Control and Cybernetics

vol. **35** (2006) No. 4

Hybrid robust stabilization in the Martinet case

by

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Abstract: In a previous work, Prieur, Trélat (2006), we derived a result of semi-global minimal time robust stabilization for analytic control systems with controls entering linearly, by means of a hybrid state feedback law, under the main assumption of the absence of minimal time singular trajectories. In this paper, we investigate the Martinet case, which is a model case in $\mathbb{R}^3$, where singular minimizers appear, and show that such a stabilization result still holds. Namely, we prove that the solutions of the closed-loop system converge to the origin in quasi minimal time (for a given bound on the controller) with a robustness property with respect to small measurement noise, external disturbances and actuator errors.

Keywords: Martinet case, hybrid feedback, robust stabilization, measurement errors, actuator noise, external disturbances, optimal control, singular trajectory, sub-Riemannian geometry.

1. Introduction

Consider the so-called *Martinet system* in $\mathbb{R}^3$

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \tag{1}$$

where, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial}{\partial x_3},\ f_2 = \frac{\partial}{\partial x_2}, \tag{2}$$

and the control function $u = (u_1, u_2)$ satisfies the constraint

$$u_1^2 + u_2^2 \leqslant 1. \tag{3}$$

System (1), together with the constraint (3), is said to be *globally asymptotically stabilizable* at the origin, if there exists a control law $x \mapsto u(x)$, satisfying the constraint (3), such that every solution of (1), associated to this control law, tends to 0 as t tends to $+\infty$ (and if a stability property holds, see Definition 5 below for a precise statement).

According to Brockett's condition (see Brockett, 1983, Theorem 1, (iii)), there does not exist any continuous stabilizing feedback law $x \mapsto u(x)$ for (1). The *robust asymptotic stabilization problem* is an active research topic. Many notions of controllers exist to handle this problem, such as discontinuous sampling feedbacks (Clarke et al., 2000; Sontag, 1999), time varying control laws (Closkey, Murray, 1997; Coron, 1992; Morin, Samson, 2003), patchy feedbacks (Ancona, Bressan, 2002), SRS feedbacks (Rifford, 2004), enjoying different robustness properties depending on the errors under consideration. We consider here feedback laws having both discrete and continuous components, which generate closed-loop systems with *hybrid* terms (see Bensoussan, Menaldi, 1997; Prieur, Trélat, 2005c; Tavernini, 1987). Such feedbacks appeared first in Prieur (2001), to stabilize nonlinear systems having a priori no discrete state. Many results on the stabilization problem of nonlinear systems by means of hybrid controllers have been recently established (see, for instance, Branicky, 1998; Goebel, Teel, 2006; Goebel et all., 2004; Liberzon, 2003; Lygeros et all., 2003; Ye, Michel, Hou, 1998). The aim is to define a *switching strategy* between several smooth control laws defined on a partition of the state space. The notion of solution, connected with the robustness problem, is by now well defined in the hybrid context (see, for instance, Goebel, Teel, 2006; Prieur, Astolfi, 2003; Prieur, Trélat, 2006).

In Prieur, Trélat (2006), we proved the following general result. Let m and n be two positive integers. Consider on $\mathbb{R}^n$ the control system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{4}$$

where $f_1, \dots, f_m$ are analytic vector fields in $\mathbb{R}^n$, and where the control function $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ satisfies the constraint

$$\sum_{i=1}^{m} u_i(t)^2 \leqslant 1. \tag{5}$$

Let $\bar{x} \in \mathbb{R}^n$. System (4), together with the constraint (5), is said to be *globally asymptotically stabilizable* at the point $\bar{x}$, if, for each point $x \in \mathbb{R}^n$, there exists a control law satisfying the constraint (5) such that the solution of (4) associated to this control law and starting from x tends to $\bar{x}$ as t tends to $+\infty$. Consider the minimal time problem for system (4) with the constraint (5), of steering a point $x \in \mathbb{R}^n$ to the point $\bar{x}$. Note that this problem is solvable when the *Lie Algebra Rank Condition* holds for the m-tuple of vector fields $(f_1, \dots, f_m)$. In general, it is impossible to compute explicitly the minimal time feedback controllers for this problem. Moreover, Brockett's condition implies that such control laws are not smooth whenever $m < n$ and the vector fields $f_1, \ldots, f_m$ are independent. This leads to investigation of the regularity of optimal feedback laws. In an analytic setting, the problem of determining the analytic regularity of the value function for a given optimal control problem, has been investigated in Sussmann (1979), in particular. It is related to the existence of singular minimizing trajectories (see Agrachev, 1998; Agrachev, Gauthier, 2001; Trélat, 2000). More precisely, if there does not exist any nontrivial singular minimizing trajectory starting from $\bar{x}$, then the minimal time function to $\bar{x}$ is subanalytic outside $\bar{x}$ (see Hardt, 1975; Hironaka, 1973, for a definition of subanalyticity). In particular, this function is analytic outside a stratified submanifold $\mathcal{S}$ of $\mathbb{R}^n$, of codimension greater than or equal to 1 (see Tamm, 1981). As a consequence, outside this submanifold, it is possible to provide an analytic minimal time feedback controller for system (4) with the constraint (5). Note that the analytic context is used so as to ensure stratification properties, which do not hold a priori if the system is smooth only. These properties are related to the notion of *o-minimal category* (see van den Dries, Miller, 1996). Then, in a second step, in order to achieve a minimal time robust stabilization procedure, using a hybrid feedback law, a suitable switching strategy (more precisely, a hysteresis) is defined between this minimal time feedback controller and other controllers defined on a neighborhood of $\mathcal{S}$. The main result of Prieur, Trélat (2006) then asserts that, in these conditions, the point $\bar{x}$ is semi-globally robustly asymptotically stabilizable, with a minimal time property.

The strategy is to combine a minimal time controller that is smooth on a part of the state space, and other controllers defined on the complement of this part, so as to provide a quasi minimal time hybrid controller by defining a switching strategy between all control laws. The resulting hybrid law enjoys a quasi minimal time property, and robustness with respect to (small) measurement noise, actuator errors and external disturbances.

In the present paper, we investigate the Martinet system (1), (3), for which there exist singular minimizing trajectories, and thus Prieur, Trélat (2006) can not be applied. However, the previous procedure can be used, for two main reasons. First, the minimal time function can be proved to belong to the *log-exp* class (see van den Dries, 1994), which is a o-minimal extension of the subanalytic class, and thus, its singular set $\mathcal{S}$ is a stratified submanifold of codimension greater than or equal to one. This stratification property allows to define a switching strategy near the manifold $\mathcal{S}$. Second, the set of extremities of singular trajectories is small in $\mathcal{S}$, and invariance properties for the optimal flow thus still hold in $\mathbb{R}^3 \setminus \mathcal{S}$. This fact is, however, far from being general.

The paper is organized as follows. In Section 2, we first recall some facts about the minimal time problem for system (1) with (3), and recall the definition of a singular trajectory. Then, we recall a notion of solution adapted to hybrid feedback laws, and the concept of stabilization via a minimal time hybrid feedback law. The main result, Theorem 1, states that the origin is a semi-globally minimal time robustly stabilizable equilibrium for the system (1), (3). The remainder of the paper is devoted to the proof of this result. In Section 3, we gather some known results for the minimal time problem in the Martinet case, and in particular, explain that the minimal time function belongs to the *log-exp class*. A definition of a log-exp function is also provided, as well as some crucial properties of *o-minimal* classes. In Section 4, we define the components of the hysteresis. The first component consists of the minimal time feedback controller, defined on the whole $\mathbb{R}^3$, except on a stratified submanifold. We then make precise the second component of this hysteresis, using Lie brackets of the vector fields f_1 and f_2. Finally, an hybrid feedback law is defined, using a hysteresis to connect both components. The main result is proved in Section 4.4. Section 5 is devoted to a conclusion and further comments.

2. Definitions and main result

2.1. The minimal time problem

Consider the minimal time problem for system (1) with the constraint (3). Since the *Lie Algebra Rank Condition* holds for the pair (f_1, f_2), any two points of $\mathbb{R}^3$ can be joined by a minimal time trajectory of (1), (3). Denote by $T(x)$ the minimal time needed to steer system (1) with the constraint (3) from a point $x \in \mathbb{R}^3$ to the origin 0 of $\mathbb{R}^3$.

Note that, obviously, the control function associated to a minimal time trajectory of (1), (3), actually satisfies $u_1^2 + u_2^2 = 1$.

