$$\left\| z(t) - \Phi_N\left(x(t), u(t), \ldots, u^{(N-2)}(t)\right)\right\| \le \theta^{N-1}\beta \exp\left[-\alpha\left(\frac{\theta}{2} - \omega_{\mathcal{X},B}\right)t\right] \cdot \cdot \left\| z(0) - \Phi_N(x(0), u(0), \ldots, u^{(N-2)}(0))\right\|,$$

where α, β are some positive constants, $\omega_{\mathcal{X},B}$ is a constant which depends on $\mathcal{X}$ and B. The inequality holds for all t such that $\chi_u(t, x_0) \subset \mathcal{X}$.

Proof. The proof is the same as in Jouan and Gauthier (1996), where it was assumed that φ is a smooth function, while in the proof only the Lipschitz property is used. Therefore the proof can be repeated under the assumption that φ is a Lipschitz function. ■

REMARK 4 The solution of the system $\mathcal{M}_{\Sigma,\theta,\mathcal{X}}$ gives an exponential estimation of the outputs and their successive derivatives up to order $N-1$, based on the knowledge of the output y, which is observed, and the input u and its derivatives $u', \ldots, u^{(N-1)}$, which is applied to the system.

If the system Σ is observable, i.e. every two different states are distinguishable, then a continuous estimation of the unknown state of the system Σ can be found in the same way as in Gauthier and Kupka (2001) and Jouan and Gauthier (1996). If the system Σ is not observable, it is not possible to find an approximation of unknown state $x(t)$, but we can find the whole class of states that are indistinguishable.

Now, on the basis of the output y, input u and their successive derivatives up to order $N-1$ we want to reconstruct the whole class of indistinguishable states by the input u applied to the considered system.

In order to do it, let us consider the family of the analytic maps $\Phi_{N,j^{N-2}u} : \mathcal{X} \to \mathbb{R}^{Np}$ parameterized by $j^{N-2}u \in \mathbb{U}_{N-1} \subset \mathbb{R}^{N-1}$ given by the formula (7). They determine the indistinguishability relation on the set $\mathcal{X}$ (from Proposition 2).

Consider the map $\mathrm{S}\widehat{\Phi}_N$, which was defined by (8) in the proof of Proposition 2. Because $\mathrm{S}\widehat{\Phi}_N$ is a homeomorphism, so $\widehat{\Psi} := \mathrm{S}\widehat{\Phi}_N^{-1}$ is a continuous map from $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ to $\bigcup_{\boldsymbol{u} \in \mathbb{U}_{N-1}} \widetilde{\mathcal{X}}_{\boldsymbol{u}} \times \{\boldsymbol{u}\}$ such that

$$\widehat{\Psi}\left(h(x), \mathcal{L}_f h(x,u), \ldots, \mathcal{L}_f^{N-1} h(x, j^{N-2}u), j^{N-2}u\right) = \left([x]_{\boldsymbol{u}}, j^{N-2}u\right). \quad (15)$$

$\widehat{\Psi}$ is well defined only on $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ and a natural problem is now to extend $\widehat{\Psi}$ to the entire space $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$. As the values of $\widehat{\Psi}$ belong to $\bigcup_{\boldsymbol{u} \in \mathbb{U}_{N-1}} \widetilde{\mathcal{X}}_{\boldsymbol{u}} \times \{\boldsymbol{u}\}$, the Tietze theorem cannot be used. Moreover such extensions may not exist. We shall use the theory of retracts to deal with this problem.

Proposition 3 *If* $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ *is a retract of* $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$*, then there is a continuous extension* Ψ^* *of the map* $\widehat{\Psi} : \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1}) \to \bigcup_{\boldsymbol{u} \in \mathbb{U}_{N-1}} \widetilde{\mathcal{X}}_{\boldsymbol{u}} \times \{\boldsymbol{u}\}$ *over* $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$.

Proof. If $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ is a retract of the space $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$ with a retraction $r : \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$, then the composed map

$$\Psi^* := \widehat{\Psi} \circ r : \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$$

is a continuous extension of $\widehat{\Psi}$ over $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$. ■

Let $\widetilde{\mathcal{X}}$ be the disjoint union of $\widetilde{\mathcal{X}}_{\boldsymbol{u}}$, where $\boldsymbol{u} \in \mathbb{U}_{N-1}$, and

$$\pi_1 : \bigcup_{\boldsymbol{u} \in \mathbb{U}_{N-1}} \widetilde{\mathcal{X}}_{\boldsymbol{u}} \times \{\boldsymbol{u}\} \to \widetilde{\mathcal{X}}$$

be the projection on the first component, i.e. $\pi_1([x]_{\boldsymbol{u}}, j^{N-2}u) = [x]_{\boldsymbol{u}}$.