For $T > 0$, let $\mathcal{U}_T$ denote the (open) subset of $u(\cdot)$ in $L^\infty([0,T], \mathbb{R}^2)$ such that the solution of (1), starting from 0 and associated to a control $u(\cdot) \in \mathcal{U}_T$, is well defined on $[0,T]$. The mapping

$$E_T : \begin{array}{ccc} \mathcal{U}_T & \longrightarrow & \mathbb{R}^3 \\ u(\cdot) & \longmapsto & x(T), \end{array}$$

which to a control $u(\cdot)$ associates the end-point $x(T)$ of the corresponding solution $x(\cdot)$ of (1) starting at 0, is called *end-point mapping* at time T; it is a smooth mapping.

Definition 1 *A trajectory $x(\cdot)$ of (1), with $x(0) = 0$, is said singular on $[0,T]$ if its associated control $u(\cdot)$ is a singular point of the end-point mapping E_T (i.e., if the Fréchet derivative of E_T at $u(\cdot)$ is not onto). In that case, the control $u(\cdot)$ is said to be singular.*

2.2. Class of controllers and notion of hybrid solution

In this section, we recall the general setting for hybrid systems.

Let $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, u) = u_1 f_1(x) + u_2 f_2(x)$. System (4) writes

$$\dot{x}(t) = f(x(t), u(t)). \tag{6}$$

The controllers under consideration in this paper depend on the continuous state $x \in \mathbb{R}^3$ and also on a discrete variable $s_d \in \mathcal{N}$, where $\mathcal{N}$ is a nonempty countable set. According to the concept of a hybrid system of Goebel, Teel, (2006), we introduce the following definition (see also Prieur, Trélat, 2006).

Definition 2 *A hybrid feedback is a 4-tuple (C, D, k, k_d), where*
- *C and D are subsets of $\mathbb{R}^3 \times \mathcal{N}$;*
- *$k : \mathbb{R}^3 \times \mathcal{N} \to \mathbb{R}^2$ is a function;*
- *$k_d : \mathbb{R}^3 \times \mathcal{N} \to \mathcal{N}$ is a function.*

The sets C and D are, respectively, called the controlled continuous evolution set *and the* controlled discrete evolution set.

We next recall the notion of robustness to small noise (see Sontag, 1999). Consider two functions e and d satisfying the following *regularity assumptions*:

$$\begin{aligned} &e(\cdot,\cdot), d(\cdot,\cdot) \in L^\infty_{loc}(\mathbb{R}^3 \times [0,+\infty); \mathbb{R}^3),\\ &e(\cdot,t), d(\cdot,t) \in C^0(\mathbb{R}^3, \mathbb{R}^3),\ \forall t \in [0,+\infty). \end{aligned} \tag{7}$$

We introduce these functions as a measurement noise e and an external disturbance d. Below, we define the perturbed hybrid system $\mathcal{H}_{(e,d)}$. The notion of solution of such hybrid perturbed systems has been well studied in the literature (see, e.g., Bensoussan, Menaldi, 1997; Branicky, 1998; Litsyn, Nepomnyashchikh, Ponosov, 2000; Prieur, 2005; Prieur, Astolfi, 2003; Tavernini, 1987). Here, we consider the notion of solution given in Goebel, Teel (2006), Goebel et al. (2004).

Definition 3 *Let $S = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$, where $J \in \mathbb{N} \cup \{+\infty\}$ and $(x_0, s_0) \in \mathbb{R}^3 \times \mathcal{N}$. The domain S is said to be a hybrid time domain. A map $(x, s_d) : S \to \mathbb{R}^3 \times \mathcal{N}$ is said to be a solution of $\mathcal{H}_{(e,d)}$ with the initial condition (x_0, s_0) if*
- *the map x is continuous on S;*
- *for every j, $0 \leqslant j \leqslant J-1$, the map $x : t \in (t_j, t_{j+1}) \mapsto x(t, j)$ is absolutely continuous;*
- *for every j, $0 \leqslant j \leqslant J-1$ and almost every $t \geqslant 0$, $(t, j) \in S$, we have*

$$(x(t,j) + e(x(t,j),t), s_d(t,j)) \in C, \tag{8}$$

and

$$\dot{x}(t,j) = f(x(t), k(x(t,j) + e(x(t,j),t), s_d(t,j))) + d(x(t,j),t), \tag{9}$$

$$\dot{s}_d(t,j) = 0; \tag{10}$$

(where the dot stands for the derivative with respect to the time variable t)

- *for every* $(t,j) \in S$*,* $(t,j+1) \in S$*, we have*

$$(x(t,j) + e(x(t,j),t), s_d(t,j)) \in D, \tag{11}$$

and

$$x(t,j+1) = x(t,j), \tag{12}$$
$$s_d(t,j+1) = k_d(x(t,j) + e(x(t,j),t), s_d(t,j)); \tag{13}$$

- $(x(0,0), s_d(0,0)) = (x_0, s_0)$.

In this context, we next recall the concept of stabilization of (6) by a minimal time hybrid feedback law sharing a robustness property with respect to measurement noise and external disturbances (see Prieur, Trélat, 2005a). The usual Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$, and the open ball centered at 0 with radius R is denoted $B(0,R)$. Recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0,+\infty) \to [0,+\infty)$ which is continuous, increasing, satisfying $\delta(0)=0$ and $\lim_{R\to+\infty} \delta(R) = +\infty$.

Definition 4 *Let* $\rho : \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function satisfying*

$$\rho(x) > 0, \ \forall x \neq 0. \tag{14}$$

We say that the completeness assumption for ρ *holds if, for all* (e,d) *satisfying the regularity assumptions (7), and such that,*

$$\sup_{[0,+\infty)} |e(x,\cdot)| \leqslant \rho(x), \ \operatorname{esssup}_{[0,+\infty)} |d(x,\cdot)| \leqslant \rho(x), \ \forall x \in \mathbb{R}^3, \tag{15}$$

for every $(x_0, s_0) \in \mathbb{R}^3 \times \mathcal{N}$*, there exists a maximal solution on* $[0,+\infty)$ *of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0)*.*

Definition 5 *We say that the* uniform finite time convergence property *holds if there exists a continuous function* $\rho : \mathbb{R}^3 \to \mathbb{R}$ *satisfying (14), such that the completeness assumption for* ρ *holds, and if there exists a function* $\delta : [0,+\infty) \to [0,+\infty)$ *of class* $\mathcal{K}_\infty$ *such that, for every* $R > 0$*, there exists* $\tau = \tau(R) > 0$*, for all functions* e, d *satisfying the regularity assumptions (7) and inequalities (15) for this function* ρ*, for every* $x_0 \in B(0,R)$*, and every* $s_0 \in \mathcal{N}$*, the maximal solution* (x, s_d) *of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0) *satisfies*

$$|x(t,j)| \leqslant \delta(R), \ \forall t \geqslant 0, \ (t,j) \in S, \tag{16}$$

and

$$x(t,j) = 0, \ \forall t \geqslant \tau, \ (t,j) \in S. \tag{17}$$

Definition 6 *The point* 0 *is said to be a* semi-globally minimal time robustly stabilizable equilibrium *for system (6) if, for every* $\varepsilon > 0$ *and every compact* *subset* $K \subset \mathbb{R}^3$*, there exists a hybrid feedback law* (C, D, k, k_d) *satisfying the constraint*

$$\|k(x, s_d)\| \leqslant 1, \tag{18}$$

where $\|\cdot\|$ *stands for the Euclidean norm in* $\mathbb{R}^2$*, such that:*

- *the uniform finite time convergence property holds;*
- *there exists a continuous function* $\rho_{\varepsilon,K} : \mathbb{R}^3 \to \mathbb{R}$ *satisfying (14) for* $\rho = \rho_{\varepsilon,K}$*, such that, for every* $(x_0, s_0) \in K \times \mathcal{N}$*, all functions* e, d *satisfying the regularity assumptions (7) and inequalities (15) for* $\rho = \rho_{\varepsilon,K}$*, the maximal solution of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0) *reaches* 0 *within time* $T(x_0) + \varepsilon$*, where* $T(x_0)$ *denotes the minimal time to steer system (6) from* x_0 *to* 0*, under the constraint* $\|u\| \leqslant 1$*.*

2.3. Main result

Theorem 1 *The origin is a semi-globally minimal time robustly stabilizable equilibrium for system (1) with the constraint (3).*

The strategy is the following. The minimal time function $T(x)$ to steer system (1) from x to 0, under the constraint (3), belongs to the *log-exp* class, and thus, is stratifiable (see next section). Hence, the corresponding minimal time feedback controller is continuous (even analytic) on $\mathbb{R}^3 \setminus \mathcal{S}$, where $\mathcal{S}$ is the set of points of $\mathbb{R}^3$ at which T is not analytic. Since T is log-exp, $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^3$, of codimension greater than or equal to one. In a neighborhood of $\mathcal{S}$, it is therefore necessary to use other controllers, and to define an adequate switching strategy. Notice that this neighborhood can be chosen arbitrarily thin, and thus, the time ε needed for its traversing is arbitrarily small. Therefore, starting from an initial point x_0, the time needed to join 0, using this hybrid strategy, is equal to $T(x_0) + \varepsilon$.

3. The minimal time problem in the Martinet case

In this section, we briefly recall some known results for the minimal time problem in the Martinet case, gathered from Agrachev et al., 1997; Bonnard, Chyba, 2003; Bonnard, Launay, Trélat, 2001; Bonnard, Trélat, 2001; Trélat, 2000.

3.1. Parametrization of extremals

It follows from the Pontryagin Maximum Principle (see Pontryagin et al., 1962) that every minimal time trajectory of (1), (3), starting from 0, is the projection of an *extremal*, that is, a 4-tuple $(x(\cdot), p(\cdot), p^0, u(\cdot))$ solution of the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, p^0, u), \ \dot{p} = -\frac{\partial H}{\partial x}(x, p, p^0, u), \ \frac{\partial H}{\partial u}(x, p, p^0, u) = 0,$$

where $H(x,p,p^0,u) = \langle p, u_1 f_1(x) + u_2 f_2(x)\rangle + p^0(u_1^2 + u_2^2)$ is the Hamiltonian function, $p(\cdot)$ is an absolutely continuous function called *adjoint vector*, and p^0 is a nonpositive constant. If $p^0 \neq 0$, the extremal is said to be *normal*, and we normalize to $p^0 = -\frac{1}{2}$. Otherwise it is said *abnormal*. Note that every abnormal extremal projects onto a singular trajectory (and conversely).

In the Martinet case, the abnormal extremals (that is, $p^0 = 0$) correspond to $u_1 = \pm 1$, $u_2 = 0$. Their projections are singular trajectories, solutions of the vector field $\frac{\partial}{\partial x_1}$, contained in the plane $x_2 = 0$. There exists a unique singular direction passing through 0, given by $x_1(t) = \pm t$, $x_2(t) = x_3(t) = 0$. These singular trajectories are indeed minimal time (see Agrachev et al., 1997).

Normal extremals are computed with $p^0 = -1/2$. They are parametrized in Agrachev et al. (1997) using *elliptic functions* (see also Bonnard, Chyba, 2003, for details).

3.2. The log-exp class

For the sake of completeness, we recall the definition of a subanalytic function (see Hardt, 1975; Hironaka, 1973), then the one of a log-exp function (see van den Dries, Macintyre, Marker, 1994), and some properties that are used in a crucial way in the present paper.

Let M be a real analytic finite dimensional manifold. A subset A of M is said to be *semi-analytic* if and only if, for every $x \in M$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} ($1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$), such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0,\ j = 1 \dots q\}.$$

Let SEM(M) denote the set of semi-analytic subsets of M. The image of a semi-analytic subset by a proper analytic mapping is not in general semi-analytic, and thus this class has to be enlarged.

A subset A of M is said to be *subanalytic* if and only if, for every $x \in M$, there exists a neighborhood U of x in M and $2p$ couples $(\Phi_i^\delta, A_i^\delta)$ ($1 \leqslant i \leqslant p$ and $\delta = 1, 2$), where $A_i^\delta \in SEM(M_i^\delta)$, and where the mappings $\Phi_i^\delta : M_i^\delta \to M$ are proper analytic, for real analytic manifolds M_i^δ, such that

$$A \cap U = \bigcup_{i=1}^{p} (\Phi_i^1(A_i^1) \backslash \Phi_i^2(A_i^2)).$$

Let SUB(M) denote the set of subanalytic subsets of M.

The subanalytic class is closed by union, intersection, complementary, inverse image by an analytic mapping, image by a proper analytic mapping. In brief, the subanalytic class is *o-minimal* (see van den Dries, Miller, 1996). Moreover, subanalytic sets are *stratifiable* in the following sense. A *stratum* of a differentiable manifold M is a locally closed sub-manifold of M. A locally finite partition $\mathcal{S}$ of M is a *stratification* of M if any $S \in \mathcal{S}$ is a stratum such that

$$\forall T \in \mathcal{S} \quad T \cap \partial S \neq \emptyset \Rightarrow T \subset \partial S \text{ and } \dim T < \dim S.$$

Finally, a mapping between two analytic manifolds M and N is said to be *subanalytic* if its graph is a subanalytic subset of $M \times N$.

The *log-exp class*, defined in van den Dries, Macintyre, Marker (1994), is an extension of the subanalytic class with functions log and exp, sharing the same properties with the one of subanalytic sets (namely, it is an o-minimal class). More precisely, a log-exp function is defined by a finite composition of subanalytic functions, of exponentials and logarithms; if $g_1, \ldots, g_m$, are log-exp functions in $\mathbb{R}^n$, and if F is a log-exp function in $\mathbb{R}^m$, then the composition $F \circ (g_1, \ldots, g_m)$ is a log-exp function in $\mathbb{R}^n$. A log-exp set is defined by a finite number of equalities and inequalities using log-exp functions.

Let M be an analytic manifold, and F be a log-exp function on M. The *analytic singular support* of F is defined as the complement of the set of points x in M such that the restriction of F to some neighborhood of x is analytic. The following property is of great interest in the present paper.

PROPOSITION 1 *(van den Dries, Macintyre, Marker, 1994; Tamm, 1981) The analytic singular support of F is log-exp (and thus, in particular, is stratifiable). If F is, moreover, locally bounded on M, then it is also of codimension greater than or equal to one.*

3.3. Regularity of the minimal time function in the Martinet case

The following crucial result has been proved in Bonnard, Launay, Trélat (2001) (see also Agrachev et al., 1997; Bonnard, Trélat, 2001; Trélat, 2000a, and the textbook Bonnard, Chyba, 2003, which contains a survey on the Martinet case).

PROPOSITION 2 *The minimal time function T to 0 of system (1), (3), belongs to the log-exp class.*

It has been proved in Agrachev et al. (1997) that T is not subanalytic, by analyzing the extremal flow given by the Pontryagin Maximum Principle, and using a parameterization with elliptic functions. Note that this loss of subanalyticity is due to the presence of the singular minimizing direction $(0x_1)$. A careful analysis then allows to show that the minimal time function is expressed as an analytic function of some specific monomials, themselves being particular log-exp functions (see Bonnard, Launay, Trélat, 2001, for a general result, and an algorithm of computation in the log-exp class).

4. Components of the hysteresis, and hybrid strategy

It follows from Proposition 2 that the minimal function $T(\cdot)$ is log-exp, and hence, from Proposition 1, its singular set $\mathcal{S} = \text{Sing}\, T(\cdot)$ (i.e., the analytic singular support of $T(\cdot)$) is a stratified submanifold of $\mathbb{R}^3$, of codimension greater than or equal to one. The objective is to construct neighborhoods of $\mathcal{S}$ in $\mathbb{R}^3$ whose complements share invariance properties for the optimal flow.

4.1. The optimal controller

In this section, we define the *optimal controller*, and give some properties of the Carathéodory solutions of (1) with this feedback law.

Outside the set $\mathcal{S}$, the function T is smooth. It follows from the Pontryagin maximum principle and the Hamilton-Jacobi theory (see Pontryagin et al., 1962) that the minimal time control functions, steering a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \mathcal{S}$ to the origin, are given by the closed-loop formula

$$\begin{aligned} u_1(x) &= -\frac{1}{2}\langle \nabla T(x), f_1(x)\rangle = -\frac{1}{2}\left(\frac{\partial T}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial T}{\partial x_3}\right), \\ u_2(x) &= -\frac{1}{2}\langle \nabla T(x), f_2(x)\rangle = -\frac{1}{2}\frac{\partial T}{\partial x_2}. \end{aligned} \tag{19}$$

The set $\mathcal{S}$ actually consists of the union of the axis $(0x_1)$ (i.e., the singular direction) and of the cut locus. Recall that, by definition, a point $x \in \mathbb{R}^3$ is not a *cut point* with respect to 0 if there exists a minimizing trajectory joining 0 to x, which is the strict restriction of a minimizing trajectory starting from 0. The *cut locus* of 0, denoted by $\mathcal{L}(0)$, is defined as the set of all cut points with respect to 0. Note that a general result of Jacquet (2001) implies that the minimal time function is analytic at the point x, provided that x is not joined from 0 by a singular minimizing trajectory, and that there exists only one minimizing trajectory steering 0 to x.