Let $g : \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to \widetilde{\mathcal{X}}$ be the map defined as follows $g := \pi_1 \circ \Psi^*$. The map g will then serve as the output of the multiobserver of Σ, whose dynamics is given by (13). Hence

$$\hat{x}(t) = g(z(t), u(t), u'(t), \dots, u^{(N-2)}(t)) \tag{16}$$

is an estimation of the class $[x(t)]_{\boldsymbol{u}} \in \widetilde{\mathcal{X}}$ that consists of all the states indistinguishable from $x(t)$ by $\boldsymbol{u}$.

Let $\tilde{g} := g|_{\mathrm{S}\Phi_N(\mathcal{X}\times\mathbb{U}_{N-1})} = \pi_1 \circ \widehat{\Psi}$. Then $\tilde{g}$ may be treated as a multifunction from $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ whose values are subsets of $\mathcal{X}$, i.e. elements of $2^{\mathcal{X}}$. It is continuous as a map $\tilde{g} : \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1}) \subset \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to \widetilde{\mathcal{X}}$ with the quotient topology on $\widetilde{\mathcal{X}}$ that is defined similarly as in the proof of Proposition 2. Namely a set $\widetilde{V} \subset \widetilde{\mathcal{X}}$ is open in $\widetilde{\mathcal{X}}$ if and only if the set $(\pi_1 \circ \pi)^{-1}(\widetilde{V})$ is open in $\mathcal{X} \times \mathbb{U}_{N-1}$. Additionally, the following proposition holds:

Proposition 4 *The multifunction $\tilde{g} : \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1}) \to 2^{\mathcal{X}}$ is upper semicontinuous, i.e. for every $(\mathbf{y}_0, \mathbf{u}_0) \in \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ and for every $\epsilon > 0$ there is $\delta > 0$ such that for all $(\mathbf{y}, \mathbf{u}) \in \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ the condition $\| (\mathbf{y}, \mathbf{u}) - (\mathbf{y}_0, \mathbf{u}_0) \| < \delta$ implies $\tilde{g}(\mathbf{y}, \mathbf{u}) \subset \mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \epsilon)$, where*

$$\mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \varepsilon) := \bigcup_{\tilde{x} \in \tilde{g}(\mathbf{y}_0, \mathbf{u}_0)} \{x \in \mathcal{X} \mid \ \|x - \tilde{x}\| < \varepsilon\},$$

and $\| \cdot \|$ is the Euclidean norm.

Proof. Suppose that $\tilde{g}$ is not upper semicontinuous, i.e. there is $(\mathbf{y}_0, \mathbf{u}_0) \in \mathrm{S}\Phi_N(\mathcal{X}\times\mathbb{U}_{N-1})$ and $\epsilon > 0$ such that for each $\delta = \frac{1}{n}$ there is $(\mathbf{y}_n, \mathbf{u}_n) \in \mathrm{S}\Phi_N(\mathcal{X}\times \mathbb{U}_{N-1})$ such that $\|(\mathbf{y}_n, \mathbf{u}_n) - (\mathbf{y}_0, \mathbf{u}_0)\| < \frac{1}{n}$ and $\tilde{g}(\mathbf{y}_n, \mathbf{u}_n) \not\subset \mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \epsilon)$. Let $x_n \in \tilde{g}(\mathbf{y}_n, \mathbf{u}_n)$ and $x_n \notin \mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \epsilon)$. Let $A := \mathcal{X} \setminus \mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \epsilon)$. A is a compact set and $x_n \in A$, $n \in \mathbb{N}$. Hence there is a convergent subsequence of (x_n). We can assume that $x_n \to x_0 \in A$. Notice that $\mathrm{S}\Phi_N(x_n, \mathbf{u}_n) \in \widehat{\mathrm{S}\Phi}_N(\tilde{g}(\mathbf{y}_n, \mathbf{u}_n), \mathbf{u}_n) = \mathrm{S}\Phi_N(\widehat{\Psi}(\mathbf{y}_n, \mathbf{u}_n)) = \mathrm{S}\Phi_N(\mathrm{S}\Phi_N^{-1}(\mathbf{y}_n, \mathbf{u}_n)) = (\mathbf{y}_n, \mathbf{u}_n)$, i.e. $\mathrm{S}\Phi_N(x_n, \mathbf{u}_n) = (\mathbf{y}_n, \mathbf{u}_n)$ and $(\mathbf{y}_n, \mathbf{u}_n) \to (\mathbf{y}_0, \mathbf{u}_0)$. Because $\mathrm{S}\Phi_N$ is continuous, then $(\mathbf{y}_0, \mathbf{u}_0) = \mathrm{S}\Phi_N(x_0, \mathbf{u}_0)$. Therefore $(x_0, \mathbf{u}_0) \in \mathrm{S}\Phi_N^{-1}(\mathbf{y}_0, \mathbf{u}_0) = \Psi(\mathbf{y}_0, \mathbf{u}_0)$ and $x_0 \in (\pi_1 \circ \Psi)(\mathbf{y}_0, \mathbf{u}_0) = \tilde{g}(\mathbf{y}_0, \mathbf{u}_0)$. But $x_0 \in \mathcal{X} \setminus \mathcal{B}(\tilde{g}(\mathbf{y}_0, \mathbf{u}_0), \epsilon)$, so we have the contradiction with the fact that $x_0 \in A$. ∎