Moreover, it follows from the computations of Agrachev et al. (1997) that the plane $x_2 = 0$ is contained in $\mathcal{S}$ (see also Bonnard, Trélat, 2001), and that the axis $(0x_1)$ is contained in the adherence of the cut locus $\mathcal{L}(0)$.

Outside $\mathcal{S}$, the smoothness of this optimal controller ensures a robustness property of the stability. A switching strategy is then necessary between this optimal controller, denoted u_{opt}, and other controllers defined in a neighborhood of $\mathcal{S}$ (see Section 4.2). The switching strategy is achieved by adding a dynamical discrete variable s_d and using a hybrid feedback law (see Section 4.3).

LEMMA 1 *For every neighborhood Ω of $\mathcal{S}$ in $\mathbb{R}^3$, there exists a neighborhood Ω' of $\mathcal{S}$, satisfying*

$$\Omega' \subsetneq \operatorname{clos}(\Omega') \subsetneq \Omega, \tag{20}$$

such that every trajectory of the closed-loop system (4) with the optimal controller, joining a point $x \in \mathbb{R}^3 \setminus \Omega$ to 0, is contained in $\mathbb{R}^3 \setminus \Omega'$.

The proof of this lemma follows Prieur, Trélat (2006, Lemma 4.2). However, in this latter reference, one has $\mathcal{S} = \mathcal{L}(0)$. Here, $\mathcal{S}$ is the union of $\mathcal{L}(0)$ and of the axis $(0x_1)$.

Proof. It suffices to prove that, for every compact subset K of $\mathbb{R}^3$, for every neighborhood Ω of $\mathcal{S} \setminus \{0\}$ in $\mathbb{R}^3$, there exists a neighborhood Ω' of $\mathcal{S} \setminus \{0\}$ in $\mathbb{R}^3$, satisfying (20), such that every trajectory of the closed-loop system (4) with the optimal controller, joining a point $x \in (\mathbb{R}^3 \setminus \Omega) \cap \mathrm{K}$ to 0, is contained in $\mathbb{R}^3 \setminus \Omega'$.

Let $x \in (\mathbb{R}^3 \setminus \Omega) \cap \mathrm{K}$. By definition of the cut locus $\mathcal{L}(0)$, every optimal trajectory joining x to 0 does not intersect $\mathcal{L}(0)$. Moreover, it does not intersect the axis $(0x_1)$ too; indeed, it follows from Agrachev et al. (1997) that the unique optimal trajectory joining 0 to a point $(a, 0, 0)$ of the axis $(0x_1)$ is necessarily associated to the control $u_1 = 1, u_2 = 0$ if $a > 0$ (resp., $u_1 = -1, u_2 = 0$ if $a < 0$), thus, is singular, and contained in the axis $(0x_1)$. Finally, every optimal trajectory joining x to 0 does not intersect $\mathcal{S}$, and thus has a positive distance to the stratified manifold $\mathcal{S}$.

Since there does not exist any nontrivial singular minimizing trajectory starting from 0 and joining a point of $(\mathbb{R}^3 \setminus \Omega) \cap \mathrm{K}$, it follows that the optimal flow joining points of the compact set $(\mathbb{R}^3 \setminus \Omega) \cap \mathrm{K}$ to 0 is parameterized by a compact set (for the details of this general reasoning, we refer the reader to Agrachev, 1998; Trélat, 2000a,b, see also Prieur, Trélat, 2006, where it is used in a crucial way). Hence, there exists a positive real number $\delta > 0$ so that every optimal trajectory joining a point $x \in (\mathbb{R}^3 \setminus \Omega) \cap \mathrm{K}$ to 0 has a distance to the set $\mathcal{S}$ which is greater than or equal to δ. The existence of Ω' follows. ■

Now, as this optimal controller has been defined, we investigate the robustness properties of system (1) in closed-loop with this controller. Given $e, d : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$, the perturbed closed-loop system under consideration in this section has the form

$$\dot{x}(t) = f(x(t), u_{opt}(x(t) + e(x(t), t))) + d(x(t), t). \tag{21}$$

Below, a robust version of Lemma 1 is stated for every noise vanishing along the discontinuous set of the optimal controller.

Lemma 2 *There exist a continuous function $\rho_{opt} : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\rho_{opt}(\xi) > 0, \ \forall \xi \neq 0, \tag{22}$$

and a continuous function $\delta_{opt} : [0, +\infty) \to [0, +\infty)$ of class $\mathcal{K}_\infty$ such that the following three properties hold:

- Robust Stability
  For every neighborhood Ω of $\mathcal{S}$, there exists a neighborhood $\Omega' \subset \Omega$ of $\mathcal{S}$, such that, for all $e, d : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ satisfying the regularity assumptions (7) and, for every $x \in \mathbb{R}^3$,

$$\sup_{[0,+\infty)} |e(x, \cdot)| \leqslant \rho_{opt}(d(x, \mathcal{S})), \ \operatorname{esssup}_{[0,+\infty)} |d(x, \cdot)| \leqslant \rho_{opt}(d(x, \mathcal{S})), \tag{23}$$

and for every $x_0 \in \mathbb{R}^3 \setminus \Omega$*, there exists a unique Carathéodory solution* $x(\cdot)$ *of (21) starting from* x_0*, maximally defined on* $[0, +\infty)$*, and satisfying* $x(t) \in \mathbb{R}^3 \setminus \Omega'$*, for every* $t > 0$*.*

- Finite time convergence
  For every $R > 0$*, there exists* $\tau_{opt} = \tau_{opt}(R) > 0$ *such that, for all* $e, d : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ *satisfying the regularity assumptions (7) and (23), for every* $x_0 \in \mathbb{R}^3$ *with* $|x_0| \leqslant R$*, and every maximal solution* $x(\cdot)$ *of (21) starting from* x_0*, one has*

$$|x(t)| \leqslant \delta_{opt}(R),\ \forall t \geqslant 0, \tag{24}$$

$$x(t) = 0,\ \forall t \geqslant \tau_{opt}, \tag{25}$$

  and

$$\|u_{opt}(x(t))\| \leqslant 1,\ \forall t \geqslant 0. \tag{26}$$

- Optimality
  For every neighborhood Ω *of* $\mathcal{S}$*, every* $\varepsilon > 0$*, and every compact subset* K *of* $\mathbb{R}^3$*, there exists a continuous function* $\rho_{\varepsilon,K} : \mathbb{R}^3 \to \mathbb{R}$ *satisfying (22) such that, for all* $e, d : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ *satisfying the regularity assumptions (7) and*

$$\begin{aligned} &\sup_{[0,+\infty)} |e(x,\cdot)| \leqslant \min(\rho_{opt}(d(x,\mathcal{S})), \rho_{\varepsilon,K}(x)), \\ &\operatorname{esssup}_{[0,+\infty)} |d(x,\cdot)| \leqslant \min(\rho_{opt}(d(x,\mathcal{S}), \rho_{\varepsilon,K}(x)),\ \forall x \in \mathbb{R}^3, \end{aligned} \tag{27}$$

  and for every $x_0 \in K \cap (\mathbb{R}^3 \setminus \Omega)$*, the solution of (21), starting from* x_0*, reaches* 0 *within time* $T(x_0) + \varepsilon$*.*

Proof. Since Carathéodory conditions hold for system (21), the existence of a unique forward Carathéodory solution of (21), for every initial condition, is ensured. Note that, since the controller u_{opt} is the minimal time control steering x to the origin under the constraint (3), the inequality (26) holds. Since the optimal controller u_{opt} defined by (19) is continuous on $\mathbb{R}^3 \setminus \mathcal{S}$, Lemma 1 implies the existence of $\rho_{opt} : [0, +\infty) \to [0, +\infty)$. The last part of the result follows from (19) and from the continuity of solutions with respect to disturbances. ■

4.2. The second component of the hysteresis

In this section, we define the second component of the hysteresis, which consists of a set of controllers, defined in a neighborhood of $\mathcal{S}$.

Since $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^3$ of codimension greater than or equal to one (see Proposition 1), there exists a partition $(M_i)_{i\in\mathbb{N}}$ of $\mathcal{S}$, where M_i is a stratum, i.e. a locally closed submanifold of $\mathbb{R}^3$. Recall that, if $M_i \cap \partial M_j \neq \emptyset$, then $M_i \subset M_j$ and $\dim(M_i) < \dim(M_j)$.