Theorem 2 *Assume that the set $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ is a retract of the space $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$. Let $\varphi : \mathbb{R}^{Np} \times \mathbb{U}_N \to \mathbb{R}^p$ be the continuous function taken from Proposition 2 and $g = \pi_1 \circ \Psi^*$. If the analytic system Σ is defined on the compact subset $\mathcal{X}$ and the function φ is globally Lipschitz then the system*

$$\dot{z} = (\mathrm{A} - \mathrm{K}_\theta \mathrm{C})z + \mathrm{K}_\theta y + \mathrm{b}\varphi\left(z, u, u', \dots, u^{(N-1)}\right) \tag{17}$$

$$\hat{x} = g\left(z, u, u', \dots, u^{(N-2)}\right) \tag{18}$$

is a multiobserver of the system Σ, *where* $z \in \mathbb{R}^{Np}$ *and*

$$\mathrm{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I}_p \end{pmatrix}_{Np\times p}, \ \mathrm{A} = \begin{pmatrix} 0 & \mathrm{I}_p & 0 & \dots & 0 \\ 0 & 0 & \mathrm{I}_p & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{I}_p \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{Np\times Np}, \ \mathrm{C} = \begin{pmatrix} \mathrm{I}_p & 0 \dots 0 \end{pmatrix}_{p\times Np},$$

$$\Delta_\theta = \begin{pmatrix} \theta \mathrm{I}_p & 0 & 0 & \dots & 0 \\ 0 & \theta^2 \mathrm{I}_p & 0 & \dots & 0 \\ 0 & 0 & \theta^3 \mathrm{I}_p & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \theta^N \mathrm{I}_p \end{pmatrix}_{Np\times Np}, \ \theta > 1,\ \mathrm{K}_\theta = \Delta_\theta \mathrm{K}, \text{ the matrix } \mathrm{K}$$

is such that $(\mathrm{A} - \mathrm{KC})$ *is a Hurwitz matrix.*

Proof. From Theorem 1 the following inequality holds

$$\left\| z(t) - \Phi_N\left(x(t), u(t), \dots, u^{(N-2)}(t)\right) \right\| \le \theta^{N-1} \beta \exp\left[-\frac{1}{2}\left(\theta\alpha - 2\omega_{\mathcal{X},B}\right) t \right] \cdot \\ \cdot \left\| z(0) - \Phi_N(x(0), u(0), \dots, u^{(N-2)}(0)) \right\|,$$

for all t such that $x(t) \in \mathcal{X}$ and for all inputs $u(\cdot) \in \mathcal{U}_{N-1,B}$, where α, β are some positive constants, $\omega_{\mathcal{X},B}$ is a constant which depends on $\mathcal{X}$ and B. This inequality means that for every $z(0) \in \mathbb{R}^{Np}$, $x(0) \in \mathcal{X}$ and $u(\cdot) \in \mathcal{U}_{N-1,B}$ the solution of the system (17) approximates the output and its successive derivatives up to order $N-1$ in the exponential way and

$$\lim_{t\to+\infty} \|z(t) - \Phi_N(x(t), u(t), \dots, u^{(N-2)}(t))\| = 0. \tag{19}$$

It remains to show that the distance between the solution $x(t)$ of (1) and the subset $\hat{x}(t)$ goes to zero when $t \to +\infty$, for the fixed $\boldsymbol{u} \in \mathbb{U}_{N-1}$.