LEMMA 3 *For every $\varepsilon > 0$, there exists a neighborhood Ω of $\mathcal{S}$ such that, for every stratum M_i of $\mathcal{S}$, there exist a nonempty subset $\mathcal{N}_i$ of $\mathbb{N}$, a locally finite family $(\Omega_{i,p})_{p\in\mathcal{N}_i}$ of open subsets of Ω, a sequence of smooth controllers $u_{i,p}$ defined in a neighborhood of $\Omega_{i,p}$, satisfying $\|u_{i,p}\| \leqslant 1$, and there exists a continuous function $\rho_{i,p} : \mathbb{R}^3 \to [0,+\infty)$ satisfying $\rho_{i,p}(x) > 0$ whenever $x \neq 0$, such that every solution of*

$$\dot{x}(t) = f(x(t), u_{i,p}(x(t) + e(x(t),t))) + d(x(t),t), \tag{28}$$

where $e, d : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{R}^3$ are two functions satisfying the regularity assumptions (7) and

$$\sup_{[0,+\infty)}|e(x,\cdot)| \leqslant \rho_{i,p}(x), \quad \operatorname{esssup}_{[0,+\infty)}|d(x,\cdot)| \leqslant \rho_{i,p}(x), \tag{29}$$

starting from $\Omega_{i,p}$ and maximally defined on $[0,T)$, leaves Ω within time ε; moreover, there exists a function $\delta_{i,p}$ of class $\mathcal{K}_\infty$ such that, for every $R > 0$, every such solution starting from $\Omega_{i,p} \cap B(0,R)$ satisfies

$$|x(t)| \leqslant \delta_{i,p}(R), \quad \forall t \in [0,T). \tag{30}$$

This lemma is proved in Prieur, Trélat (2006). It can be applied here, due to the crucial fact that $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^3$. For the sake of completeness, we however give a proof below.

Proof. First of all, recall that, under the Lie Algebra Rank Condition, the topology defined by the sub-Riemannian distance d_{SR} coincides with the Euclidean topology of $\mathbb{R}^3$, and that, since $\mathbb{R}^3$ is complete, any two points of $\mathbb{R}^3$ can be joined by a minimizing path.

Let $\varepsilon > 0$ be fixed. Since $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^3$, there exists a neighborhood Ω of $\mathcal{S}$ satisfying the following property: for every $y \in \mathcal{S}$, there exists $z \in \mathbb{R}^3 \setminus \operatorname{clos}(\Omega)$ such that $d_{SR}(y,z) < \varepsilon$.

Consider a stratum M_i of $\mathcal{S}$. For every $y \in M_i$, let $z \in \mathbb{R}^3 \setminus \operatorname{clos}(\Omega)$ such that $d_{SR}(y,z) < \varepsilon$. The Lie Algebra Rank Condition implies that there exists an open-loop control $t \mapsto u_y(t)$, defined on $[0,T)$ for a $T > \varepsilon$, satisfying the constraint $\|u_y\| \leqslant 1$, such that the associated trajectory $x_y(\cdot)$ (which can be assumed to be one-to-one), a solution of the Martinet system, starting from y, reaches z (and thus, leaves $\operatorname{clos}(\Omega)$) within time ε. Using a density argument, the control u_y can be, moreover, chosen as a smooth function. Since the trajectory is one-to-one, the open-loop control $t \mapsto u_y(t)$ can be considered as a feedback $t \mapsto u_y(x_y(t))$ along $x_y(\cdot)$. Consider a smooth extension of u_y on $\mathbb{R}^3$, still denoted u_y, satisfying the constraint $\|u_y(x)\| \leqslant 1$, for every $x \in \mathbb{R}^3$. By continuity, there exists a neighborhood Ω_y of y, and positive real numbers δ_y and ρ_y, such that every solution of

$$\dot{x}(t) = f(x(t), u_y(x(t) + e(x(t),t))) + d(x(t),t), \tag{31}$$

where $e, d : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{R}^3$ are two functions satisfying the regularity assumptions (7) and

$$\sup_{[0,+\infty)}|e(x,\cdot)| \leqslant \rho_y, \quad \operatorname{esssup}_{[0,+\infty)}|d(x,\cdot)| \leqslant \rho_y ,$$

starting from Ω_y and maximally defined on $[0,T)$, leaves Ω within time ε; moreover,

$$|x(t)| \leqslant \delta_y, \ \forall t \in [0,T).$$

Repeat this construction for each $y \in M_i$.

Now, on the one hand, let $(y_p)_{p\in\mathcal{N}_i}$ be a sequence of points of M_i such that the family $(\Omega_{y_p})_{p\in\mathcal{N}_i}$ is a locally finite covering of M_i, where $\mathcal{N}_i$ is a subset of $\mathbb{N}$. Define $\Omega_{i,p} = \Omega_{y_p}$ and $u_{i,p} = u_{y_p}$.

On the other hand, the existence of a continuous function $\rho_{i,p} : \mathbb{R}^3 \to [0,+\infty)$, satisfying $\rho_{i,p}(x) > 0$ whenever $x \neq 0$, follows for the continuity of solutions with respect to disturbances. The existence of a function $\delta_{i,p}$ of class $\mathcal{K}_\infty$ such that (30) holds, is obvious.

Repeat this construction for every stratum M_i of $\mathcal{S}$. Then, the statement of the lemma follows. ∎

4.3. Definition of the hybrid feedback law

The following construction of the hybrid feedback law, using an hysteresis, is already known. The definitions and properties recalled in this section readily follow those of Prieur, Trélat (2006), and are given hereafter for the sake of completeness.

Let $\mathcal{F} = \{1,\ldots,7\}$, and $\mathcal{N}$ be a countable set. In the sequel, Greek letters refer to elements of $\mathcal{N}$. Fix ω, an element of $\mathcal{N}$, considered as the *largest* element of $\mathcal{N}$, *i.e.*, ω is *greater* than any other element of $\mathcal{N}$. We, however, do not introduce any order in $\mathcal{N}$. This element ω has actually a particular role in the sequel, since it will refer to the optimal controller in the hybrid feedback law defined below.

Given a set-valued map $F : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$, the solutions $x(\cdot)$ of the differential inclusion $\dot{x} \in F(x)$ consist of all absolutely continuous functions satisfying $\dot{x}(t) \in F(x(t))$ almost everywhere.

Definition 7 *The family* $(\mathbb{R}^3 \setminus \{0\}, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{N}})$ *is said to satisfy the property* $(\mathcal{P})$ *if:*

1. *for every* $(\alpha, l) \in \mathcal{N}\times\mathcal{F}$, $\Omega_{\alpha,l}$ *is an open subset of* $\mathbb{R}^3$*;*
2. *for every* $\alpha \in \mathcal{N}$, *and every* $m > l \in \mathcal{F}$,

$$\Omega_{\alpha,l} \subsetneq \operatorname{clos}(\Omega_{\alpha,l}) \subsetneq \Omega_{\alpha,m}; \qquad (32)$$

3. *for every* α *in* $\mathcal{N}$, g_α *is a smooth vector field, defined in a neighborhood of* $\mathrm{clos}(\Omega_{\alpha,7})$*, taking values in* $\mathbb{R}^3$*;*
4. *for every* $(\alpha, l) \in \mathcal{N} \times \mathcal{F}$*,* $l \leqslant 6$*, there exists a continuous function* $\rho_{\alpha,l} : \mathbb{R}^3 \to [0, +\infty)$ *satisfying* $\rho_{\alpha,l}(x) \neq 0$ *whenever* $x \neq 0$ *such that every maximal solution* $x(\cdot)$ *of*

$$\dot{x} \in g_\alpha(x) + B(0, \rho_{\alpha,l}(x)); \tag{33}$$

defined on $[0, T)$ *and starting from* $\partial\Omega_{\alpha,l}$*, is such that*

$$x(t) \in \mathrm{clos}(\Omega_{\alpha,l+1}) \ , \ \forall t \in [0, T);$$

5. *for every* $l \in \mathcal{F}$*, the sets* $(\Omega_{\alpha,l})_{\alpha\in\mathcal{N}}$ *form a locally finite covering of* $\mathbb{R}^3 \setminus \{0\}$*.*

We next define a class of hybrid controllers as those considered in Section 2 (see also Prieur, Goebel, Teel, 2005).