From Proposition 4 the multifunction $\tilde{g} := \pi_1 \circ \widehat{\Psi} : \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1}) \subset \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to 2^{\mathcal{X}}$ is upper semicontinuous. Because the set $\mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ is a retract of the space $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$, so there is a retraction $r : \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to \mathrm{S}\Phi_N(\mathcal{X} \times \mathbb{U}_{N-1})$ and the following multifunction $g := \tilde{g} \circ r : \mathbb{R}^{Np} \times \mathbb{U}_{N-1} \to 2^{\mathcal{X}}$ is a continuous extension of $\tilde{g}$ on the whole space $\mathbb{R}^{Np} \times \mathbb{U}_{N-1}$ and $g = \pi_1 \circ \Psi^* = \tilde{g} \circ r$. The function g is a composition of an upper semicontinuous multifunction $\tilde{g}$ and a continuous function r so g is upper semicontinuous. Therefore, for the fixed $\boldsymbol{u} \in \mathbb{U}_{N-1}$ the multifunction $g(\cdot, \boldsymbol{u})$ is upper semicontinuous, too. Hence, $\forall \epsilon > 0 \exists \eta > 0\, \forall z' \left[\|z'(t) - z(t)\| < \eta \Rightarrow g(z', j^{N-2}u(t)) \in \mathcal{B}(g(z(t), j^{N-2}u(t)), \epsilon) \right]$. Taking $z' = \Phi_N(x(t), j^{n-2}u(t))$ and using (19) we obtain

$$g(\mathrm{S}\Phi_N(x(t), j^{n-2}u(t))) \subset \mathcal{B}(g(z(t), j^{N-2}u(t)), \epsilon),$$

for big enough time t. Note that $x(t) \in g(\mathrm{S}\Phi_N(x(t), j^{n-2}u(t)))$. Hence $\forall \epsilon > 0$ $\exists T\ \forall t > T\ \ d(x(t); g(z(t), j^{N-2}u(t))) < \epsilon$, so

$$\lim_{t\to+\infty} d(x(t); g(z(t), j^{N-2}u(t))) = 0. \qquad \blacksquare$$

Remark 5 The dynamical part of the multiobserver is the same as in Bartosiewicz, Wyrwas (2005), Wyrwas (2004) and it approximates the output and its successive derivatives up to order $N-1$. The observability property does not affect this approximation. It influences only the output of a multiobserver. In Bartosiewicz, Wyrwas (2005), Wyrwas (2004) the values of the output map g were finite subsets of $\mathbb{R}^n$. By rejecting local observability we obtain the indistinguishable classes that are no longer discrete subsets of $\mathbb{R}^n$. Hence, the values of the output map g are not necessarily discrete.

Remark 6 If we use an universal control then the values of g are subsets of $\mathbb{R}^n$ that approximate the whole class of states that are indistinguishable from the unknown state of original system Σ. However, if a control that is not universal is applied, then the output of the multiobserver is a set that consists of states that may be indistinguishable or distinguishable. This set may be quite big, but the distance between this set and the unknown state of original system still goes to zero when the time goes to infinity.

Now we show how to use our approach in concrete examples.

Example 1 Let Σ be the system defined on $\mathbb{R}^2$ in the following way:

$$\begin{aligned} \dot{x}_1 &= x_1 \cdot u \\ \dot{x}_2 &= x_2 \cdot u \\ y &= x_1^2. \end{aligned}$$

The function $h(x_1, x_2) = x_1^2$ determines the indistinguishability relation and

$$\mathcal{L}_f h(x_1, x_2) = 2h(x_1, x_2) \cdot u.$$

Then $\varphi_1(z, u) = 2zu$ and all controls are universal.

The map

$$\Phi_1(x_1, x_2) = x_1^2$$

determines the indistinguishability relation. The map Φ_1 is not injective. States (x_1, a) and $(-x_1, b)$ are indistinguishable for all $a, b \in \mathbb{R}$, therefore

$$[(x_1, x_2)] = \{(\pm x_1, a) |\ a \in \mathbb{R}\}.$$

So, the system is not locally observable and the assumptions of Bartosiewicz, Wyrwas (2005), Wyrwas (2004) are not satisfied.