Definition 8 *Let* $(\mathbb{R}^3 \setminus \{0\}, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{N}})$ *satisfy the property* $(\mathcal{P})$ *as in Definition 7. Assume that, for every* α *in* $\mathcal{N}$*, there exists* $k_\alpha \in \mathbb{R}^2$*, satisfying the constraint (3), such that, for every* x *in a neighborhood of* $\Omega_{\alpha,7}$*,*

$$g_\alpha(x) = f(x, k_\alpha). \tag{34}$$

Set

$$D_1 = \Omega_{\omega,2}, \tag{35}$$

$$D_{\alpha,2} = \mathbb{R}^3 \setminus \Omega_{\alpha,6}. \tag{36}$$

Let (C, D, k, k_d) *be the hybrid feedback defined by*

$$C = \Big\{(x,\alpha) \mid x \in \Big(\mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1}\Big)\Big\}, \tag{37}$$

$$D = \{(x, \alpha) \mid x \in D_1 \cup D_{\alpha,2}\}, \tag{38}$$

$$\begin{array}{rcll} k : \mathbb{R}^3 \times \mathcal{N} & \to & \mathbb{R}^2 & \\ (x,\alpha) & \mapsto & k_\alpha(x) & \text{if} \quad x \in \Omega_{\alpha,7}, \\ & & 0 & \text{else,} \end{array} \tag{39}$$

and

$$\begin{array}{rcll} k_d : \mathbb{R}^3 \times \mathcal{N} & \rightrightarrows & \mathcal{N} & \\ (x,\alpha) & \mapsto & \alpha' > \alpha, & \text{if } x \in \mathrm{clos}(\Omega_{\alpha',1} \cap D_{\alpha,1}) \text{ and if } x \notin D_{\alpha,2}, \\ & & \alpha, & \text{if } x \in \mathrm{clos}(\Omega_{\alpha,1} \cap D_{\alpha,2}). \end{array} \tag{40}$$

The 4-tuple (C, D, k, k_d) *is a hybrid feedback law on* $\mathbb{R}^3$ *as considered in Section 2.2. We denote by* $\mathcal{H}_{(e,d)}$ *the system (6) in closed-loop with such a feedback with the perturbations* e *and* d *as measurement noise and external disturbance respectively.*

To investigate the robustness properties, we introduce the following definition (see Prieur, Trélat, 2006).

Definition 9 *Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ be a continuous map such that $\chi(x) > 0$, for every $x \neq 0$.*

- *We say that χ is an admissible radius for the measurement noise, if, for every $x \in \mathbb{R}^3$ and every $\alpha \in \mathcal{N}$, such that $x \in \Omega_{\alpha,7}$,*

$$\chi(x) < \frac{1}{2} \min_{l \in \{1,\dots,6\}} d(\mathbb{R}^3 \setminus \Omega_{\alpha,l+1}, \Omega_{\alpha,l}). \tag{41}$$

- *We say that χ is an admissible radius for the external disturbances if, for every $x \in \mathbb{R}^3$, we have*

$$\chi(x) \leqslant \max_{(\alpha,l),\ x \in \Omega_{\alpha,l}} \rho_{\alpha,l}(x).$$

There exists an admissible radius for the measurement noise and for the external disturbances (note that, from (32), the right-hand side of the inequality (41) is positive).

Consider an admissible radius χ for the measurement noise and the external disturbances. Let e and d be a measurement noise and an external disturbance respectively, such that, for all $(x,t) \in \mathbb{R}^3 \times [0, +\infty)$,

$$e(x,t) \leqslant \chi(x), \quad d(x,t) \leqslant \chi(x). \tag{42}$$

The properties of the solutions of the closed-loop with the hybrid feedback law defined in Definition 8 have been stated in Prieur, Trélat (2006) (see also Prieur, Goebel, Teel, 2005; Prieur, Trélat, 2005a), and we briefly recall them.

1. For all $(x_0, s_0) \in \mathbb{R}^3 \times \mathcal{N}$, there exists a solution of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0). Recall that a Zeno-solution is a complete solution, whose domain of definition is bounded in the t-direction. A solution (x, s_d), defined on a hybrid domain S, is an instantaneous Zeno-solution, if there exist $t \geqslant 0$ and an infinite number of $j \in \mathbb{N}$ such that $(t,j) \in S$.
2. There do not exist instantaneous Zeno-solutions, although a finite number of switches may occur at the same time.
3. Let (x, s_d) be a maximal solution of $\mathcal{H}_{(e,d)}$ defined on a hybrid time S. Suppose that the supremum T of S in the t-direction is finite. Then, $\limsup_{t \to T, (t,l) \in S} |x(t,l)| = +\infty$.
4. For every $\alpha \in \mathcal{A}$, set

$$\tau_\alpha = \sup\big\{T \mid x \text{ is a Carathéodory solution of } \dot{x} \in f(x, k_\alpha) + B(0, \chi(x)) \text{ with } x(t) \in \Omega_{\alpha,7},\ \forall t \in [0,T)\big\}\ . \tag{43}$$

Note that, at this stage, there may exist $\alpha \in \mathcal{A}$ such that $\tau_\alpha = +\infty$. Let (x, s_d) be a solution of $\mathcal{H}_{(e,d)}$ defined on a hybrid time domain S and starting in $\mathbb{R}^3 \setminus \{0\} \times \mathcal{N}$. Let T be the supremum in the t-direction of S. Then, one of the two following cases may occur:

- either there exists no positive jump time, more precisely there exists $\alpha \in \mathcal{N}$ such that,
  (a) for almost every $t \in (0, T)$ and for every l such that $(t, l) \in S$, one has $k(s_d(t, l)) = k_\alpha$;
  (b) the map x is a Carathéodory solution of $\dot{x} = f(x, k_\alpha) + d$ on $(0, T)$;
  (c) for every $t \in (0, T)$, and every l such that $(t, l) \in S$, one has $x(t, l) + e(x(t, l), t) \in \mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1}$;
  (d) for all $(t, l) \in S$, $t > 0$, one has $x(t, l) + e(x(t, l), t) \notin D$, where D is defined by (38);
  (e) the inequality $T < \tau_\alpha$ holds.
- or there exists a unique positive jump time, more precisely there exist $\alpha \in \mathcal{N} \setminus \{\omega\}$ and $t_1 \in (0, T)$ such that, letting $t_0 = 0$, $t_2 = T$, $\alpha_0 = \alpha$, and $\alpha_1 = \omega$, for every $j = 0, 1$, the following properties hold:
  (f) for almost every $t \in (t_j, t_{j+1})$ and for every l such that $(t, l) \in S$, one has $k(s_d(t, l)) = k_{\alpha_j}$;
  (g) the map x is a Carathéodory solution of $\dot{x} = f(x, k_{\alpha_j}) + d$ on (t_j, t_{j+1});
  (h) for every $t \in (t_0, t_1)$, and every l such that $(t, l) \in S$, one has $x(t, l) + e(x(t, l), t) \in \mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1}$;
  (i) for every t in (t_j, t_{j+1}), and every l such that $(t, l) \in S$, one has $x(t, l) + e(x(t, l), t) \notin D_{\alpha_j,2}$, where $D_{\alpha_j,2}$ is defined by (36);
  (j) the inequality $t_1 < \tau_{\alpha_0}$ holds.

We next define our hybrid feedback law. Let $\varepsilon > 0$ and K be a compact subset of $\mathbb{R}^3$. Let Ω be the neighborhood of $\mathcal{S}$ given by Lemma 3. For this neighborhood Ω, let $\Omega' \subset \Omega$ be the neighborhood of $\mathcal{S}$ yielded by Lemma 1.

Let $\mathcal{N}$ be the countable set defined by

$$\mathcal{N} = \{(i, p),\ i \in \mathbb{N},\ p \in \mathcal{N}_i\} \cup \{\omega\},$$

where ω is an element of $\mathbb{N} \times \mathbb{N}$, distinct from every (i, p), $i \in \mathbb{N}$, $p \in \mathcal{N}_i$.

We proceed in two steps.

Step 1: Definition of k_α and $\Omega_{\alpha,l}$, where $\alpha \in \mathcal{N} \setminus \{\omega\}$ and $l \in \mathcal{F}$

Let $i \in \mathbb{N}$. Lemma 3, applied with the stratum M_i, implies the existence of a family of smooth controllers $(k_{i,p})_{p \in \mathcal{N}_i}$ satisfying the constraint (5), and of a family of neighborhoods $(\Omega_{i,p,7})_{p \in \mathcal{N}_i}$. The existence of the families $(\Omega_{i,p,1})_{p \in \mathcal{N}_i}, \ldots, (\Omega_{i,p,6})_{p \in \mathcal{N}_i}$, satisfying

$$\Omega_{i,p,l} \subsetneq \mathrm{clos}(\Omega_{i,p,l}) \subsetneq \Omega_{i,p,m},$$

for every $m > l \in \mathcal{F}$, follows from a finite induction argument, using Lemma 3.