Let $\widetilde{\Phi} : \mathbb{R}^2/_\sim \to \mathbb{R}$, $\widetilde{\Phi}([(x_1, x_2)]) := x_1^2$, where $[(x_1, x_2)] \in \mathbb{R}^2/_\sim$. Then $\widetilde{\Phi}$ is injective and

$$\operatorname{im} \widetilde{\Phi} = \{y \in \mathbb{R} : y \geqslant 0\}.$$

Let $\Psi := \widetilde{\Phi}^{-1} : \text{im}\, \widetilde{\Phi} \to \mathbb{R}^2/_\sim$, then

$$\Psi(y) = [(\sqrt{y}, 0)] = \{(\pm\sqrt{y}, a)|\ a \in \mathbb{R}\}.$$

Ψ is well defined only for $y \geqslant 0$. The set $\Phi_1(\mathbb{R}^2) = [0, +\infty)$ is a retract of $\mathbb{R}$, because there is a retraction $r : \mathbb{R} \to [0, +\infty)$ of the form $r(x) = |x|$, so we can find a continuous extension of Ψ over $\mathbb{R}$. Let $g := \Psi \circ r : \mathbb{R} \to \mathbb{R}/_\sim$. Then

$$g(z) = \left[(\sqrt{|z|}, 0)\right] = \left\{(\pm\sqrt{|z|}, a)|\ a \in \mathbb{R}\right\}.$$

Then, g is a continuous extension of the multifunction Ψ.

The system $\mathcal{M}_{\Sigma,\theta}$, given by

$$\begin{aligned} \dot{z} &= -k\theta z + k\theta y + 2zu \\ \hat{x} &= g(z) \end{aligned}$$

where $k > 0$ and $\theta \in \mathbb{R}_+$ ($\theta \gg 1$), is a multiobserver of the system Σ. Because

$$\lim_{t\to\infty} d(x(t); \hat{x}(t)) = 0,$$

so $\hat{x}(t)$ is an estimation of the class $[x(t)] \in \mathbb{R}^2/_\sim$, that contains all states indistinguishable from $x(t)$. Observe that the values of g are sets that are not discrete, but they are closed.

The constructed multiobservers can be used in the problem of output stabilization.

3.1. An application of multiobservers to output stabilization

Assume that the point $(x_0, u) = (0, 0)$ is a unique equilibrium point of the system Σ, i.e. $f(0, 0) = 0$. Let also $h(0) = 0$ and $x_0 \in \mathcal{X}$.

Let us recall some definitions concerning a stabilization problem.

Definition 4 (Jouan, Gauthier, 1996) We say that the system (1) is *stabilizable* within K if there is a smooth feedback $\alpha_K : \mathcal{X} \to \mathbb{R}$ such that $\alpha_K(0) = 0$ and 0 is an asymptotically stable equilibrium point of $\dot{x} = f(x, \alpha_K(x))$ such that the basin of attraction of 0 contains K.

Definition 5 (Jouan, Gauthier, 1996) We say that the system (1) is *semi-globally stabilizable* if for any compact subset $K \subset \mathcal{X}$ it is stabilizable within K.

Definition 6 We say that the system (1) is *globally stabilizable* if it is stabilizable within $\mathcal{X}$.

Remark 7 If the system (1) is globally stabilizable then it is semi-globally stabilizable.

As the following example shows the property of (semi-)global stabilizability depends on the choice of $\mathcal{X}$.

EXAMPLE 2 Let Σ be the system defined on $\mathcal{X} \subseteq \mathbb{R}^2$ as follows

$$\begin{aligned} \dot{x}_1 &= x_1^2 u \\ \dot{x}_2 &= x_1 x_2 u \\ y &= x_1^2 + x_2^2, \end{aligned}$$

where $u \in U = (0, +\infty)$ and $y \in \mathbb{R}$. Let $a \in \mathbb{R}$ and $a > 0$. Notice that even if $\mathcal{X} = \{(x_1, x_2) : \ |x_1| \leqslant a \ \wedge \ |x_2| \leqslant a\}$ (or $\mathcal{X} = \{(x_1, x_2) : \ x_1^2 + x_2^2 \leqslant a^2\}$) then the considered system is neither globally stabilizable nor semi-globally stabilizable on $\mathcal{X}$. The set $\mathcal{X}$ contains infinitely many equilibrium points of the form $(0, x_2)$, $x_2 \in [-a, a]$ that do not change after applying the feedback. But there are other sets for which the system Σ is globally stabilizable (and then semi-globally stabilizable, too).

Let $\mathcal{X}$ be the set defined as follows

$$\mathcal{X} = \mathbb{R}^2 \setminus \{(x_1, x_2) : \ x_1 = 0 \ \wedge \ x_2 \neq 0\}.$$

The set $\mathcal{X}$ contains only one equilibrium point $(0, 0)$. Then there is feedback $\alpha(x) = -x_1$ such that $x_0 = (0, 0)$ is an asymptotically stable equilibrium point of the following system

$$\begin{aligned} \dot{x}_1 &= x_1^2 \cdot (-x_1) \\ \dot{x}_2 &= x_1 x_2 \cdot (-x_1). \end{aligned}$$

The set $\mathcal{X}$ is invariant with respect to the dynamics of the above system and for every initial condition $x_0 = (x_{10}, x_{20}) \in \mathcal{X}$ we have the following solution

$$x(t) = (x_{10}, x_{20}) \cdot \frac{1}{1 + 2t x_{10}^2},$$

for $t \geq 0$. Because for every initial condition $x_0 \in \mathcal{X}$ we have $\lim\limits_{t \to +\infty} x_i(t) = 0$, for $i = 1, 2$, so the system Σ is globally stabilizable on $\mathcal{X}$ and the basin of attraction of $(0, 0)$ contains $\mathcal{X}$.

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a state space of the system Σ given by (1) and (2). Assume that $x_0 = 0 \in \mathcal{X}$ is a unique equilibrium point of Σ. Based on the results of Jouan and Gauthier (1996) we have the following property for Σ and its extension.

LEMMA 1 *If system Σ is semi-globally stabilizable by some smooth feedback then* *the following system*

$$\Sigma_k : \begin{cases} \dot{x}_1 = f(x,u) \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{k-1} = u_k \\ y = (h(x), u) \end{cases}$$

is semi-globally stabilizable too.

Proof. The proof of this lemma is the first part of the proof of Lemma 5.1 in Jouan and Gauthier (1996) (see also Lemma 2.1 given in Chapter 7 in Gauthier and Kupka, 2001). ■

DEFINITION 7 We say the system (1), (2) is *semi-globally dynamic-output-feedback stabilizable* if there is a positive integer M such that, for every compact $K \subset \mathcal{X} \times \mathbb{R}^M$, there exists a $\mathcal{C}^2$ vector field $\varphi(\cdot, y)$ on $\mathbb{R}^M$, $\mathcal{C}^2$ depending on $y \in \mathbb{R}^p$ and a $\mathcal{C}^2$ mapping $\beta : \mathbb{R}^M \to \mathbb{R}$ such that the equilibrium point of the system

$$\begin{cases} \dot{x} = f(x, \beta(z)), \\ \dot{z} = \varphi(z, h(x)) \end{cases} \tag{20}$$

is asymptotically stable within K.

Jouan and Gauthier (1996) show that the system is semi-globally dynamic output feedback stabilizable if it is globally observable, stabilizable by a state feedback that can be expressed as a smooth function that depends on the output and its successive derivatives up to order $N-1$ and if it satisfies the ACP. Some of these assumptions can be weakened.

Let the system Σ satisfy the following assumptions:

ASSUMPTION 2 *The function φ given in Proposition 2 is of class $\mathcal{C}^2$.*

ASSUMPTION 3 *$\Sigma_0 := \Sigma$ and Σ_i are semi-globally stabilizable by feedbacks $\alpha_0 : \mathcal{X} \to \mathbb{R}$, $\alpha_i : \mathcal{X} \times \mathbb{R}^{N-1} \to \mathbb{R}$ (respectively), $\alpha_i(0) = 0$, for $i = 0, 1, 2, \ldots, N-1$ such that*

$$\alpha_i(x, j^{N-2}u^0) = G_i(h(x), \mathcal{L}_f h(x, u^0), \ldots, \mathcal{L}_f^{N-1} h(x, j^{N-2}u^0)),$$

for all $x \in \mathcal{X}$, $j^{N-2}u^0 \in \mathbb{R}^{N-1}$, where G_i is a smooth function and $N \geqslant 1$. Set $(x, j^{-1}u^0) = x$.

REMARK 8 Jouan and Gauthier (1996) assumed a stronger (than Assumption 2) property of the nonlinear system that was called ACP. This property is equivalent to the fact that it is possible to express locally the N-th derivative of the output as an analytic function that depends on the earlier derivatives of the output, input and its derivatives up to order $N-1$. If we assume that the system satisfies ACP then the function φ taken form Proposition 2 is smooth. Hence ACP implies Assumption 2.

Remark 9 If Assumption 3 holds then

$$\alpha_i(x, j^{N-2}u) = \alpha_i(\tilde{x}, j^{N-2}u),$$

for all $\tilde{x} \in [x]_{\boldsymbol{u}}$ and for $i = 0, 1, 2, \ldots, N-1$.