Step 2: Definition of k_ω and $\Omega_{\omega,l}$, where $l \in \mathcal{F}$

Let $\Omega_{\omega,1}$ be an open set of $\mathbb{R}^3$ containing $\mathbb{R}^3 \setminus \bigcup_{\alpha\in\mathcal{N}\setminus\{\omega\}} \Omega_{\alpha,1}$ and contained in $\mathbb{R}^3 \setminus \mathcal{S}$. The point 0 belongs to $\mathrm{clos}(\Omega_{\omega,1})$. Lemma 1, applied with $\Omega = \mathbb{R}^3 \setminus \mathrm{clos}(\Omega_{\omega,1})$, allows to define k_ω as k_{opt}, and Ω' as a closed subset of $\mathbb{R}^3$ such that

$$\Omega' \subsetneq \Omega, \tag{44}$$

and such that Ω' is a neighborhood of $\mathcal{S}$. Set $\Omega_{\omega,2} = \mathbb{R}^3 \setminus \Omega'$; it is an open subset of $\mathbb{R}^3$, contained in $\mathbb{R}^3 \setminus \mathcal{S}$. Moreover, from (44),

$$\Omega_{\omega,1} \subsetneq \mathrm{clos}(\Omega_{\omega,1}) \subsetneq \Omega_{\omega,2}.$$

The existence of the sets $\Omega_{\omega,3}$, ... $\Omega_{\omega,7}$ follows from a finite induction argument, using Lemma 1. Moreover, from Lemma 2, we have the following property: for every $l \in \{1, \dots, 6\}$, for every $x_0 \in \Omega_{\omega,l}$, the unique Carathéodory solution $x(\cdot)$ of (21), with $x(0) = x_0$, satisfies $x(t) \in \Omega_{\omega,l+1}$, for every $t \geqslant 0$.

From the two previous steps, we can easily check that all requirements of Definition 7 are satisfied: The family $(\mathbb{R}^3 \setminus \{0\}, ((\Omega_{\alpha,\mathrm{l}})_{\mathrm{l}\in\mathcal{F}}, \mathrm{g}_\alpha)_{\alpha\in\mathcal{N}})$ satisfies the property $(\mathcal{P})$, where g_α is a function defined in a neighborhood of $\Omega_{\alpha,7}$ by

$$g_\alpha(x) = f(x, k_\alpha).$$

The hybrid feedback law (C, D, k, k_d) is then defined according to Definition 8.

4.4. Proof of the main result

Now that all ingredients have been introduced, the main result follows.

Let $\varepsilon > 0$, and K be a compact subset of $\mathbb{R}^3$. Consider the hybrid feedback law (C, D, k, k_d), defined previously. Let $\chi : \mathbb{R}^3 \to \mathbb{R}_{\geqslant 0}$ be an admissible radius for the external disturbances and the measurement noise (see Definition 9). So as to reduce this function, we assume that, for every $\alpha \in \mathcal{N} \setminus \{\omega\}$,

$$\chi(x) \leqslant \rho_{opt}(d(x, \mathcal{S})), \quad \forall x \in \Omega_{\omega,7}, \tag{45}$$

$$\chi(x) \leqslant \rho_\alpha(x), \quad \forall x \in \Omega_{\alpha,7}. \tag{46}$$

Note that, from the choice of the components of the hybrid feedback law, and from Lemma 3, for every $\alpha \in \mathcal{N} \setminus \{\omega\}$, the constant τ_α defined by (4) satisfies $\tau_\alpha < \varepsilon$.

Let us prove that the point 0 is a semi-globally minimal time robustly stable equilibrium for the system $\mathcal{H}_{(e,d)}$ in a closed-loop with the hybrid feedback law (C, D, k, k_d) as stated in Theorem 1.

Step 1: Completeness and global stability

Let $R > 0$ and $\delta : [0, +\infty) \to [0, +\infty)$ of class $\mathcal{K}_\infty$ be such that, for every $\alpha \in \mathcal{N} \setminus \{\omega\}$,

$$\delta(x) \leqslant \delta_{opt}(R), \quad \forall x \in \Omega_{\omega,7}, \tag{47}$$

$$\delta(x) \leqslant \delta_{\alpha}(R), \quad \forall x \in \Omega_{\alpha,7}. \tag{48}$$

Let e, d be two functions satisfying the regularity assumptions and (42). Let (x, s_d) be a maximal solution of $\mathcal{H}_{(e,d)}$ on a hybrid domain S starting from (x_0, s_0), with $|x_0| < R$. From Lemmas 2 and 3, we have, for every $(t, l) \in S$,

$$|x(t, j)| \leqslant \delta(R). \tag{49}$$

Therefore, since $\limsup_{t \to T, (t,l) \in S} |x(t, l)| \neq +\infty$, the supremum T of S in the t-direction is infinite, and the maximality property follows. The stability property follows from (49).

Step 2: Uniform finite time convergence property

Let $x_0 \in B(0, R)$, and $s_0 \in \mathcal{N}$. Let (x, s_d) denote the solution of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0).

If $x_0 = 0$, then, using (39) and the fact that $\chi(0) = 0$, the solution remains at the point 0.

We next assume that $x_0 \neq 0$. Let $\alpha_0 \in \mathcal{N}$ such that $x(\cdot)$ is a solution of $\dot{x} = f(x, k_{\alpha_0}(x)) + d$ on $(0, t_1)$ for a $t_1 > 0$.

If $\alpha_0 = \omega$, then the feedback law under consideration coincides with the optimal controller and there does not exist any switching time $t > 0$. Then, from Lemma 2, the solution reaches 0 within time $T(x_0) + \varepsilon$.

If $\alpha_0 \neq \omega$, then, from Lemma 3, the solution leaves $\Omega_{\alpha_0,7}$ within time ε and then enters the set $\Omega_{\omega,7}$. Therefore, with (4), the solution reaches 0 within time $T(x_1) + \varepsilon$, where x_1 denotes the point of the solution $x(\cdot)$ when entering $\Omega_{\omega,7}$.

Let $\tau(R) = \max_{x \in \delta(R)} T(x) + \varepsilon$. With (49), we get (17) and the uniform finite time property. Note that, from Lemma 3, the constraint (18) is satisfied.

Step 3: Quasi-optimality

Let K be a compact subset of $\mathbb{R}^3$, and $(x_0, s_0) \in K \times \mathcal{N}$. Let $R > 0$ be such that $K \subset B(0, R)$. From the previous arguments, two cases occur:

- the solution starting from (x_0, s_0) reaches 0 within time $T(x_0)+\varepsilon$ whenever $\alpha_0 = \omega$;
- the solution starting from (x_0, s_0) reaches 0 within time $T(x_1) + \varepsilon$, whenever $\alpha_0 \neq \omega$, where x_1 denotes the point of the solution $x(\cdot)$ when entering $\Omega_{\omega,7}$. Up to reducing the neighborhoods $\Omega_{\alpha,l}$, one has $|T(x_0)-T(x_1)| \leqslant \varepsilon$. Indeed, T is uniformly continuous on the compact K.

Hence, the maximal solution starting from (x_0, s_0) reaches 0 within time $T(x_0)+2\varepsilon$. This is the quasi-optimality property.

Theorem 1 is proved.

5. Conclusion and further comments

In our main result, we proved that the origin is semi-globally minimal time robustly stabilizable, for the Martinet system (1), (3). This proves that the main assumption of Prieur, Trélat (2006), namely, the absence of singular minimizing trajectories, is not necessary to ensure such a stabilization result. Actually, the crucial fact used in our proof relies on the stratification properties of the minimal time function. This holds whenever the minimal time function belongs to the subanalytic class, or to the log-exp class. More generally, this holds in a o-minimal class. For general analytic control systems of the form (4), (5), in the absence of singular minimizing trajectory, the minimal time function to a point can be proved to be subanalytic outside this point (see Prieur, Trélat, 2006). In the Martinet case, the minimal time function is not subanalytic, due to the presence of a singular minimizing trajectory, however, it belongs to the log-exp class, which is also o-minimal, and hence, is still stratifiable.

This situation extends to the so-called Martinet integrable case (see Bonnard, Launay, Trélat, 2001). In a neighborhood of 0, a model of this latter case is given by the two vector fields

$$f_1 = g_1(x_2)\left(\frac{\partial}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial}{\partial x_3}\right),\ f_2 = g_2(x_2)\frac{\partial}{\partial x_2},$$

where g_1 and g_2 are germs of analytic functions at 0 such that $g_i(0) = 1$. It is proved in Bonnard, Launay, Trélat (2001) that the minimal time function still belongs to the log-exp class in this case. This is, however, no longer true whenever the functions g_1, g_2 also depend on x_1 and x_3. In this case, it is conjectured in Bonnard, Trélat (2001), Trélat (2000a) that the minimal time function does not belong to the log-exp class. In this latter case, a larger class is due for describing the regularity of the minimal time function, but it is not clear if it is possible to find a suitable o-minimal class. This problem is very intricate and is actually related to the Hilbert 16th problem (see Trélat, 2000a). Hence, in the general case where singular minimizing trajectories exist, this problem of regularity is widely open. Although less general, it is, however, intimately related to the problem of robust stabilization.

References

Agrachev, A. (1998) Compactness for SR minimizers and subanalyticity. *Rend. Sem. Mat. Pol. Univ. Torino,* **56**.

Agrachev, A., Bonnard, B., Chyba, M. and Kupka, I. (1997) Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.* **2**, 377–448.

Agrachev, A. and Gauthier, J.P. (2001) On subanalyticity of Carnot-Caratheodory distances. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (3).

ANCONA, F. and BRESSAN, A. (2002) Flow stability of patchy vector fields and robust feedback stabilization. *SIAM J. Cont. Opt.* **41** (5), 1455–1476.

BENSOUSSAN, A. and MENALDI, J.L. (1997) Hybrid control and dynamic programming. *Dyn. Cont. Discrete Impulsive Syst.* **3** (3), 395–442.

BONNARD, B. and CHYBA, M. (2003) Singular trajectories and their role in control theory. *Math. & Appl.* **40** (Berlin).

BONNARD, B., LAUNAY, G. and TRÉLAT, E. (2001) The transcendance needed to compute the sphere and wave front in Martinet sub-Riemannian geometry. *Geometric control theory* (Moscow, 1998), 82–117, *Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz.* **64**, Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 1999. English version in *J. Math. Sciences* **103** (6), 686–708.

BONNARD, B. and TRÉLAT, E. (2001) On the role of abnormal minimizers in sub-Riemannian geometry. *Ann. Fac. Sci. Toulouse Math.* (6) **10**, 3, 405–491.

BRANICKY, M.S. (1998) Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automat. Cont.* **43**, 475–482.

BROCKETT, R.W. (1983) Asymptotic stability and feedback stabilization. In: R.W. Brockett, R.S. Millman and H.J. Sussmann, eds., *Differential geometric control theory.* Birkhäuser, Boston, 181–191.

CLARKE, F.H., LEDYAEV, YU.S., RIFFORD, L. and STERN, R.J. (2000) Feedback stabilization and Lyapunov functions. *SIAM J. Cont. Opt.* **39** (1), 25–48.

CLOSKEY, R.T. and MURRAY, R.M. (1997) Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. *IEEE Trans. Automat. Control* **42** (5), 614–628.

CORON, J.-M. (1992) Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Syst.* **5**, 295–312.

VAN DEN DRIES, L., MACINTYRE, A. and MARKER, D. (1994) The elementary theory of restricted analytic fields with exponentiation. *Ann. of Math.* **140**, 183–205.

VAN DEN DRIES, L. and MILLER, C. (1996) Geometric categories and o-minimal structures. *Duke Math. Journal* **84** (2).

GOEBEL, R. and TEEL, A.R. (2006) Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica* **42**, 573–587.

GOEBEL, R., HESPANHA, J., TEEL, A.R., CAI, C. and SANFELICE, R. (2004) Hybrid systems: generalized solutions and robust stability. *IFAC Symp. on Nonlinear Control Systems*, Stuttgart, Germany.

HARDT, R.M. (1975) Stratification of real analytic mappings and images. *Invent. Math.* **28**.

HIRONAKA, H. (1973) Subanalytic sets. In: *Number theory, algebraic geometry*

and commutative algebra, in honor of Y. Akizuki, Tokyo.
Jacquet, S. (2001) Regularity of sub-Riemannian distance and cut locus. *Lecture Notes in Control and Information Sciences* **258**. In: A. Isidori, F. Lamnabhi Lagarrigue and W. Respondek, eds., *Nonlinear Control in the Year 2000.* Springer.
Ledyaev, Yu.S. and Sontag, E.D. (1997) A remark on robust stabilization of general asymptotically controllable systems. *Proc. Conf. on Information Sciences and Systems*, Johns Hopkins, Baltimore, 246–251.
Liberzon, D. (2003) Switching in systems and control. In: *Systems and control: foundations and applications,* Birkhäuser.
Litsyn, E., Nepomnyashchikh, Y.V. and Ponosov, A. (2000) Stabilization of linear differential systems via hybrid feedback controls. *SIAM J. Cont. Opt.* **38**, 1468–1480.
Lygeros, J., Johansson, K.H., Simic, S.N., Zhang, J. and Sastry, S.S. (2003) Dynamical properties of hybrid automata. *IEEE Trans. Automat. Control* **48** (1), 2–17.
Morin, P. and Samson, C. (2003) Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Trans. Automat. Control* **48** (9), 1496–1508.
Pontryagin, L.S., Boltyanskij, V.G., Gamkrelidze, R.V. and Mishchenko, E.F. (1962)] *The Mathematical Theory of Optimal Processes.* Interscience Publishers, John Wiley & Sons, New York.
Prieur, C. (2001) Uniting local and global controllers with robustness to vanishing noise. *Math. Control Signals Systems* **14**, 143–172.
Prieur, C. (2005) Asymptotic controllability and robust asymptotic stabilizability. *SIAM J. Cont. Opt.* **43** (5), 1888–1912.
Prieur, C. and Astolfi, A. (2003) Robust stabilization of chained systems via hybrid control. *IEEE Trans. Automat. Control* **48** (10), 1768–1772.
Prieur, C., Goebel, R. and Teel, A. (2005) Results on robust stabilization of asymptotically controllable systems by hybrid feedback. *IEEE Conf. Dec. Contr. and Eur. Cont. Conf. (CDC-ECC'05)*, Seville, Spain.
Prieur, C. and Trélat, E. (2005a) Robust optimal stabilization of the Brockett integrator via a hybrid feedback. *Math. Control Signals Syst.* **17** (3), 201–216.
Prieur, C. and Trélat, E. (2005b) Semi-global minimal time hybrid robust stabilization of analytic driftless control-affine systems. *IEEE Conf. Dec. Contr. and Eur. Cont. Conf. (CDC-ECC'05)*, Seville, Spain.
Prieur, C. and Trélat, E. (2006) Quasi-optimal robust stabilization of control systems. *SIAM J. Control Optim.* **45** (5), 1875-1997.
Rifford, L. (2004) The stabilization problem: AGAS and SRS feedbacks. In: *Optimal Control, Stabilization, and Nonsmooth Analysis, Lectures Notes in Control and Information Sciences* **301**, Springer-Verlag, Heidelberg, 173–184.
Sontag, E.D. (1999) Clocks and insensitivity to small measurement errors.

ESAIM Cont. Opt. Calc. Var. **4**, 537–557.

SONTAG, E.D. (1999) Stability and stabilization: Discontinuities and the effect of disturbances. In: F.H. Clarke and R.J. Stern, eds., *Nonlinear Analysis, Differential Equations, and Control, Proc. NATO Advanced Study Institute*. Kluwer, Montreal, 551–598.

SUSSMANN, H.J. (1979) Subanalytic sets and feedback control. *J. Diff. Eq.* **31** (1), 31–52.

TAMM, M. (1981) Subanalytic sets in the calculus of variation. *Acta Math.* **146**.

TAVERNINI, L. (1987) Differential automata and their discrete simulators. *Nonlinear Anal.* **11**, 665–683.

TRÉLAT, E. (2000a) Etude asymptotique et transcendance de la fonction valeur en contrôle optimal ; catégorie log-exp en géométrie sous-riemannienne dans le cas Martinet. PhD Thesis, Univ. Dijon, France.

TRÉLAT, E. (2000b) Some properties of the value function and its level sets for affine control systems with quadratic cost. *J. Dyn. Cont. Syst.* **6** (4), 511–541.

YE, H., MICHEL, A.N. and HOU, L. (1998) Stability theory for hybrid dynamical systems. *IEEE Trans. Automat. Control* **43** (4), 461–474.