Additionally

$$\alpha_i(x, j^{N-2}u) = \alpha_i\left(g\left(y, y', \ldots, y^{(N-1)}, j^{N-2}u\right), j^{N-2}u\right),$$

for all $x \in g\left(y, y', \ldots, y^{(N-1)}, j^{N-2}u\right)$ and for $i = 0, 1, 2, \ldots, N-1$.

When we assume that ACP holds, it is enough to take only the feedback α_0 in order to have semi-globally stabilization of Σ_k given in Lemma 1 for $k = 1, 2, \ldots$. If we have the feedback α_0 then the following feedback $u_1 = L_f\alpha_0 - r(u - \alpha_0)$, for some constant $r > 0$, stabilizes Σ_1, where L_f denotes the Lie operator in direction of x and because the ACP holds, so this feedback can be expressed as a smooth function depending on output and its successive derivatives up to order $N-1$. This procedure can be repeated and finally we obtain the feedback u_{N-1} that semi-globally stabilizes the system Σ_{N-1} and u_{N-1} can be expressed as a smooth function that depends on the output and its derivatives up to order $N-1$, the input and its derivatives up to order $N-2$ (see Gauthier and Kupka, 2001; Jouan and Gauthier, 1996).

Theorem 3 *If system Σ satisfies* Assumption 2 *and* Assumption 3, *then it is semi-globally dynamic output feedback stabilizable.*

Proof. We start as in Jouan and Gauthier (1996). Let $K \subset \mathcal{X}$, $K' \subset \mathbb{R}^{Np}$ and $K'' \subset \mathbb{R}^{N-1}$ be three arbitrary compact sets. Take a feedback α_{N-1} given in Lemma 1 that stabilizes the $(N-1)^{\text{st}}$ dynamical extension Σ_{N-1} of Σ within $K \times K''$. Then, the closed loop system

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{N-2} = \alpha_{N-1}(x, u, u_1, \ldots, u_{N-2}) \end{cases} \tag{21}$$

is asymptotically stable at $(x_0, 0\ldots, 0)$ within $K \times K''$. Let $\mathfrak{B}_{N-1}$ be the basin of attraction of $(x_0, 0, \ldots, 0)$ for (21).

Let V be a proper Lyapunov function for (21) on $\mathfrak{B}_{N-1}$. The function V has a maximum m over $K \times K'$. Define $D_k := \{s|\ V(s) \leqslant k\}$, for $k \geqslant 0$. Consider D_m and D_{m+1}. Then, $K \times K'' \subset D_m \subset Int(D_{m+1})$.

From Assumption 2 we have

$$\alpha_{N-1}(x, u, u_1, \ldots, u_{N-2}) = G_{N-1}(\mathrm{S}\Phi_N(x, u, u_1, \ldots, u_{N-2})) \tag{22}$$

for all $(x, u, u_1, \ldots, u_{N-2}) \in D_{m+1}$. Moreover, we may assume that the support of G_{N-1} is compact. Hence G_{N-1} reaches its maximum over $\mathbb{R}^{Np} \times \mathbb{R}^{N-1}$.

Define $\tilde{K} := \pi_1(D_{m+1})$, where $\pi_1 : \mathcal{X} \times \mathbb{R}^{N-1} \to \mathcal{X}$ is the projection. Then $\tilde{K}$ is compact.

From Assumption 1 we have the function φ of class $\mathcal{C}^2$ such that

$$\mathcal{L}_f^N h(x, u, u_1, \ldots, u_{N-1}) = \varphi(h(x), \ldots, \mathcal{L}_f^{N-1} h(x, u, \ldots, u_{N-2}), u, \ldots, u_{N-1}),$$

for all $x \in \tilde{K}$ and all $u, u_1, \ldots, u_{N-1}$.

Consider the system Σ_θ defined on $\mathcal{X} \times \mathbb{R}^{N-1} \times \mathbb{R}^{Np}$ of the following form

$$\begin{cases} \dot{x} = f(x, u) \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{N-2} = G_{N-1}(z, u, u_1, \ldots, u_{N-2}) \\ \dot{z} = (\mathrm{A} - \mathrm{K}_\theta \mathrm{C})z + \mathrm{K}_\theta h(x) + \mathrm{b}\varphi\left(z, u, \ldots, u_{N-2}, G_{N-1}(z, u, \ldots, u_{N-2})\right), \end{cases} \tag{23}$$

where matrices A, b, C, K_θ are the same as in Theorem 2.

It is sufficient to show that the basin of attraction of $(x_0, 0, \ldots, 0, 0)$ for Σ_θ contains $K \times K'' \times K'$.

The proof of this fact is similar to the proof of Theorem 5.1 of Jouan and Gauthier (1996) (see also Theorem 2.3 in Chapter 7 of Gauthier and Kupka, 2001) and it consists of three steps like in Jouan and Gauthier (1996):

Step 1. First we show that there is a stable center manifold for Σ_θ at 0 and the linearized system of Σ_θ has no unstable eigenvalue. From the center manifold theorem, Σ_θ is locally asymptotically stable at $(x_0, 0, \ldots, 0, 0)$. Observe that the center manifold theorem holds for φ of class $\mathcal{C}^2$ (see Carr, 1981). The $\mathcal{C}^\infty$ property used in Jouan and Gauthier (1996) is not necessary.

Step 2. In the second step it is shown that all semitrajectories that remain in $D_{m+1} \times \mathbb{R}^{Np}$ are contained in the basin of attraction of $(x_0, 0, \ldots, 0, 0)$.

Step 3. In the last step it is proved that if θ is large enough then all semitrajectories starting from $K \times K'' \times K'$ stay in $D_{m+1} \times \mathbb{R}^{Np}$. ■

REMARK 10 If the solution of the dynamical part of the multiobserver belongs to the image of the map Φ_N, i.e. $z(t) \in \mathrm{Im}\Phi_N$, then the output of multiobserver can be used. Namely, instead of considering the system (23) we can study the following system

$$\begin{cases} \dot{x} = f(x,u) \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{N-2} = \alpha_{N-1}(\hat{x}, u, u_1, \ldots, u_{N-2}) \\ \dot{z} = (\mathrm{A} - \mathrm{K}_\theta \mathrm{C})z + \mathrm{K}_\theta h(x) + \mathrm{b}\varphi\left(z, u, \ldots, u_{N-2}, \alpha_{N-1}(\hat{x}, u, \ldots, u_{N-2})\right) \\ \hat{x} = g(z, u, u_1, \ldots, u_{N-2}). \end{cases}$$

The following example shows how to use a dynamical part of a multiobserver to stabilize the system by dynamic output feedback.

EXAMPLE 3 Let us consider the system Σ defined on $\mathcal{X} \subset \mathbb{R}^2$ in Example 1. The state $x_0 = (0,0) \in \mathcal{X}$ is a unique equilibrium point. As it was shown in Example 1 it is possible to construct a multiobserver and its dynamical part is as follows:

$$\dot{z} = -\theta z + \theta y + 2zu. \tag{24}$$

The solution of (24) approximates the output of the system Σ. We use the equation (24) to show that the considered system is semi-globally, dynamically output-feedback stabilizable. There is a feedback $\alpha : \mathcal{X} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\alpha(x_1, x_2) = G(x_1^2) = -x_1^2$ (where $G(z) = -z$ is a smooth map) such that $x = (0,0)$ is an asymptotically stable equilibrium point of the system

$$\begin{cases} \dot{x}_1 = x_1 \cdot \left(-x_1^2\right) \\ \dot{x}_2 = x_2 \cdot \left(-x_1^2\right) \end{cases} \tag{25}$$

and the basin of attraction of $(0,0)$ contains $\mathcal{X}$ (because $\lim\limits_{t\to+\infty} x(t,(x_{10},x_{20})) = \lim\limits_{t\to+\infty} (x_{10},x_{20})\frac{1}{1+2x_{10}^2 t} = 0$, for every initial condition $(x_{10},x_{20}) \in \mathbb{R}^2$).

Let $\varphi(z,y,u) = -\theta z + \theta y + 2zu$ and $\beta(z) = -z$. Then the system Σ_θ

$$\begin{cases} \dot{x}_1 = -x_1 z \\ \dot{x}_2 = -x_2 z \\ \dot{z} = -\theta z + \theta x_1^2 - 2z^2 \end{cases} \tag{26}$$

has the equilibrium point $(0,0,0)$.

The following set

$$\mathfrak{M} := \{(x_1, x_2, z) |\; z = x_1^2\} \subset \mathcal{X} \times \mathbb{R}$$

is the center manifold for (26). On the center manifold we have the system (25) for which $x_0 = (0,0)$ is an asymptotically stable equilibrium point. Then

$(x_0, z_0) = (0, 0, 0)$ is an asymptotically stable equilibrium point of the system Σ_θ, too. From Theorem 3 the equilibrium point $(0, 0, 0)$ of the system (26) is asymptotically stable within every compact subset $K \subset \mathcal{X} \times \mathbb{R}$ and the basin of attraction of $(0, 0)$ contains K. Hence, the system Σ is semi-globally dynamic output feedback stabilizable.

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